Student Notes and Problems

ANSWERS AND SOLUTIONS

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RADICAL EXPRESSIONS AND EQUATIONS

Lesson 1—Simplifying Radical Expressions

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. a) $\sqrt{150}$
$= \sqrt{25\times 6}$
$= \sqrt{25}\times\sqrt{6}$
$= 5\sqrt{6}$

b) $12a\sqrt{32a^3}$
$= 12a\sqrt{16}\sqrt{2}\sqrt{a^2}\sqrt{a}$
$= 12a(4)\left(\sqrt{2}\right)(a)\left(\sqrt{a}\right)$
$= 48a^2\sqrt{2a}$

c) $5x^2z\sqrt{9x^2y^5z}L$
$= 5x^2z\sqrt{(9)\left(x^2\right)\left(y^2\right)\left(y^2\right)(y)(z)}$
$= 5x^2z\sqrt{9}\sqrt{x^2}\sqrt{y^2}\sqrt{y^2}\sqrt{y}\sqrt{z}$
$= 5x^2z(3)(x)(y)(y)\sqrt{y}\sqrt{z}$
$= 15x^3y^2z\sqrt{yz}$

d) $\sqrt[3]{128m^4n^{10}}$
$= \sqrt[3]{(64)(2)\left(m^3\right)(m)\left(n^3\right)\left(n^3\right)\left(n^3\right)(n)}$
$= \left(\sqrt[3]{64\times 2}\right)\left(\sqrt[3]{m^3\times m}\right)\left(\sqrt[3]{n^3\times n^3\times n^3\times n}\right)$
$= 4\times\sqrt[3]{2}\times m\times\sqrt[3]{m}\times n\times n\times n\times\sqrt[3]{n}$
$= 4mn^3\sqrt[3]{2mn}$

2. a) $12\sqrt{3}$
$= \sqrt{12^2\times 3}$
$= \sqrt{432}$

b) $\frac{1}{3}\sqrt{45}$
$= \sqrt{\left(\frac{1}{3}\right)^2\times 45}$
$= \sqrt{5}$

c) $2\sqrt[3]{8}$
$= \sqrt[3]{2^3\times 8}$
$= \sqrt[3]{64}$

d) $8a\sqrt{a^3b}$
$= \sqrt{\left(8^2\right)\left(a^2\right)\left(a^3b\right)}$
$= \sqrt{64a^5b}$

3. Step 1
Express each mixed radical as an entire radical.

$3\sqrt{8} = \sqrt{72}$ $\quad\sqrt{59}$
$2\sqrt{10} = \sqrt{40}$ $\quad 4\sqrt{7} = \sqrt{112}$
$7\sqrt{2} = \sqrt{98}$

Step 2
Compare the size of the radicands, and order them from largest to smallest.
$\sqrt{112}, \sqrt{98}, \sqrt{72}, \sqrt{59}, \sqrt{40}$

Step 3
Order the radicals in their original form from largest to smallest.

The radicals in order from largest to smallest are $4\sqrt{7}, 7\sqrt{2}, 3\sqrt{8}, \sqrt{59},$ and $2\sqrt{10}$.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. $\sqrt{90}$
$= \sqrt{9\times 10}$
$= 3\sqrt{10}$

2. $3\sqrt{16}$
$= 3(4)$
$= 12$

3. $18\sqrt{2x^5}$
$= 18\sqrt{2\left(x^2\right)\left(x^2\right)(x)}$
$= 18(x)(x)\sqrt{2x}$
$= 18x^2\sqrt{2x}$

4. $7a\sqrt{68a^2b^2}$
$= 7a\sqrt{(4)(17)a^2b^2}$
$= 7a(2)(a)(b)\sqrt{17}$
$= 14a^2b\sqrt{17}$

5. $5\sqrt[3]{297}$
$= 5\sqrt[3]{27\times 11}$
$= 15\sqrt[3]{11}$

6. $2\sqrt{108x^8}$
$= 2\sqrt{(36)(3)(x^2)(x^2)(x^2)(x^2)}$
$= 2(6)(x)(x)(x)(x)\sqrt{3}$
$= 12x^4\sqrt{3}$

7. $2\sqrt{7}$
$= \sqrt{2^2\times 7}$
$= \sqrt{28}$

8. $12x\sqrt{2}$
$= \sqrt{(12^2)(x^2)(2)}$
$= \sqrt{288x^2}$

9. $3b^2\sqrt{13b}$
$= \sqrt{(3b^2)^2(13b)}$
$= \sqrt{(9b^4)(13b)}$
$= \sqrt{117b^5}$

10. $\frac{1}{4}\sqrt[3]{320}$
$= \sqrt[3]{\left(\frac{1}{4}\right)^3\times 320}$
$= \sqrt[3]{5}$

11. Step 1
Find decimal approximations, or express the mixed radicals in entire form.

$2\sqrt{13}$
$= \sqrt{52}$
$\doteq$ [redacted]

$8\sqrt{2}$
$= \sqrt{128}$
$\doteq$ [redacted]

$7\sqrt{8}$
$= \sqrt{392}$
$\doteq$ [redacted]

$\sqrt{55}$
$\doteq$ [redacted]

$5\sqrt{4}$
$= \sqrt{100}$
$= 10$

Step 2
Compare the size of the radicands, and order them from smallest to largest.
$\sqrt{52}$, $\sqrt{55}$, $\sqrt{100}$, $\sqrt{128}$, $\sqrt{392}$

Step 3
Order the radicals in their original form from smallest to largest.

In order from smallest to largest, the radicals are $2\sqrt{13}$, $\sqrt{55}$, $5\sqrt{4}$, $8\sqrt{2}$, and $7\sqrt{8}$.

Lesson 2—Adding and Subtracting Radicals

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **a)** $3\sqrt{6}+\sqrt{6}=4\sqrt{6}$

b) $8\sqrt{12}-\sqrt{27}$
$= 16\sqrt{3}-3\sqrt{3}$
$= 13\sqrt{3}$

c) $6\sqrt[3]{64}-4$
$= 24-4$
$= 20$

2. **a)** $3\sqrt{18}+\sqrt{3}-\sqrt{2}+6$
$= 9\sqrt{2}+\sqrt{3}-\sqrt{2}+6$
$= 8\sqrt{2}+\sqrt{3}+6$

b) $2\sqrt[3]{24}+\sqrt{27}+\sqrt{5}-6\sqrt{5}$
$= 2\sqrt[3]{(8)(3)}+\sqrt{(9)(3)}+\sqrt{5}-6\sqrt{5}$
$= 4\sqrt[3]{3}+3\sqrt{3}-5\sqrt{5}$

3. **a)** $4\sqrt{9}-3\sqrt{9}$
$=\sqrt{9}$
$=3$

Alternate solution:
$4\sqrt{9}-3\sqrt{9}$
$=4(3)-3(3)$
$=12-9$
$=3$

b) $4\sqrt{12}+\sqrt{12}$
$=5\sqrt{12}$
$=5\sqrt{(4)(3)}$
$=10\sqrt{3}$

Alternate solution:
$4\sqrt{12}+\sqrt{12}$
$=4\sqrt{(4)(3)}+\sqrt{(4)(3)}$
$=8\sqrt{3}+2\sqrt{3}$
$=10\sqrt{3}$

4. The perimeter of a rectangle can be found by adding the two lengths and two widths together.
$\sqrt{80}+\sqrt{80}+2\sqrt{5}+2\sqrt{5}$
$=2\sqrt{80}+4\sqrt{5}$
$=2\sqrt{(16)(5)}+4\sqrt{5}$
$=8\sqrt{5}+4\sqrt{5}$
$=12\sqrt{5}$

The perimeter of the rectangle is $12\sqrt{5}$ cm.

5. $\sqrt{12x^7}-7x^2\sqrt{3x^3}+3x^3\sqrt{27x}$
$=2x^3\sqrt{3x}-7x^3\sqrt{3x}+9x^3\sqrt{3x}$
$=4x^3\sqrt{3x}$

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. $2\sqrt{25}+\sqrt{25}-\sqrt{35}-8$
$=2(5)+5-\sqrt{35}-8$
$=7-\sqrt{35}$

2. $-9\sqrt{32}+2\sqrt{7}-5\sqrt[3]{54}-\sqrt[3]{56}$
$=-9\sqrt{(16)(2)}+2\sqrt{7}-5\sqrt[3]{(27)(2)}-\sqrt[3]{(8)(7)}$
$=-36\sqrt{2}+2\sqrt{7}-15\sqrt[3]{2}-2\sqrt[3]{7}$

3. $\sqrt{99x^4}+2\sqrt{11x^8}$
$=\sqrt{(9)(11)(x^2)(x^2)}+2\sqrt{11(x^2)(x^2)(x^2)(x^2)}$
$=3x^2\sqrt{11}+2x^4\sqrt{11}$

4. $\sqrt{12}+\sqrt{27}-5\sqrt{x}=0$
$2\sqrt{3}+3\sqrt{3}-5\sqrt{x}=0$
$5\sqrt{3}-5\sqrt{x}=0$

For the equation to equal zero, $5\sqrt{3}$ will be subtracted from $5\sqrt{3}$.
$5\sqrt{3}-5\sqrt{x}=0$
$5\sqrt{3}-5\sqrt{3}=0$

Therefore, $x = 3$.

5. $\sqrt{(16)(2)}-\sqrt{(9)(2)}$
$=4\sqrt{2}-3\sqrt{2}$
$=\sqrt{2}$

6. $y\sqrt{72x^5}+2x\sqrt{18x^3y^2}$
$=6x^2y\sqrt{2x}+6x^2y\sqrt{2x}$
$=12x^2y\sqrt{2x}$

7. The student's work is not correct. To combine radicals, only the coefficients of like radicals are added.
$\sqrt{25}+\sqrt{16}$
$=5+4$
$=9$

Lesson 3—Multiplying and Dividing Radicals

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. a) $\left(3\sqrt{10}\right)\left(8\sqrt{5}\right)$
$= 24\sqrt{50}$
$= 120\sqrt{2}$

b) $\left(2x^4\sqrt{3}\right)\left(4x\sqrt{18}\right)$
$= 8x^5\sqrt{54}$
$= 24x^5\sqrt{6}$

c) $2\sqrt[3]{250} \times 2\sqrt[3]{81}$
$= 2\sqrt[3]{(125)(2)} \times 2\sqrt[3]{(27)(3)}$
$= 10\sqrt[3]{2} \times 6\sqrt[3]{3}$
$= 60\sqrt[3]{6}$

d) $\left(5\sqrt{36x^5}\right)\left(\sqrt{162x^2}\right)$
$= \left(30x^2\sqrt{x}\right)\left(9x\sqrt{2}\right)$
$= 270x^3\sqrt{2x}$

2. $4\sqrt{3}\left(\sqrt{6} - 2\sqrt{15}\right)$
$= 4\sqrt{3}\left(\sqrt{6}\right) - 4\sqrt{3}\left(2\sqrt{15}\right)$
$= 4\sqrt{18} - 8\sqrt{45}$
$= 12\sqrt{2} - 24\sqrt{5}$

3. $\left(5\sqrt{3} + 3\sqrt{6}\right)\left(\sqrt{5} + 2\sqrt{10}\right)$
$= \left(5\sqrt{3}\right)\left(\sqrt{5}\right) + \left(5\sqrt{3}\right)\left(2\sqrt{10}\right)$
$+ \left(3\sqrt{6}\right)\left(\sqrt{5}\right) + \left(3\sqrt{6}\right)\left(2\sqrt{10}\right)$
$= 5\sqrt{15} + 10\sqrt{30} + 3\sqrt{30} + 6\sqrt{60}$
$= 5\sqrt{15} + 13\sqrt{30} + 6\sqrt{60}$
$= 5\sqrt{15} + 13\sqrt{30} + 6\sqrt{4 \times 15}$
$= 5\sqrt{15} + 13\sqrt{30} + 12\sqrt{15}$
$= 17\sqrt{15} + 13\sqrt{30}$

4. a) $\dfrac{12\sqrt{18}}{4\sqrt{2}}$
$= \dfrac{12}{4}\sqrt{\dfrac{18}{2}}$
$= 3\sqrt{9}$
$= 9$

b) $\dfrac{28\sqrt{15}}{56\sqrt{60}}$
$= \dfrac{28}{56}\sqrt{\dfrac{15}{60}}$
$= \dfrac{1}{2}\sqrt{\dfrac{1}{4}}$
$= \left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right)$
$= \dfrac{1}{4}$

c) $\dfrac{\sqrt{x^8}}{\sqrt{x^{12}}}$
$= \dfrac{1}{\sqrt{x^4}}$
$= \dfrac{1}{x^2}$

5. a) $\dfrac{12}{\sqrt{3}}$
$= \dfrac{12}{\sqrt{3}}\left(\dfrac{\sqrt{3}}{\sqrt{3}}\right)$
$= \dfrac{12\sqrt{3}}{\sqrt{9}}$
$= \dfrac{12\sqrt{3}}{3}$
$= 4\sqrt{3}$

b) $\dfrac{2\sqrt[3]{8}}{\sqrt[3]{4}}$
$= \dfrac{4}{\sqrt[3]{4}}\left(\dfrac{\sqrt[3]{2}}{\sqrt[3]{2}}\right)$
$= \dfrac{4\sqrt[3]{2}}{\sqrt[3]{8}}$
$= \dfrac{4\sqrt[3]{2}}{2}$
$= 2\sqrt[3]{2}$

c) $\frac{3\sqrt{x^7}}{12\sqrt{x}}$
$= \frac{\sqrt{x^6}}{4}$
$= \frac{x^3}{4}$

If you simplify first, you can sometimes avoid rationalizing the denominator.

d) $\frac{6}{3-\sqrt{5}}$
$= \frac{6}{3-\sqrt{5}}\left(\frac{3+\sqrt{5}}{3+\sqrt{5}}\right)$
$= \frac{18+6\sqrt{5}}{9+3\sqrt{5}-3\sqrt{5}-5}$
$= \frac{18+6\sqrt{5}}{4}$
$= \frac{9+3\sqrt{5}}{2}$

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. $\left(2\sqrt{100}\right)\left(3\sqrt{100}\right)\left(\frac{1}{25}\sqrt{54}\right)$
$=(20)(30)\left(\frac{3}{25}\sqrt{6}\right)$
$=72\sqrt{6}$

2. **Step 1**
Simplify the left side of the equation.
$\left(a\sqrt{2}\right)\left(2\sqrt{5}\right)=8\sqrt{10}$
$2a\sqrt{10}=8\sqrt{10}$

Step 2
Solve for a.
$2a=8$
$a=4$

3. $\left(3\sqrt{8x^2}\right)\left(4\sqrt{9x^5}\right)$
$=12\sqrt{72x^7}$
$=12\left(\sqrt{36}\sqrt{2}\sqrt{x^2}\sqrt{x^2}\sqrt{x^2}\sqrt{x^1}\right)$
$=72x^3\sqrt{2x}$

4. The conjugate of $\left(2\sqrt{7}-3\sqrt{4}\right)$ is $\left(2\sqrt{7}+3\sqrt{4}\right)$.
$2\sqrt{7}\left(2\sqrt{7}+3\sqrt{4}\right)$
$=2\sqrt{7}\left(2\sqrt{7}\right)+2\sqrt{7}\left(3\sqrt{4}\right)$
$=4(7)+6\sqrt{28}$
$=28+12\sqrt{7}$

5. $\frac{\sqrt{72}}{2\sqrt{5}}\times\frac{3\sqrt{54}}{2\sqrt{8}}$
$=\frac{6\sqrt{2}}{2\sqrt{5}}\times\frac{9\sqrt{6}}{4\sqrt{2}}$
$=\frac{54\sqrt{12}}{8\sqrt{10}}$
$=\frac{27\sqrt{6}}{4\sqrt{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right)$
$=\frac{27\sqrt{30}}{20}$

6. Substitute $x = 2$, and simplify.
$\frac{\sqrt{48}}{2\sqrt{3x}}$
$=\frac{\sqrt{48}}{2\sqrt{3(2)}}$
$=\frac{4\sqrt{3}}{2\sqrt{6}}$
$=\frac{2}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)$
$=\frac{2\sqrt{2}}{2}$
$=\sqrt{2}$

7. $\frac{2-\sqrt{3}}{\sqrt{5}+1}$
$=\frac{2-\sqrt{3}}{\sqrt{5}+1}\left(\frac{\sqrt{5}-1}{\sqrt{5}-1}\right)$
$=\frac{2\sqrt{5}-2-\sqrt{15}+\sqrt{3}}{5-\sqrt{5}+\sqrt{5}-1}$
$=\frac{2\sqrt{5}-2-\sqrt{15}+\sqrt{3}}{4}$

Lesson 4—Solving Radical Equations

CLASS EXERCISES
ANSWERS AND SOLUTIONS

1. $x+2 \geq 0$
$x \geq -2$

2. **Step 1**
Isolate the radical on one side of the equation.
$\sqrt{x+1}-3=-6$
$\sqrt{x+1}=-3$

Step 2
Square both sides of the equation, and solve for x.
$\left(\sqrt{x+1}\right)^2=(-3)^2$
$x+1=9$
$x=8$

Step 3
Verify the solution by substituting it into the original equation.

x = **8**	
LHS	**RHS**
$\sqrt{8+1}-3$	-6
$\sqrt{9}-3$	-6
0	-6

Since LHS ≠ RHS, x = 8 is not a solution.
Therefore, there is no solution, or $\{\ \}$.

3. **Step 1**
Since the radical is isolated, square both sides of the equation and expand.
$\sqrt{4x+4}=x+1$
$\left(\sqrt{4x+4}\right)^2=(x+1)^2$
$4x+4=(x+1)(x+1)$
$4x+4=x^2+2x+1$

Step 2
Gather like terms, and write a quadratic equation equal to 0.
$4x+4=x^2+2x+1$
$0=x^2-2x-3$

Step 3
Factor the quadratic equation, and solve for x.
$0=x^2-2x-3$
$0=(x+1)(x-3)$
$x+1=0$ or $x-3=0$
$x=-1$ $\quad$ $x=3$

Step 4
Verify the solutions by substituting them into the original equation.

x = **–1**	
LHS	**RHS**
$\sqrt{4(-1)+4}$	–1 + 1
$\sqrt{0}$	0
0	0

x = **3**	
LHS	**RHS**
$\sqrt{4(3)+4}$	3 + 1
$\sqrt{16}$	4
4	4

Therefore, the solutions are x = –1 and x = 3, or {–1, 3}.

4. **Step 1**
Isolate the radical on one side of the equation.
$x+2-\sqrt{x^2+12}=0$
$x+2=\sqrt{x^2+12}$

Step 2
Square both sides of the equation, and expand.
$(x+2)^2=\left(\sqrt{x^2+12}\right)^2$
$(x+2)(x+2)=x^2+12$
$x^2+4x+4=x^2+12$

Step 3
Gather like terms, and solve for x.
$x^2+4x+4=x^2+12$
$4x=8$
$x=2$

Step 4
Verify the solution by substituting it into the original equation.

x = **2**	
LHS	**RHS**
$2+2-\sqrt{(2)^2+12}$	0
$4-\sqrt{16}$	0
4 – 4	0
0	0

Therefore, the solution is $x = 2$, or {2}.

5. **Step 1**
Isolate the more complex radical.

$$\sqrt{8+x}+4=8-\sqrt{x}$$
$$\sqrt{8+x}=4-\sqrt{x}$$

Step 2
Square both sides of the equation, and expand.

$$\left(\sqrt{8+x}\right)^2=\left(4-\sqrt{x}\right)^2$$
$$8+x=\left(4-\sqrt{x}\right)\left(4-\sqrt{x}\right)$$
$$8+x=16-4\sqrt{x}-4\sqrt{x}+x$$

Step 3
Gather like terms, and isolate the radical on one side.

$$8+x=16-4\sqrt{x}-4\sqrt{x}+x$$
$$8+\cancel{x}=16-8\sqrt{x}+\cancel{x}$$
$$8\sqrt{x}=8$$
$$\sqrt{x}=\frac{8}{8}$$
$$\sqrt{x}=1$$

Step 4
Square both sides of the equation, and solve for x.

$$\sqrt{x}=1$$
$$\left(\sqrt{x}\right)^2=(1)^2$$
$$x=1$$

x = **1**	
LHS	**RHS**
$\sqrt{8+1}+4$	$8-\sqrt{1}$
$\sqrt{9}+4$	8 – 1
7	7

Step 5
Verify the solution by substituting it into the original equation.

Therefore, the solution is $x = 1$, or {1}.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Square both sides of the equation.

$$\sqrt{x-2}=4$$
$$\left(\sqrt{x-2}\right)^2=4^2$$
$$x-2=16$$

Step 2
Solve for x.

$$x-2=16$$
$$x=18$$

Step 3
Verify the solution by substituting it into the original equation.

x = **18**	
LHS	**RHS**
$\sqrt{18-2}$	4
$\sqrt{16}$	4
4	4

Therefore, the solution is $x = 18$, or {18}.

2. **Step 1**
Square both sides of the equation, and expand.

$$\sqrt{x^2+5}=5-x$$
$$\left(\sqrt{x^2+5}\right)^2=(5-x)^2$$
$$x^2+5=(5-x)(5-x)$$
$$x^2+5=25-10x+x^2$$

Step 2
Gather like terms, and solve for x.

$$\cancel{x^2}+5=25-10x+\cancel{x^2}$$
$$10x=20$$
$$x=2$$

Step 3
Verify the solution by substituting it into the original equation.

x = **2**	
LHS	**RHS**
$\sqrt{2^2+5}$	$5-2$
$\sqrt{9}$	3
3	3

Therefore, the solution is $x = 2$, or {2}.

3. Step 1
Square both sides of the equation.

$$\sqrt{x+2} = x$$
$$\left(\sqrt{x+2}\right)^2 = (x)^2$$
$$x+2 = x^2$$

Write a quadratic equation equal to 0.

$$x+2 = x^2$$
$$0 = x^2 - x - 2$$

4
Factor the quadratic equation, and solve for x.

$$0 = x^2 - x - 2$$
$$0 = (x+1)(x-2)$$
$$x+1=0 \quad \text{or} \quad x-2=0$$
$$x=-1 \qquad\qquad x=2$$

Step 4
Verify the solutions by substituting them into the original equation.

x = **–1**	
LHS	**RHS**
$\sqrt{-1+2}$	–1
$\sqrt{1}$	–1
1	–1

x = **2**	
LHS	**RHS**
$\sqrt{2+2}$	2
$\sqrt{4}$	2
2	2

Therefore, the solution is $x = 2$, or {2}.

4. Step 1
Isolate the radical on one side of the equation.

$$\sqrt{a^2+6} - a + 3 = 0$$
$$\sqrt{a^2+6} = a - 3$$

Step 2
Square both sides of the equation, and expand.

$$\sqrt{a^2+6} = a - 3$$
$$\left(\sqrt{a^2+6}\right)^2 = (a-3)^2$$
$$a^2 + 6 = a^2 - 6a + 9$$

Step 3
Gather like terms, and solve for x.

$$\cancel{a^2} + 6 = \cancel{a^2} - 6a + 9$$
$$6a = 3$$
$$a = \frac{1}{2}$$

Step 4
Verify the solution by substituting it into the original equation.

$\boldsymbol{a = \frac{1}{2}}$	
LHS	**RHS**
$\sqrt{\left(\frac{1}{2}\right)^2+6} - \frac{1}{2} + 3$	0
$\sqrt{\frac{25}{4}} + \frac{5}{2}$	0
$\frac{5}{2} + \frac{5}{2}$	0
5	0

Therefore, there is no solution, or $\varnothing$.

5. Step 1
Isolate the radical on one side of the equation.

$$2 - \sqrt{2x-5} = x$$
$$2 - x = \sqrt{2x-5}$$

Step 2
Square both sides of the equation, and expand.

$$2 - x = \sqrt{2x-5}$$
$$(2-x)^2 = \left(\sqrt{2x-5}\right)^2$$
$$4 - 4x + x^2 = 2x - 5$$

Step 3
Write a quadratic equation equal to 0.

$$4-4x+x^2=2x-5$$
$$x^2-6x+9=0$$

Step 4
Factor the quadratic equation, and solve for x.

$$\begin{aligned} x^2-6x+9&=0\\ (x-3)(x-3)&=0\\ x-3&=0\\ x&=3 \end{aligned}$$

Step 5
Verify the solution by substituting it into the original equation.

$x = 3$	
LHS	**RHS**
$2-\sqrt{2(3)-5}$	3
$2-\sqrt{1}$	3
2 – 1	3
1	3

Therefore, there is no solution, or $\varnothing$.

6. **Step 1**
Since one of the radicals is already isolated, square both sides of the equation and expand.

$$\begin{aligned} \sqrt{4x+5}&=2+\sqrt{2x-1}\\ \left(\sqrt{4x+5}\right)^2&=\left(2+\sqrt{2x-1}\right)^2\\ 4x+5&=4+2\sqrt{2x-1}+2\sqrt{2x-1}+2x-1 \end{aligned}$$

Step 2
Gather like terms, and isolate the second radical on one side.

$$\begin{aligned} 4x+5&=4+2\sqrt{2x-1}+2\sqrt{2x-1}+2x-1\\ 4x+5&=3+2x+4\sqrt{2x-1}\\ 2x+2&=4\sqrt{2x-1}\\ x+1&=2\sqrt{2x-1} \end{aligned}$$

Step 3
Square both sides of the equation, and solve for x.

$$\begin{aligned} x+1&=2\sqrt{2x-1}\\ (x+1)^2&=\left(2\sqrt{2x-1}\right)^2\\ x^2+2x+1&=4(2x-1)\\ x^2+2x+1&=8x-4\\ x^2-6x+5&=0 \end{aligned}$$

Step 4
Factor the quadratic equation, and solve for x.

$$\begin{aligned} x^2-6x+5&=0\\ (x-1)(x-5)&=0 \end{aligned}$$

$$x-1=0 \quad \text{or} \quad x-5=0$$
$$x=1 \qquad\qquad x=5$$

$x = 1$	
LHS	**RHS**
$\sqrt{4(1)+5}$	$2+\sqrt{2(1)-1}$
$\sqrt{9}$	$2+\sqrt{1}$
3	3

Step 5
Verify the solutions by substituting them into the original equation.

$x = 5$	
LHS	**RHS**
$\sqrt{4(5)+5}$	$2+\sqrt{2(5)-1}$
$\sqrt{25}$	$2+\sqrt{9}$
5	5

Therefore, the solutions are $x = 1$ and $x = 5$, or $\{1, 5\}$.

7. **Step 1**
Translate the word problem into an equation.

$$x=\sqrt{x}+6$$

Step 2
Solve the radical equation.

$$\begin{aligned} x&=\sqrt{x}+6\\ (x-6)^2&=\left(\sqrt{x}\right)^2\\ x^2-12x+36&=x\\ x^2-13x+36&=0\\ (x-4)(x-9)&=0 \end{aligned}$$

$$x-4=0 \quad \text{or} \quad x-9=0$$
$$x=4 \qquad\qquad x=9$$

Step 3
Verify the solutions by substituting them into the original equation.

$x = 4$	
LHS	**RHS**
4	$\sqrt{4}+6$
4	2 + 6
4	8

$x = 9$	
LHS	**RHS**
9	$\sqrt{9}+6$
9	3 + 6
9	9

Therefore, the solution is $x = 9$, or {9}.

Practice Test

ANSWERS AND SOLUTIONS

1. $3ab^2\sqrt[3]{2}$
$= \sqrt[3]{\left(3ab^2\right)^3(2)}$
$= \sqrt[3]{\left(27a^3b^6\right)(2)}$
$= \sqrt[3]{54a^3b^6}$

2. $\sqrt{28x^3y^5}$
$= \sqrt{4\times 7\times x^2\times x\times y^2\times y^2\times y}$
$= 2xy^2\sqrt{7xy}$

3. The equation $\sqrt{25}-\sqrt{9}$ does not equal $\sqrt{16}$.
If each radical is simplified, the result is as follows:
$\sqrt{25}-\sqrt{9}=\sqrt{16}$
$5-3=4$
$2\neq 4$

The LHS does not equal the RHS of the equation, which means you cannot add or subtract radicals by combining the radicands. The correct way to combine radicals is to add or subtract the coefficients of like radicals:
$a\sqrt{x}-b\sqrt{x}=(a-b)\sqrt{x}$.

4. To write an equivalent form of the number 11 as an entire radical under a square root, use the concept of writing a mixed radical as an entire radical.
$11=\sqrt{11^2}$
$=\sqrt{121}$

5. $2\sqrt{98}+\sqrt{10}-5\sqrt{8}-3\sqrt{40}$
$=2\sqrt{49}\sqrt{2}+\sqrt{10}-5\sqrt{4}\sqrt{2}-3\sqrt{4}\sqrt{10}$
$=14\sqrt{2}+\sqrt{10}-10\sqrt{2}-6\sqrt{10}$
$=4\sqrt{2}-5\sqrt{10}$

6. $\sqrt{125+71}$
$=\sqrt{196}$
$=14$

7. $12\sqrt{2}\times 2\sqrt{3}\times 2\sqrt{2}$
$=48\sqrt{12}$
$=48\sqrt{4}\sqrt{3}$
$=96\sqrt{3}$

8. $\sqrt{3}\left(2x^2\sqrt{6x}+\sqrt{6x^5}\right)$
$=\sqrt{3}\left(2x^2\sqrt{6x}\right)+\sqrt{3}\left(\sqrt{6x^5}\right)$
$=2x^2\sqrt{18x}+\sqrt{18x^5}$
$=2x^2\sqrt{9}\sqrt{2x}+\sqrt{9x^4}\sqrt{2x}$
$=6x^2\sqrt{2x}+3x^2\sqrt{2x}$
$=9x^2\sqrt{2x}$

As an alternate solution, you could simplify the terms in the brackets first.
$\sqrt{3}\left(2x^2\sqrt{6x}+\sqrt{6x^5}\right)$
$=\sqrt{3}\left(2x^2\sqrt{6x}+x^2\sqrt{6x}\right)$
$=\sqrt{3}\left(3x^2\sqrt{6x}\right)$
$=3x^2\sqrt{18x}$
$=9x^2\sqrt{2x}$

9. $\dfrac{\sqrt{5}+1}{\sqrt{5}-1}$
$=\left(\dfrac{\sqrt{5}+1}{\sqrt{5}-1}\right)\left(\dfrac{\sqrt{5}+1}{\sqrt{5}+1}\right)$
$=\dfrac{5+\sqrt{5}+\sqrt{5}+1}{5+\sqrt{5}-\sqrt{5}-1}$
$=\dfrac{6+2\sqrt{5}}{4}$
$=\dfrac{3+\sqrt{5}}{2}$

10. $\dfrac{8\sqrt{18}}{24\sqrt{21}}$

$= \dfrac{\sqrt{6}}{3\sqrt{7}}$

$= \dfrac{\sqrt{6}}{3\sqrt{7}}\left(\dfrac{\sqrt{7}}{\sqrt{7}}\right)$

$= \dfrac{\sqrt{42}}{21}$

11. Step 1

Square both sides of the equation.

$$\sqrt{1-x} = 3$$
$$\left(\sqrt{1-x}\right)^2 = (3)^2$$
$$1-x = 9$$

Step 2

Solve for x.

$$1-x = 9$$
$$-8 = x$$

Step 3

Verify the solution by substituting it into the original equation.

x = **–8**	
LHS	**RHS**
$\sqrt{1-(-8)}$	3
$\sqrt{9}$	3
3	3

Therefore, the solution is x = –8, or {–8}.

12. Step 1

Square both sides of the equation, and expand.

$$\sqrt{a-3} = a-5$$
$$\left(\sqrt{a-3}\right)^2 = (a-5)^2$$
$$a-3 = a^2-10a+25$$

Step 2

Write a quadratic equation equal to 0.

$$a-3 = a^2-10a+25$$
$$0 = a^2-11a+28$$

Step 3

Factor the quadratic equation, and solve for x.

$$0 = a^2-11a+28$$
$$0 = (a-4)(a-7)$$

$a-4=0$ or $a-7=0$

$a=4$ $\qquad a=7$

Step 4

Verify the solutions by substituting them into the original equation.

a = **4**	
LHS	**RHS**
$\sqrt{4-3}$	4 – 5
$\sqrt{1}$	–1
1	–1

a = **7**	
LHS	**RHS**
$\sqrt{7-3}$	7 – 5
$\sqrt{4}$	2
2	2

Therefore, the solution is a = 7, or {7}.

13. Step 1

Isolate the radical on one side of the equation.

$$\sqrt{7y+2}-y = -4$$
$$\sqrt{7y+2} = -4+y$$

Step 2

Square both sides of the equation, and expand.

$$\left(\sqrt{7y+2}\right)^2 = (-4+y)^2$$
$$7y+2 = 16-8y+y^2$$

Step 3

Gather like terms, and solve for y.

$$7y+2 = 16-8y+y^2$$
$$0 = y^2-15y+14$$
$$0 = (y-1)(y-14)$$

$y-1=0$ or $y-14=0$

$y=1$ $\qquad y=14$

Step 4

Verify the solutions by substituting them into the original equation.

y = **1**y = 1	
LHS	**RHS**
$\sqrt{7(1)+2}-(1)$	–4
$\sqrt{9}-1$	–4
2	–4

y = **14**	
LHS	**RHS**
$\sqrt{7(14)+2}-(14)$	–4
$\sqrt{100}-14$	–4
10 – 14	–4
–4	–4

Therefore, the solution is y = 14, or {14}.

14. Step 1

Isolate the radical with the variable on one side of the equation.

$$\sqrt{12}+\sqrt{27}-5\sqrt{m}=0$$
$$\sqrt{12}+\sqrt{27}=5\sqrt{m}$$

Step 2

Simplify the left side of the equation.

$$\sqrt{12}+\sqrt{27}=5\sqrt{m}$$
$$2\sqrt{3}+3\sqrt{3}=5\sqrt{m}$$
$$5\sqrt{3}=5\sqrt{m}$$

Step 3

Solve for *m*.

Since $5\sqrt{3}=5\sqrt{m}$, then $m = 3$.

15. The formula for the volume of a cube is $V = s^3$, where V is the volume and s is the side length. Substitute the volume measure into the formula, and solve for s.

$$V=s^3$$
$$200=s^3$$
$$\sqrt[3]{200}=s$$
$$\sqrt[3]{8}\times\sqrt[3]{25}=s$$
$$2\sqrt[3]{25}=s$$

The measure of one side of a cube with a volume of 200 cm^3 is $2\sqrt[3]{25}$ cm .

16. Step 1

Isolate the more complex radical.

$$\left(\sqrt{8x+20}\right)-\left(\sqrt{2x+5}\right)-3=0$$
$$\left(\sqrt{8x+20}\right)=\sqrt{2x+5}+3$$

Step 2

Square both sides of the equation, and expand.

$$\left(\sqrt{8x+20}\right)=\sqrt{2x+5}+3$$
$$\left(\sqrt{8x+20}\right)^2=\left(\sqrt{2x+5}+3\right)^2$$
$$8x+20=\left(\sqrt{2x+5}+3\right)\left(\sqrt{2x+5}+3\right)$$
$$8x+20=(2x+5)+6\sqrt{2x+5}+9$$

Step 3

Collect like terms, and isolate the radical on one side.

$$8x+20=(2x+5)+6\sqrt{2x+5}+9$$
$$8x+20=2x+14+6\sqrt{2x+5}$$
$$6x+6=6\sqrt{2x+5}$$
$$x+1=\sqrt{2x+5}$$

Step 4

Square both sides of the equation, and set up a quadratic equation.

$$x+1=\sqrt{2x+5}$$
$$(x+1)^2=\left(\sqrt{2x+5}\right)^2$$
$$x^2+\cancel{2x}+1=\cancel{2x}+5$$
$$x^2-4=0$$

Step 5

Factor the difference of squares, and solve for *x*.

$$x^2-4=0$$
$$(x+2)(x-2)=0$$
$$x+2=0 \quad \text{or} \quad x-2=0$$
$$x=-2 \qquad\qquad x=2$$

Step 6

Verify the solutions by substituting them into the original equation.

$x = -2$	
LHS	**RHS**
$\left(\sqrt{8(-2)+20}\right)-\left(\sqrt{2(-2)+5}\right)-3$	0
$\left(\sqrt{4}\right)-\left(\sqrt{1}\right)-3$	0
–2	0

$x = 2$	
LHS	**RHS**
$\left(\sqrt{8(2)+20}\right)-\left(\sqrt{2(2)+5}\right)-3$	0
$\left(\sqrt{36}\right)-\left(\sqrt{9}\right)-3$	0
0	0

Therefore, the solution is $x = 2$, or {2}.

RATIONAL EXPRESSIONS AND EQUATIONS

Lesson 1—Introduction to Rational Expressions and Non-Permissible Values

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **a)** The expression $\frac{4x^3+1}{x}$ is a rational expression since the numerator and denominator are polynomials.

b) The expression $\sqrt{6}p$ is a polynomial because the radical sign ($\sqrt{\ }$) does not include the variable. The expression $\frac{m-n}{\sqrt{6}p}$ is a rational expression since the numerator and denominator are polynomials.

c) The expression $\frac{x^2+2\sqrt{x}}{3x-1}$ is not a rational expression because there is a variable with a fractional degree ($\sqrt{x}$ is $x^{\frac{1}{2}}$ as a power) in the numerator. Therefore, the numerator is not a polynomial.

d) The expression $9c^5+8^d$ is not a rational expression because the numerator in $\frac{9c^5+8^d}{1}$ contains a term with a variable exponent. Therefore, the numerator is not a polynomial.

2. **a)** If –4*x* in the denominator equals 0, the expression $\frac{3x^2}{-4x}$ is undefined.
Therefore, the NPV is $x=0$.

b) If (2*x* – 6) in the denominator equals 0, the expression $\frac{x^2-1}{2x-6}$ is undefined.

$$\begin{aligned}2x-6&=0\\2x&=6\\x&=3\end{aligned}$$

Therefore, the NPV is $x = 3$.

3. **a)** If (x^2 – 16) in the denominator equals 0, the expression $\frac{4x}{x^2-16}$ is undefined.

$$\begin{aligned}x^2-16&=0\\x^2&=16\\x&=\pm\sqrt{16}\\x&=\pm4\end{aligned}$$

Therefore, the NPVs are $x = 4$ and $x = –4$.

Alternate solution:

$$\begin{aligned}x^2-16&=0\\(x-4)(x+4)&=0\end{aligned}$$

$$\begin{aligned}x-4=0 &\quad\text{or}\quad x+4=0\\x=4 &\quad\quad\quad\ x=-4\end{aligned}$$

b) If (a^2 – 9*a* +18) in the denominator equals 0, the expression $\frac{a^2-9}{a^2-9a+18}$ is undefined.

$$\begin{aligned}a^2-9a+18&=0\\(a-3)(a-6)&=0\end{aligned}$$

$$\begin{aligned}a-3=0 &\quad\text{or}\quad a-6=0\\a=3 &\quad\quad\quad\ a=6\end{aligned}$$

Therefore, the NPVs are $a = 3$ and $a = 6$.

c) If (*m* + 3*m*) in the denominator equals 0, the expression $\frac{m^2-2m+1}{m+3m}$ is undefined.

$$\begin{aligned}m+3m&=0\\4m&=0\\m&=\frac{0}{4}\\m&=0\end{aligned}$$

Therefore, the NPV is $m = 0$.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. B

The expression $\sqrt{2}x$ is rational because the radical sign ($\sqrt{}$) does not include the variable, and the expression could be written as a fraction, $\frac{\sqrt{2}x}{1}$.

The expressions $\frac{\sqrt{x}}{y}$, $\frac{24}{x^{\frac{3}{2}}}$, and $\sqrt[3]{x}+\sqrt[3]{8}$ each contain terms where the variables have fractional degrees.

2. C

The expression $\frac{1}{x^{3x}}$ is not rational because there is a variable exponent in the denominator. Therefore, the denominator is not a polynomial.

The expression $x^{(2+2)}$ has an unusual notation for the exponent, but the exponent is still an integer, making it rational. The expressions $\frac{x-1}{x^2}$ and $2x+7$ are quotients of two polynomials, where the latter has a denominator of 1.

3. If $(x-2)$ or (x) in the denominator equals 0, the expression $\frac{(x+2)(2x)}{3(x-2)(x)}$ is undefined.

Therefore, $x \neq 0$ and $x \neq 2$.

4. If $\left(x^2-9\right)$ in the denominator equals 0, the expression $\frac{5\left(x^2-1\right)}{x^2-9}$ is undefined.

$$x^2-9=0$$
$$(x-3)(x+3)=0$$
$$x-3=0 \quad \text{or} \quad x+3=0$$
$$x=3 \qquad\qquad x=-3$$

Therefore, $x \neq 3$ and $x \neq -3$.

5. If $(6x^2-15x)$ in the denominator equals 0, the expression $\frac{x+6}{6x^2-15x}$ is undefined.

$$6x^2-15x=0$$
$$3x(2x-5)=0$$

Therefore, $x \neq 0$ and $x \neq \frac{5}{2}$.

6. If $(x+2)$ or (x^2+5x+6) in the denominator equals 0, the expression $\frac{2+3x}{2(x+2)\left(x^2+5x+6\right)}$ is undefined.

$$x+2=0 \quad \text{or} \quad x^2+5x+6=0$$
$$x=-2 \qquad (x+2)(x+3)=0$$
$$x=-2,-3$$

Therefore, $x \neq -2$ and $x \neq -3$. It is not necessary to list the –2 value twice.

Lesson 2—Simplifying Rational Expressions

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. a) Factor the denominator, and reduce by a factor of 3 to simplify the expression.

The NPVs can be identified using the original denominator in factored form.

$$\frac{12x}{3x-9}$$
$$=\frac{\cancel{12}^{4}x}{\cancel{3}(x-3)}$$
$$=\frac{4x}{x-3}, x \neq 3$$

b) Reduce the coefficients in the monomials by a common factor of 8, and use the quotient law of exponents to simplify the expression.

The NPVs can be identified using the original denominator.

$$\frac{8x^3}{48x}$$
$$=\frac{\cancel{8}x^{3-1}}{\underset{6}{\cancel{48}}}$$
$$=\frac{x^2}{6},\ x\neq 0$$

c) Factor where possible, and eliminate common factors to simplify the expression.

The NPVs can be identified using the original denominator in factored form.

$$\frac{10x^2+2x}{6x^2+4x}$$
$$=\frac{\cancel{2x}(5x+1)}{\cancel{2x}(3x+2)}$$
$$=\frac{5x+1}{3x+2},\ x\neq 0,-\frac{2}{3}$$

2. a) Factor where possible, and eliminate common factors to simplify the expression.

The NPVs can be identified using the original denominator. The binomial factors in the numerator and denominator are the same even though the terms are written in different orders.

$$\frac{18-4m}{-2m+9}$$
$$=\frac{2(9-2m)}{-2m+9}$$
$$=\frac{2\cancel{(9-2m)}}{\cancel{-2m+9}}$$
$$=2,\ m\neq\frac{9}{2}$$

b) Factor where possible, and eliminate common factors to simplify the expression.

The NPVs can be identified using the original denominator in factored form.

$$\frac{x^2-25}{2x^2+10x}$$
$$=\frac{(x-5)\cancel{(x+5)}}{2x\cancel{(x+5)}}$$
$$=\frac{x-5}{2x},\ x\neq 0,-5$$

c) Factor where possible, and eliminate common factors to simplify the expression.

The NPVs can be identified using the original denominator in factored form.

$$\frac{b^2+b}{b^2+2b+1}$$
$$=\frac{b\cancel{(b+1)}}{\cancel{(b+1)}(b+1)}$$
$$=\frac{b}{b+1},\ b\neq -1$$

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. B

If $(x + 1)$ or $(x + 3)$ in the denominator equals 0, the expression $\frac{(x+1)(x+2)}{(x+1)(x+3)}$ is undefined.

Therefore, the NPVs are $x = -1$ and $x = -3$.

2. There are no common factors to eliminate, so the expression $\frac{3x^2-1}{x}$ is in its simplest form already, where $x\neq 0$.

3. Factor where possible, and eliminate common factors to simplify the expression.

The NPVs can be identified using the original denominator in factored form.

$$\frac{(x-2)}{(x^2+x-6)}$$
$$=\frac{\cancel{x-2}}{(\cancel{x-2})(x+3)}$$
$$=\frac{1}{x+3},\ x\neq 2,-3$$

4. Factor where possible, and eliminate common factors to simplify the expression.

The NPVs can be identified using the original denominator in factored form.

$$\frac{x-1}{(x^2-1)}$$
$$=\frac{\cancel{x-1}}{(\cancel{x-1})(x+1)}$$
$$=\frac{1}{x+1},\ x\neq \pm 1$$

Comparing the NPVs of the original and simplified forms, you can see that the NPVs of the original denominator are 1 and –1, but the NPV of the simplified form is just –1. However, both 1 and –1 must be stated as restrictions in the solution for the expressions to be equivalent. The original and simplified forms are equivalent only when all the NPVs are identified.

5. The strategies are the same for writing equivalent forms of rational numbers and expressions.

One method is to multiply the numerator and denominator by the same value.

Rational numbers:

$$\frac{2}{7}\left(\frac{3}{3}\right)=\frac{6}{21}$$

Rational expressions:

$$\frac{x^2}{2+x}\left(\frac{3x}{3x}\right)=\frac{3x^3}{6x+3x^2},\ x\neq -2,0$$

In these examples, $\frac{2}{7}$ is equivalent to $\frac{6}{21}$, and $\frac{x^2}{2+x}$ is equivalent to $\frac{3x^3}{6x+3x^2}$.

Another method is to simplify the numerator and denominator by dividing (reducing) by a common factor.

Rational numbers:

$$\frac{8}{36}=\frac{(\cancel{4})(2)}{(\cancel{4})(9)}$$
$$=\frac{2}{9}$$

Rational expressions:

$$\frac{12x-3}{3x^2}$$
$$=\frac{\cancel{3}(4x-1)}{\cancel{3}x^2}$$
$$=\frac{4x-1}{x^2},\ x\neq 0$$

In these examples, $\frac{8}{36}$ is equivalent to $\frac{2}{9}$, and $\frac{12x-3}{3x^2}$ is equivalent to $\frac{4x-1}{x^2}$.

The NPVs must be stated for rational expressions.

6. Factor where possible, and eliminate common factors to simplify the expression.

The NPVs can be identified using the original denominator in factored form.

$$\frac{18-50y^2}{20y+12}$$
$$=\frac{2(9-25y^2)}{4(5y+3)}$$
$$=\frac{\cancel{2}(3-5y)(\cancel{3+5y})}{\underset{2}{\cancel{4}}(\cancel{5y+3})}$$
$$=\frac{3-5y}{2},\ y\neq -\frac{3}{5}$$

In this question, it is important to recognize that the numerator contains a difference of squares, and that the factors $(3+5y)$ and $(5y+3)$ are the same.

7. Write a ratio comparing gas volume per unit of space.

The gas volume $\left(x^3+2x^2\right)$ will be in the numerator, and the cubic space $\left(x+2\right)^3$ will be in the denominator. The $\left(x+2\right)$ in the denominator is cubed since the length and height are the same as the width in a cube. The units will cancel each other since both the numerator and denominator are now in units^3.

$$\frac{x^3+2x^2}{(x+2)^3}$$
$$=\frac{x^2\left(\cancel{x+2}\right)}{\left(x+2\right)\left(x+2\right)\left(\cancel{x+2}\right)}$$
$$=\frac{x^2}{\left(x+2\right)^2}$$

This is the most reduced form as a rational expression. It could also be written as the ratio $x^2:(x+2)^2$.

The NPVs can be identified using the original denominator in factored form. Therefore, $x\neq-2$.

Note that this is the complete answer, but the value of $\left(x+2\right)$ units is given as the width of an actual cubic space. As such, if x were less than –2, there would have been a negative dimension to the object, and this would not exist in real space.

8. Simplify the expression by factoring where possible.

$$\frac{a^2+12a}{2a}=\frac{a\left(a+12\right)}{2a}$$

Eliminate the common factors, and state any NPVs.

$$\frac{\cancel{a}\left(a+12\right)}{2\cancel{a}}=\frac{a+12}{2},\ a\neq0$$

This is in simplest form since there are no more common factors in the numerator and denominator. You cannot reduce the 12 and 2 because 2 is not a factor common to both terms in the numerator. Therefore, the final step is not correct in Marsha's solution.

The correct solution is as follows:

$$\frac{a^2+12a}{2a}$$
$$=\frac{a\left(a+12\right)}{2a}$$
$$=\frac{a+12}{2},\ a\neq0$$

Lesson 3—Multiplying Rational Expressions

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. a) Simplify the expression first by eliminating common factors, and then multiply.

The NPVs can be identified using the original denominators.

$$\frac{x^3y}{10}\times\frac{2xy}{x^2}$$
$$=\frac{\overset{x}{\cancel{x^3}}\,y}{\underset{5}{\cancel{10}}}\times\frac{\cancel{2}xy}{\cancel{x^2}}$$
$$=\frac{xy}{5}\times\frac{xy}{1}$$
$$=\frac{x^2y^2}{5},\ x\neq0$$

b) Since the expression is in factored form already, eliminate common factors and then multiply.

The NPVs can be identified using the original denominators.

$$\left(\frac{(8)(a)(b)}{(2)(2)}\right)\left(\frac{(4)(c)}{(c)(c)(c)}\right)$$
$$=\left(\frac{(8)(a)(b)}{(\cancel{2})(\cancel{2})}\right)\left(\frac{(\cancel{4})(\cancel{c})}{(\cancel{c})(c)(c)}\right)$$
$$=\frac{8ab}{c^2},\ c\neq0$$

2. Simplify the expressions, and multiply as necessary.

The NPVs can be identified using the original denominators in factored form.

$$\left(\frac{x^2+5x+6}{x^2-1}\right)\left(\frac{x+1}{x+3}\right)$$
$$=\left(\frac{(x+2)(x+3)}{(x+1)(x-1)}\right)\left(\frac{x+1}{x+3}\right)$$
$$=\left(\frac{(x+2)\cancel{(x+3)}}{\cancel{(x+1)}(x-1)}\right)\left(\frac{\cancel{x+1}}{\cancel{x+3}}\right)$$
$$=\frac{x+2}{x-1},\ x\neq\pm1,-3$$

3. Simplify the expression, and multiply as necessary. The NPVs can be identified using the original denominators.

$$(2)\left(\frac{x^2-3x-10}{2x}\right)\left(\frac{x^2-3x}{x-5}\right)$$
$$=\left(\frac{2}{1}\right)\left(\frac{(x-5)(x+2)}{2x}\right)\left(\frac{x(x-3)}{x-5}\right)$$
$$=\left(\frac{\cancel{2}}{1}\right)\left(\frac{\cancel{(x-5)}(x+2)}{\cancel{2}\,\cancel{x}}\right)\left(\frac{\cancel{x}(x-3)}{\cancel{x-5}}\right)$$
$$=(x+2)(x-3),\ x\neq0,\ 5$$

Expressions in simplest form can be left in factored form (written as a product of polynomials) rather than actually multiplied out.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. Simplify the expression, and multiply as necessary. The NPVs can be identified using the original denominators in factored form.

$$\left(\frac{(x+1)^3}{24x}\right)\left(\frac{6(2x)}{2x^2-2}\right)$$
$$=\left(\frac{\cancel{(x+1)}(x+1)(x+1)}{\underset{4}{\cancel{24}}\,\cancel{x}}\right)\left(\frac{\cancel{6}(\cancel{2}\,\cancel{x})}{\cancel{2}(x-1)\cancel{(x+1)}}\right)$$
$$=\frac{(x+1)(x+1)}{4(x-1)},\ x\neq0,\ 1,-1$$

2. Simplify the expression, and multiply as necessary. The NPVs can be identified using the original denominators in factored form.

$$\left(\frac{3(x+1)^2}{4(x-1)^2}\right)\left(\frac{4(x+1)^2}{3(x-1)^2}\right)$$
$$=\left(\frac{\cancel{3}(x+1)(x+1)}{\cancel{4}(x-1)(x-1)}\right)\left(\frac{\cancel{4}(x+1)(x+1)}{\cancel{3}(x-1)(x-1)}\right)$$
$$=\frac{(x+1)^4}{(x-1)^4},\ x\neq1$$

3. **D**

Simplify the given products by eliminating common factors and using the quotient law of exponents.

$$-\left(\frac{x}{3}\right)$$
$$=-\frac{x}{3}$$

$$\left(\frac{2}{3}\right)\left(\frac{6}{x}\right)$$
$$=\frac{12}{3x}$$
$$=\frac{4}{x},\ x\neq0$$

$$\left(\frac{3}{3x}\right)(2)$$
$$=\left(\frac{3}{3x}\right)\left(\frac{2}{1}\right)$$
$$=\frac{2}{x},\ x\neq0$$

$$\left(\frac{-x}{4x^2}\right)\left(\frac{-12x}{x}\right)$$
$$=\left(\frac{\overset{-1}{\cancel{-x}}}{\cancel{4}\,\underset{x}{\cancel{x^2}}}\right)\left(\frac{\overset{-3}{\cancel{-12}}\,\cancel{x}}{\cancel{x}}\right)$$
$$=\frac{3}{x},\ x\neq0$$

The expression $\left(\frac{-x}{4x^2}\right)\left(\frac{-12x}{x}\right)$ is equivalent to $\frac{3}{x}$, $x\neq0$.

4. Simplify the expression, and multiply as necessary. The NPVs can be identified using the original denominators in factored form.

$$\left(\frac{x^2-4}{x^2+5x-14}\right)\left(\frac{2x^2+14x}{2}\right)$$
$$=\left(\frac{(x-2)(x+2)}{(x-2)(x+7)}\right)\left(\frac{2x(x+7)}{2}\right)$$
$$=\left(\frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}\cancel{(x+7)}}\right)\left(\frac{\cancel{2}x\cancel{(x+7)}}{\cancel{2}}\right)$$
$$=x(x+2),\ x\neq2,-7$$

5. Answers will vary. The two rational expressions must simplify (after multiplication) to $\frac{2}{x}$ and should therefore be more complex than $\frac{2}{x}$ (there should be some common factors to eliminate).

An example is $\left(\frac{2(x+1)}{x}\right)\left(\frac{1}{(x+1)}\right)$.

The restrictions for this are $x \neq 0$ and $x \neq -1$.

6. Answers will vary. However, the initial denominator for one of the rational expressions must have a factor that becomes zero when $x = 4$.

An example is $\left(\frac{2(x-4)^2}{x(x-4)}\right)\left(\frac{3}{3(x-4)}\right)$ that has NPVs of $x = 0$ and $x = 4$.

7. **A**
In the given expression, the last factor in the denominator $(x+n-1)$ is one less than the last factor in the numerator $(x+n)$. As such, after reducing, all the factors in the denominator will be eliminated and the numerator will remain as $(x+n)$. There will be $(n-1)$ non-permissible values, and the denominator will be 1.

8. Simplify the expression, and multiply as necessary. The NPVs can be identified using the original denominators in factored form.

$$\frac{3x^3+2x^2-3x-2}{x+1} \times \frac{15x}{9x^2+6x}$$
$$= \frac{x^2(3x+2)-1(3x+2)}{x+1} \times \frac{15x}{3x(3x+2)}$$
$$= \frac{(x-1)\cancel{(x+1)}\cancel{(3x+2)}}{\cancel{x+1}} \times \frac{\overset{5}{\cancel{15x}}}{\cancel{3x}\cancel{(3x+2)}}$$
$$= 5(x-1),\ x \neq 0, -1, -\frac{2}{3}$$

The first numerator can be factored by grouping, which is written as $(x^2-1)(3x+2)$. Since one of these factors is a difference of squares, further factoring is required.

Lesson 4—Dividing Rational Expressions

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **a)** Rewrite the quotient as a product using the reciprocal of the second expression. Simplify the product, and multiply as necessary.

$$\frac{7}{8} \div \frac{4}{6}$$
$$= \frac{7}{8} \times \frac{6}{4}$$
$$= \frac{7}{8} \times \frac{\overset{-3}{\cancel{6}}}{\underset{2}{\cancel{4}}}$$
$$= -\frac{21}{16}$$

b) Rewrite the quotient as a product using the reciprocal of the second expression. Simplify the product, and multiply as necessary.

$$\frac{3x}{10} \div \frac{12}{30}$$
$$= \frac{3x}{10} \times \frac{30}{12}$$
$$= \frac{\cancel{3}x}{\cancel{10}} \times \frac{\overset{3}{\cancel{30}}}{\underset{4}{\cancel{12}}}$$
$$= \frac{3x}{4}$$

2. **a)** Rewrite the quotient as a product using the reciprocal of the second expression. Simplify the product, and multiply as necessary. The NPVs can be identified using the original denominators in the quotient and the new denominator in the product.

$$\frac{3(x-8)}{x^2} \div \frac{12(x-8)}{x(x+4)}$$
$$= \frac{3(x-8)}{x^2} \times \frac{x(x+4)}{12(x-8)}$$
$$= \frac{\cancel{3}\cancel{(x-8)}}{\underset{x}{\cancel{x^2}}} \times \frac{\cancel{x}(x+4)}{\underset{4}{\cancel{12}}\cancel{(x-8)}}$$
$$= \frac{x+4}{4x},\ x \neq 0, -4,\ 8$$

b) Rewrite the quotient as a product using the reciprocal of the second expression. Simplify the product and multiply as necessary. The NPVs can be identified using the original denominators in the quotient and the new denominator in the product.

$$\frac{x^2-3x-10}{2x}\div\frac{x-5}{x^2-3x}$$
$$=\frac{\cancel{(x-5)}(x+2)}{2\cancel{x}}\times\frac{\cancel{x}(x-3)}{\cancel{x-5}}$$
$$=\frac{(x+2)(x-3)}{2},\ x\neq 0,\ 3,\ 5$$

c) Rewrite the quotient as a product using the reciprocal of the second expression. Simplify the product and multiply as necessary. The NPVs can be identified using the original denominators in the quotient and the new denominator in the product.

$$\frac{x^2-16}{x^2+7x+12}\div\frac{x^2-2x-8}{2x+4}$$
$$=\frac{(x-4)(x+4)}{(x+3)(x+4)}\times\frac{2(x+2)}{(x-4)(x+2)}$$
$$=\frac{\cancel{(x-4)}\cancel{(x+4)}}{(x+3)\cancel{(x+4)}}\times\frac{2\cancel{(x+2)}}{\cancel{(x-4)}\cancel{(x+2)}}$$
$$=\frac{2}{x+3},\ x\neq -2,-3,-4,\ 4$$

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. The expression $\dfrac{5}{\left(\dfrac{20}{x^2}\right)}$ can be rewritten as $\dfrac{5}{1}\div\dfrac{20}{x^2}$ and then rewritten as the product $\left(\dfrac{5}{1}\right)\left(\dfrac{x^2}{20}\right)$.

This simplifies to $\dfrac{x^2}{4},\ x\neq 0$.

2. $\text{Area}=\dfrac{\text{base}\times\text{height}}{2}$

$$A=\left(x^2-2x+1\right)\left(\frac{2x}{x-1}\right)^2\div 2$$
$$A=(x-1)(x-1)\left(\frac{4x^2}{(x-1)(x-1)}\right)\times\frac{1}{2}$$
$$A=\cancel{(x-1)}\cancel{(x-1)}\left(\frac{\overset{2}{\cancel{4}}x^2}{\cancel{(x-1)}\cancel{(x-1)}}\right)\times\frac{1}{\cancel{2}}$$
$$A=2x^2,\ x\neq 1$$

3. A

One method for solving $\dfrac{x}{3}\div y=\dfrac{2}{x}$ is to investigate what expression must be multiplied by $\dfrac{x}{3}$ to give $\dfrac{2}{x}$. The denominator of $\dfrac{2}{x}$ is x, and the numerator of $\dfrac{x}{3}$ is x, so the unknown expression must have an x^2 in the denominator.

$$\left(\frac{x}{3}\right)\left(\frac{}{x^2}\right)=\frac{2}{x}$$

Also, the denominator of $\dfrac{x}{3}$ is 3, and the numerator of $\dfrac{2}{x}$ is 2, so the unknown expression must have a 6 in the numerator.

$$\left(\frac{x}{3}\right)\left(\frac{6}{x^2}\right)=\frac{2}{x}$$

The reciprocal of $\dfrac{6}{x^2}$ will give y for the division statement, so $y=\dfrac{x^2}{6}$ in $\dfrac{x}{3}\div y=\dfrac{2}{x}$.

$$\frac{x}{3}\div y$$
$$=\frac{x}{3}\div\frac{x^2}{6}$$
$$=\frac{x}{3}\times\frac{6}{x^2}$$
$$=\frac{2}{x}$$

4. Rewrite the quotient as a product using the reciprocal of the second expression. Simplify the product, and multiply as necessary. The NPVs can be identified using the original denominators in the quotient and the new denominator in the product.

$$\frac{x^2-x-12}{x+1}\div\frac{x^2-5x+4}{x^2-1}$$
$$=\frac{(x+3)(x-4)}{x+1}\times\frac{(x-1)(x+1)}{(x-1)(x+4)}$$
$$=\frac{(x+3)\cancel{(x-4)}}{\cancel{x+1}}\times\frac{\cancel{(x-1)}\cancel{(x+1)}}{\cancel{(x-1)}\cancel{(x+4)}}$$
$$=x+3,\ x\neq-1,\ 1,\ 4$$

5. Rewrite the quotient as a product using the reciprocal of the second expression. Simplify the product, and multiply as necessary. The NPVs can be identified using the original denominators in the quotient and the new denominator in the product.

$$\frac{4(1-x)}{x}\div\frac{x-1}{2x}$$
$$=\left(\frac{4(1-x)}{x}\right)\left(\frac{2x}{x-1}\right)$$
$$=\left(\frac{-4\cancel{(-1+x)}}{\cancel{x}}\right)\left(\frac{2\cancel{x}}{\cancel{x-1}}\right)$$
$$=-8,\ x\neq 0,\ 1$$

6. Rewrite $\frac{25-x^2}{bx}\div\frac{(5+x)}{2b}\div\frac{(x-5)}{-(bx-1)}\times\frac{x^2}{(bx-1)}$

as the product of four rational expressions. Factor and eliminate any common factors.

$$\frac{25-x^2}{bx}\div\frac{(5+x)}{2b}\div\frac{(x-5)}{-(bx-1)}\cdot\frac{x^2}{(bx-1)}$$
$$=\frac{25-x^2}{bx}\cdot\frac{2b}{(5+x)}\cdot\frac{-(bx-1)}{(x-5)}\cdot\frac{x^2}{(bx-1)}$$
$$=\frac{(5-x)(5+x)}{bx}\cdot\frac{2b}{(5+x)}\cdot\frac{(bx-1)}{-(x-5)}\cdot\frac{x^2}{(bx-1)}$$
$$=\frac{\cancel{(5-x)}\cancel{(5+x)}}{\cancel{b}\,\cancel{x}}\cdot\frac{2\cancel{b}}{\cancel{(5+x)}}\cdot\frac{\cancel{(bx-1)}}{\cancel{(5-x)}}\cdot\frac{\overset{x}{\cancel{x^2}}}{\cancel{(bx-1)}}$$
$$=2x$$

Note that the negative with (bx – 1) was written with (x – 5), which was then multiplied to give (5 – x).

State the NPVs and solve for b.

$bx=0$
$x=0$

$2b=0$
$b=0$

$(bx-1)=0$
$x=\frac{1}{b}$ and $b=\frac{1}{x}$

$(x\pm5)=0$
$x=\pm5$

Therefore, the NPVs for x are 0, $\frac{1}{b}$, 5, and –5,

and the NPVs for b are 0 and $\frac{1}{x}$.

Since the simplified form for this division is $2x$, any value of b that is not a restricted value (NPV) will produce this result.

7. **A**
When simplifying, it is true that common factors can disappear, but the NPVs are identified from the original denominators, so they are not lost or eliminated.

Restricted values are found only in the denominator. When an expression is replaced by its reciprocal, the new denominator can produce new restricted values. Simplifying is the process of eliminating factors that are common to the numerator and denominator. Any fraction with a denominator of zero is undefined.

Lesson 5—Adding and Subtracting Rational Expressions

CLASS EXERCISES
ANSWERS AND SOLUTIONS

1. Step 1
Simplify each expression where possible.

The expression $\frac{x^2-4}{(x-2)^2}$ can be simplified.

$$\frac{x^2-4}{(x-2)^2}$$
$$=\frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(x-2)}$$
$$=\frac{x+2}{x-2}, x \neq 2$$

Step 2
Identify the lowest common denominator (LCD).

The three denominators of the expressions $\frac{2}{5x}$, $\frac{3x+1}{2}$, and $\frac{x+2}{x-2}$ have no factors in common, so the LCD is $(5x)(2)(x-2)$.

2. a) Step 1
Simplify each expression where possible.

The expression $\frac{6}{10y}$ can be reduced to $\frac{3}{5y}$.

Step 2
Identify the lowest common denominator (LCD).

The LCD for the expression $\frac{3}{5y}-\frac{1}{15y^2}$ is $15y^2$.

Step 3
Write an equivalent expression with the LCD as the denominator.

$$\frac{3}{5y}-\frac{1}{15y^2}$$
$$=\frac{3(3y)}{5y(3y)}-\frac{1}{15y^2}$$
$$=\frac{9y}{15y^2}-\frac{1}{15y^2}, y \neq 0$$

b) Step 1
Simplify each expression where possible.

The expression $\frac{x-2x^2}{2x-4x^2}$ can be simplified.

$$\frac{x-2x^2}{2x-4x^2}$$
$$=\frac{\cancel{x}\cancel{(1-2x)}}{2\cancel{x}\cancel{(1-2x)}}$$
$$=\frac{1}{2}$$

Step 2
Identify the lowest common denominator (LCD).

The LCD for the expression $\frac{x-4}{2x}+\frac{1}{2}$ is $2x$.

Step 3
Write an equivalent expression with the LCD as the denominator.

$$\frac{x-4}{2x}+\frac{1}{2}$$
$$=\frac{(x-4)(1)}{2x(1)}+\frac{1(x)}{2(x)}$$
$$=\frac{x-4}{2x}+\frac{x}{2x}, x \neq 0$$

You might have noticed that if the x in the numerator and denominator of $\frac{\cancel{x}\cancel{(1-2x)}}{2\cancel{x}\cancel{(1-2x)}}$ had not been eliminated, the expressions $\frac{x-4}{2x}$ and $\frac{x}{2x}$ would already have had a common denominator of $2x$.

Noticing details like this is not essential (as shown in the solution), but could certainly save a little time.

3. a) Since the denominator is already common, combine the terms in the numerator and then simplify if necessary. Use the original expressions to state any NPVs.

$$\frac{5x^2+3x+1}{x+2}+\frac{1-4x^2}{x+2}$$
$$=\frac{x^2+3x+2}{x+2}$$
$$=\frac{\cancel{(x+2)}(x+1)}{\cancel{x+2}}$$
$$=x+1,\ x\neq -2$$

b) Since the denominator is already common, combine the terms in the numerator and then simplify if necessary. Use the original expressions to state any NPVs.

$$\frac{2a}{3a}-\frac{(5a-a^2)}{3a}$$
$$=\frac{2a-5a+a^2}{3a}$$
$$=\frac{a^2-3a}{3a}$$
$$=\frac{\cancel{a}(a-3)}{3\cancel{a}}$$
$$=\frac{a-3}{3},\ a\neq 0$$

Alternate solution:
Simplify each expression first, and then subtract accordingly.

$$\frac{2a}{3a}-\frac{(5a-a^2)}{3a}$$
$$=\frac{2\cancel{a}}{3\cancel{a}}-\frac{\cancel{a}(5-a)}{3\cancel{a}}$$
$$=\frac{2}{3}-\frac{(5-a)}{3}$$
$$=\frac{2-5+a}{3}$$
$$=\frac{a-3}{3},\ a\neq 0$$

c) Step 1
Identify the LCD.

The LCD is $(m-1)(m+3)$.

Step 2
Write equivalent expressions with the new denominator.

$$\frac{m}{m-1}-\frac{m-1}{m+3}$$
$$=\frac{m(m+3)}{(m-1)(m+3)}-\frac{(m-1)(m-1)}{(m-1)(m+3)}$$
$$=\frac{m^2+3m}{(m-1)(m+3)}-\frac{(m^2-2m+1)}{(m-1)(m+3)}$$

Step 3
Combine like terms in the numerators, keeping the common denominator unchanged.

$$\frac{m^2+3m}{(m-1)(m+3)}-\frac{(m^2-2m+1)}{(m-1)(m+3)}$$
$$=\frac{m^2+3m-m^2+2m-1}{(m-1)(m+3)}$$
$$=\frac{5m-1}{(m-1)(m+3)}$$

Step 4
Check if the expression can be reduced further, and state the NPVs.

Since the numerator is not factorable, the expression is $\dfrac{5m-1}{(m-1)(m+3)},\ m\neq 1,-3$ in simplest form.

d) Step 1
Simplify each expression where possible.

$$\frac{x}{x-1}-\frac{x}{1-x}$$
$$=\frac{x}{x-1}-\frac{x}{-1(-1+x)}$$
$$=\frac{x}{x-1}+\frac{x}{x-1}$$

Step 2
Identify the lowest common denominator (LCD).

Factoring –1 from the second denominator will make the question a sum with an LCD of $x-1$.

Step 3
Since the denominator is now common, combine the terms in the numerator and then simplify if necessary. Use the original expressions to state any NPVs.

$$\frac{x}{x-1}+\frac{x}{x-1}$$
$$=\frac{x+x}{x-1}$$
$$=\frac{2x}{x-1},\ x\neq 1$$

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Identify the LCD, and write equivalent expressions with the new denominator.

The LCD is $(3x)(4)$. Note the last term has a denominator of 1.

$$\frac{2}{3x}+\frac{3x}{4}-1$$
$$=\frac{2(4)}{(3x)(4)}+\frac{3x(3x)}{(3x)(4)}-\frac{1(3x)(4)}{(3x)(4)}$$
$$=\frac{8}{(3x)(4)}+\frac{9x^2}{(3x)(4)}-\frac{12x}{(3x)(4)}$$

Step 2
Combine like terms in the numerators, keeping the common denominator unchanged.

$$\frac{8}{(3x)(4)}+\frac{9x^2}{(3x)(4)}-\frac{12x}{(3x)(4)}$$
$$=\frac{9x^2-12x+8}{12x}$$

Step 3
Check if the expression can be reduced further, and state the NPVs.

Since the numerator is not factorable, the expression is $\frac{9x^2-12x+8}{12x},\ x\neq 0$ in simplest form.

2. **C**
The original denominator in the expression $\frac{(x+1)(x-1)}{12x^2}$ will be multiplied by $2x$, so the new numerator will also be multiplied by $2x$.

The new numerator will be $2x(x+1)(x-1)$ or $2x\left(x^2-1\right)$.

3. Isolate the term with x, and combine the like terms with y.

With a single expression on either side of the equal sign, cross-multiply and solve for x.

$$\frac{9y}{4}-\frac{x}{2}=\frac{y}{4}$$
$$\frac{9y}{4}-\frac{y}{4}=\frac{x}{2}$$
$$\frac{8y}{4}=\frac{x}{2}$$
$$16y=4x$$
$$4y=x$$

Alternate solution:
Combine the terms on the left side, and then cross-multiply. Gather like terms, and solve for x.

$$\frac{9y}{4}-\frac{x}{2}=\frac{y}{4}$$
$$\frac{9y}{4}-\frac{2x}{4}=\frac{y}{4}$$
$$\frac{9y-2x}{4}=\frac{y}{4}$$
$$\cancel{4}\left(9y-2x\right)=\cancel{4}\left(y\right)$$
$$-2x=y-9y$$
$$-2x=-8y$$
$$x=4y$$

The given solutions solve this question step by step, but at any point along the way, the solution may be stated by observation.

4. Step 1
Simplify each expression where possible.

$$\frac{a+3}{(2a+2)}+\frac{a-1}{2(a+1)}-\frac{6}{a}$$
$$=\frac{a+3}{2(a+1)}+\frac{a-1}{2(a+1)}-\frac{6}{a}$$

The expressions cannot be simplified, but the denominator of the first expression can be factored.

Step 2
Identify the lowest common denominator (LCD). The LCD will be (2)(a)(a + 1).

Step 3
Write an equivalent expression with the LCD as the denominator.

$$\frac{a+3}{2(a+1)}+\frac{a-1}{2(a+1)}-\frac{6}{a}$$
$$=\frac{(a+3)(a)}{(2)(a)(a+1)}+\frac{(a-1)(a)}{(2)(a)(a+1)}-\frac{6(2)(a+1)}{(2)(a)(a+1)}$$
$$=\frac{a^2+3a}{(2)(a)(a+1)}+\frac{a^2-a}{(2)(a)(a+1)}-\frac{12a+12}{(2)(a)(a+1)}$$

Step 4
Combine like terms in the numerators, keeping the common denominator unchanged.

$$\frac{a^2+3a}{(2)(a)(a+1)}+\frac{a^2-a}{(2)(a)(a+1)}-\frac{12a+12}{(2)(a)(a+1)}$$
$$=\frac{a^2+3a+a^2-a-12a-12}{(2)(a)(a+1)}$$
$$=\frac{2a^2-10a-12}{(2)(a)(a+1)}$$

Step 5
Check if the expression can be reduced further, and state the NPVs.

$$\frac{2a^2-10a-12}{(2)(a)(a+1)}$$
$$=\frac{\cancel{2}\left(a^2-5a-6\right)}{(\cancel{2})(a)(a+1)}$$
$$=\frac{(a-6)\cancel{(a+1)}}{(a)\cancel{(a+1)}}$$
$$=\frac{(a-6)}{a},\ a\neq 0,-1$$

5. Step 1
Simplify each expression where possible.

$$\frac{x^2-6x-7}{x+1}-\frac{2x+12}{2x+2}+\frac{x+1}{x^2-1}$$
$$=\frac{(x+1)(x-7)}{(x+1)}-\frac{\cancel{2}(x+6)}{\cancel{2}(x+1)}+\frac{(x+1)}{(x+1)(x-1)}$$

The first and third expressions can be reduced by eliminating the common factors of $(x+1)$.
However, by noting the denominators of all three expressions, it can be seen that a common denominator is going to include $(x+1)$ as a factor, so simplifying is not necessary. A factor of 2 can be eliminated in the second expression.

Step 2
Identify the lowest common denominator (LCD).

The LCD will be $(x+1)(x-1)$.

Step 3
Write an equivalent expression with the LCD as the denominator.

$$\frac{(x+1)(x-7)}{(x+1)}-\frac{(x+6)}{(x+1)}+\frac{(x+1)}{(x+1)(x-1)}$$
$$=\frac{(x-1)(x+1)(x-7)}{(x+1)(x-1)}-\frac{(x+6)(x-1)}{(x+1)(x-1)}+\frac{(x+1)}{(x+1)(x-1)}$$
$$=\frac{\left(x^2-1\right)(x-7)}{(x+1)(x-1)}-\frac{\left(x^2+5x-6\right)}{(x+1)(x-1)}+\frac{x+1}{(x+1)(x-1)}$$
$$=\frac{x^3-7x^2-x+7}{(x+1)(x-1)}-\frac{x^2+5x-6}{(x+1)(x-1)}+\frac{x+1}{(x+1)(x-1)}$$

Step 4
Combine like terms in the numerators, keeping the common denominator unchanged.

$$\frac{x^3-7x^2-x+7}{(x+1)(x-1)}-\frac{x^2+5x-6}{(x+1)(x-1)}+\frac{x+1}{(x+1)(x-1)}$$
$$=\frac{x^3-7x^2-x+7-x^2-5x+6+x+1}{(x+1)(x-1)}$$
$$=\frac{x^3-8x^2-5x+14}{(x+1)(x-1)}$$

Step 5
Check if the expression can be reduced further, and state the NPVs.

The expression is in simplest form since it cannot be factored further. The NPVs are $x = 1$ and $x = -1$, which can be written beside the expression as $\frac{x^3 - 8x^2 - 5x + 14}{(x+1)(x-1)}, x \neq 1, -1$.

6. **Step 1**
Simplify each expression where possible.

$$\frac{2}{(1-x)} + \frac{x-1}{(x-1)(x)} - \frac{3}{(x-1)(x)}$$
$$= \frac{-2}{(x-1)} + \frac{x-1}{(x-1)(x)} - \frac{3}{(x-1)(x)}$$

The second expression can be reduced by eliminating the common factor of $(x-1)$. However, by noting the denominators of all three expressions, it can be seen that a common denominator is going to include $(x-1)$ as a factor, so simplifying is not necessary. A factor of –1 can be eliminated in the first expression, leaving the denominator as $(x-1)$.

Step 2
Identify the lowest common denominator (LCD). The LCD will be $x(x-1)$.

Step 3
Write an equivalent expression with the LCD as the denominator.

$$\frac{-2}{(x-1)} + \frac{x-1}{(x-1)(x)} - \frac{3}{(x-1)(x)}$$
$$= \frac{-2(x)}{(x-1)(x)} + \frac{x-1}{(x-1)(x)} - \frac{3}{(x-1)(x)}$$

Step 4
Combine like terms in the numerators, keeping the common denominator unchanged.

$$\frac{-2(x)}{(x-1)(x)} + \frac{x-1}{(x-1)(x)} - \frac{3}{(x-1)(x)}$$
$$= \frac{-2x + x - 1 - 3}{(x-1)(x)}$$
$$= \frac{-x-4}{(x-1)(x)}, x \neq 0, 1$$

For the numerator to equal –3, set it equal to –3 and solve for *x*.

$$-x - 4 = -3$$
$$-4 + 3 = x$$
$$-1 = x$$

Therefore, the value of *x* must be –1 for the numerator to equal –3.

Lesson 6—Rational Equations and Problem Solving

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **a)** **Step 1**
Find the LCM of the denominators, and multiply each term by that LCM.

The LCM will be $(x-1)(x+1)$.

$$\frac{5}{x-1} - \frac{12}{x^2-1} = 1$$
$$\frac{5}{x-1} - \frac{12}{(x-1)(x+1)} = 1$$
$$(x-1)(x+1)\left[\frac{5}{x-1} - \frac{12}{(x-1)(x+1)} = 1\right]$$
$$\frac{5(x-1)(x+1)}{(x-1)} - \frac{12(x-1)(x+1)}{(x-1)(x+1)} = x^2 - 1$$

Step 2
Remove all denominators by eliminating common factors, and then expand where necessary.

$$\frac{5\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} - \frac{12\cancel{(x-1)}\cancel{(x+1)}}{\cancel{(x-1)}\cancel{(x+1)}} = x^2 - 1$$
$$5(x+1) - 12 = x^2 - 1$$
$$5x + 5 - 12 = x^2 - 1$$

Step 3
Gather like terms, and solve the resulting quadratic equation.

$$5x + 5 - 12 = x^2 - 1$$
$$0 = x^2 - 5x + 6$$
$$0 = (x-2)(x-3)$$

$0 = x - 2$ or $0 = x - 3$
$2 = x$ $\quad$ $3 = x$

Step 4
State any NPVs for the variable.

The NPVs from the original equation $\frac{5}{x-1}-\frac{12}{x^2-1}=1$ are $x = 1$ and $x = –1$.

Therefore, the solution set is {2, 3}.

Step 5
Verify the solution using the original equation.

x = 2	
$\frac{5}{(2)-1}-\frac{12}{(2)^2-1}$	1
$\frac{5}{1}-\frac{12}{3}$	1
5 – 4	1
1	1

x = 3	
$\frac{5}{(3)-1}-\frac{12}{(3)^2-1}$	1
$\frac{5}{2}-\frac{12}{8}$	1
$\frac{20-12}{8}$	1
$\frac{8}{8}$	1

The solutions $x = 2$ and $x = 3$ have been verified.

b) **Step 1**
Find the LCM of the denominators, and multiply each term by that LCM.

The LCM will be (3 + x).

$$3+\frac{2}{3+x}=5$$
$$(3+x)\left[3+\frac{2}{3+x}=5\right]$$
$$(3+x)(3)+(3+x)\left(\frac{2}{3+x}\right)=(3+x)(5)$$

Step 2
Remove all denominators by eliminating common factors, and then expand where necessary.

$$(3+x)(3)+\left(\cancel{3+x}\right)\left(\frac{2}{\cancel{3+x}}\right)=(3+x)(5)$$
$$9+3x+2=15+5x$$

Step 3
Gather like terms, and solve for x.

$$9+3x+2=15+5x$$
$$-4=2x$$
$$-2=x$$

Step 4
State any NPVs for the variable.

The NPV from the original equation $3+\frac{2}{3+x}=5$ is $x = –3$.

Therefore, the solution set is {–2}.

Step 5
Verify the solution using the original equation.

x = –2	
$3+\frac{2}{3+(-2)}$	5
$3+\frac{2}{1}$	5
5	5

The solution $x = –2$ has been verified.

c) **Step 1**
With a single expression on either side of the equal sign, cross-multiply to remove the denominators, and then expand where necessary.

$$\frac{2}{x+4}=\frac{5}{3x+12}$$
$$(2)(3x+12)=(5)(x+4)$$
$$6x+24=5x+20$$

Step 2
Gather like terms, and solve for x.

$$6x+24=5x+20$$
$$x=-4$$

Step 3
State any NPVs for the variable.

The NPV from the original equation $\frac{2}{x+4}=\frac{5}{3x+12}$ is $x = –4$.

The solution $x = –4$ is non-permissible, so there are no solutions and verifying is not necessary.

2. Step 1
Assign a variable to represent the unknown, and set up an equation using the given information.

Let x represent the wind speed. It takes the sailboat the same time to travel 70 km against the wind as it does 130 km with the wind.

The sailboat will travel faster with the wind, so its speed will be (20 + x) km/h.

Similarly, the sailboat will travel slower against the wind, so its speed will be (20 – x) km/h.

Therefore, set up an equation where the two times are equal, $\left(t = \frac{d}{s}\right)$.

time with the wind = time against the wind

$$\frac{130 \text{ km}}{(20+x)\text{km/h}} = \frac{70 \text{ km}}{(20-x)\text{km/h}}$$

Step 2
Solve the rational equation.

Cross-multiply, and then solve for x.

$$\begin{aligned}\frac{130 \text{ km}}{(20+x)\text{km/h}} &= \frac{70 \text{ km}}{(20-x)\text{km/h}}\\ 130(20-x) &= 70(20+x)\\ 2\ 600 - 130x &= 1\ 400 + 70x\\ 1\ 200 &= 200x\\ 6 &= x\end{aligned}$$

Therefore, the wind speed is 6 km/h.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. Step 1
With a single expression on either side of the equal sign, cross-multiply to remove the denominators, and then expand where necessary.

$$\begin{aligned}\frac{2}{x-1} &= \frac{3}{x}\\ 2(x) &= 3(x-1)\\ 2x &= 3x-3\end{aligned}$$

Step 2
Gather like terms, and solve for x.

$$\begin{aligned}2x &= 3x-3\\ 3 &= x\end{aligned}$$

Step 3
State any NPVs for the variable.

The NPVs from the original equation $\frac{2}{x-1} = \frac{3}{x}$ are x = 1 and x = 0.

Therefore, the solution is x = 3.

2. Step 1
With a single expression on either side of the equal sign, cross-multiply to remove the denominators, and then expand where necessary.

$$\begin{aligned}\frac{3x+1}{6x} &= \frac{x+1}{2x+1}\\ (3x+1)(2x+1) &= (x+1)(6x)\\ 6x^2+5x+1 &= 6x^2+6x\end{aligned}$$

Step 2
Gather like terms, and solve for x.

$$\begin{aligned}\cancel{6x^2}+5x+1 &= \cancel{6x^2}+6x\\ 1 &= x\end{aligned}$$

Step 3
State any NPVs for the variable.

The NPVs from the original equation $\frac{3x+1}{6x} = \frac{x+1}{2x+1}$ are $x = 0$ and $x = -\frac{1}{2}$.

Therefore, the solution is x = 1.

3. Step 1
Simplify where possible.

$$\begin{aligned}\frac{x}{x+1} &= \frac{-2}{2x+2}\\ \frac{x}{x+1} &= \frac{\cancelto{-1}{-2}}{\cancel{2}(x+1)}\\ \frac{x}{x+1} &= \frac{-1}{x+1}\end{aligned}$$

After factoring and eliminating a factor of 2 from the second expression, it is possible to continue solving using the set steps or to solve by observation.

Since both denominators are the same, the numerators must also be equal for the equality to be true. This means that $x = -1$. However, by noting the NPVs, it is found that $x \neq -1$. The only solution to this equation is non-permissible, so there are no solutions.

4. Step 1
Determine the LCM and the NPVs.

For the expression $\frac{2}{x}-\frac{5}{3x}=4$, the LCM of the denominators is $3x$. The only NPV is $x = 0$.

Step 2
Multiply each term by the LCM.

$$\frac{2}{x}-\frac{5}{3x}=4$$
$$(3x)\left[\frac{2}{x}-\frac{5}{3x}=4\right]$$
$$(3x)\frac{2}{x}-(3x)\frac{5}{3x}=(3x)4$$

Step 3
Simplify and solve for x.

$$(3\cancel{x})\frac{2}{\cancel{x}}-(\cancel{3x})\frac{5}{\cancel{3x}}=(3x)4$$
$$6-5=12x$$
$$1=12x$$
$$\frac{1}{12}=x$$
$$x=\frac{1}{12},\ x\neq 0$$

5. B
Solve by cross-multiplying.

$$\frac{13}{y+1}=4,\ y\neq -1$$
$$13=4(y+1)$$
$$13=4y+4$$
$$9=4y$$
$$y=\frac{9}{4}$$

The NPV from the original equation $\frac{13}{y+1}=4$ is $y\neq -1$. Therefore, the solution is $y=\frac{9}{4}$.

Alternative Solution
Use verification with each value to see if the left side equals the right side of the equation.

$y=\frac{3}{2}$	
$\frac{13}{\left(\frac{3}{2}\right)+1}$	4
$\frac{13}{\left(\frac{5}{2}\right)}$	4
$\frac{26}{5}$	4

$y=\frac{9}{4}$	
$\frac{13}{\left(\frac{9}{4}\right)+1}$	4
$\frac{13}{\left(\frac{13}{4}\right)}$	4
4	4

$y=\frac{4}{9}$	
$\frac{13}{\left(\frac{4}{9}\right)+1}$	4
$\frac{13}{\left(\frac{13}{9}\right)}$	4
9	4

$y=\frac{2}{3}$	
$\frac{13}{\left(\frac{2}{3}\right)+1}$	4
$\frac{13}{\left(\frac{5}{3}\right)}$	4
$\frac{39}{5}$	4

6. A cube has three equal dimensions (length, width, and height), and its volume is determined by multiplying these three dimensions together.

Step 1
Assign a variable to represent the unknown, and set up an equation using the given information.

Let x represent the side length of the small cube. This means the side length of the large cube, being twice as long, will be $2x$.

Since you are given the total volume of the two halves, make an equation that represents this total. Specifically, half of the small cube's volume added to half of the large cube's volume will equal 36 cm^3.

The volume of the small cube will be $V = lwh$ or $V_{small\ cube} = x^3$ since the three dimensions are equal. The large cube will have a volume of $V_{large\ cube} = (2x)^3$.

$$\frac{V_{\text{small cube}}}{2} + \frac{V_{\text{large cube}}}{2} = 36$$
$$\frac{x^3}{2} + \frac{(2x)^3}{2} = 36$$
$$\frac{x^3}{2} + \frac{8x^3}{2} = 36$$

Step 2
Solve the rational equation.

Simplify the left side, and then solve for *x*.

$$\frac{x^3}{2} + \frac{8x^3}{2} = 36$$
$$\frac{9x^3}{2} = 36$$
$$9x^3 = 72$$
$$x^3 = 8$$
$$x = 2$$

The side length of the small cube is 2 cm. There are no non-permissible values to check.

Practice Test

ANSWERS AND SOLUTIONS

1. D

The expression $\frac{4}{2x-2}$ simplifies to the expression $\frac{2}{(x-1)}$ through factoring and eliminating a common factor of 2.

The other answers reflect some common mistakes in simplifying.

2. Factor where possible, and eliminate common factors to simplify the expression.

The NPVs can be identified using the original denominator in factored form.

$$\frac{2x^2 - x - 1}{(6x+3)}$$
$$= \frac{(\cancel{2x+1})(x-1)}{3(\cancel{2x+1})}$$
$$= \frac{x-1}{3}, x \neq -\frac{1}{2}$$

3. D

After factoring, the original denominator will be $2(a)(a-1)$, so $a \neq 0$ and $a \neq 1$.

The variable *x* need not be introduced. Eliminating common factors does not eliminate the need to list NPVs from the original denominator.

4. Step 1
Find expressions for the lengths of rectangles *A* and *B*.

If the area of a rectangular object is found by multiplying the length and width $(A = lw)$, then rearranging this formula for length gives the formula $l = \frac{A}{w}$.

$$l_A = \frac{A_A}{w_A} \qquad l_B = \frac{A_B}{w_B}$$
$$l_A = \frac{x^2 + 4x - 5}{x - 1} \qquad l_B = \frac{x^2 - 4}{x - 2}$$

Step 2
Write a rational expression of the form $l_A \div l_B$, and simplify.

$$l_A \div l_B$$
$$= \frac{x^2 + 4x - 5}{x - 1} \div \frac{x^2 - 4}{x - 2}$$
$$= \frac{(\cancel{x-1})(x+5)}{\cancel{x-1}} \times \frac{(\cancel{x-2})}{(\cancel{x-2})(x+2)}$$
$$= \frac{(x+5)}{(x+2)}$$

Step 3
Identify the NPVs.

Considering each denominator before and after any simplifying, $x \neq 1$, $x \neq 2$, and $x \neq -2$. However, these rectangles would not exist in real space if either the length or width were reduced to zero or a negative value. Considering that the width of rectangle *A* is $(x-1)$ and the width of rectangle *B* is $(x-2)$, then no value of the variable less than or equal to 2 should be allowed. The full set of restrictions for the variable would then become $x > 2$.

5. Rewrite the quotient as a product using the reciprocal of the second expression.

Simplify the product and multiply as necessary. The NPVs can be identified using the original denominators in the quotient and the new denominator in the product.

$$\frac{(x+2)(x-3)}{(x-a)^2} \div \frac{x^2-x-6}{x-1}$$
$$= \left(\frac{(x+2)(x-3)}{(x-a)^2}\right)\left(\frac{x-1}{x^2-x-6}\right)$$
$$= \left(\frac{(\cancel{x+2})(\cancel{x-3})}{(x-a)(x-a)}\right)\left(\frac{x-1}{(\cancel{x+2})(\cancel{x-3})}\right)$$
$$= \frac{x-1}{(x-a)(x-a)},\ x \neq -2,\ 3$$

For the numerator to simplify to 1, there must be a common factor in the denominator to cancel the $(x-1)$ remaining in the numerator. This means that the denominator is $(x-1)^2$, and $a=1$.

The expression then simplifies to $\frac{1}{(x-1)}$, where $x \neq 1$, $x \neq -2$, and $x \neq 3$. Since *a* is not *x*, *a* = 1 is allowed even though $x \neq 1$.

6. **Step 1**
Simplify each expression where possible.

$$\frac{3}{x^2+2x}+\frac{x-1}{3x}-\frac{x^2+7x+10}{3x^2+15x}$$
$$= \frac{3}{x(x+2)}+\frac{x-1}{3x}-\frac{(\cancel{x+5})(x+2)}{3x(\cancel{x+5})}$$

Step 2
Identify the lowest common denominator (LCD). The LCD will be $3x(x+2)$.

Step 3
Write an equivalent expression with the LCD as the denominator.

$$\frac{3}{x(x+2)}+\frac{x-1}{3x}-\frac{(\cancel{x+5})(x+2)}{3x(\cancel{x+5})}$$
$$= \frac{3(3)}{3x(x+2)}+\frac{(x-1)(x+2)}{3x(x+2)}-\frac{(x+2)(x+2)}{3x(x+2)}$$
$$= \frac{9}{3x(x+2)}+\frac{x^2+x-2}{3x(x+2)}-\frac{x^2+4x+4}{3x(x+2)}$$

Step 4
Combine like terms in the numerators, keeping the common denominator unchanged.

$$\frac{9}{3x(x+2)}+\frac{x^2+x-2}{3x(x+2)}-\frac{x^2+4x+4}{3x(x+2)}$$
$$= \frac{9+x^2+x-2-x^2-4x-4}{3x(x+2)}$$
$$= \frac{-3x+3}{3x(x+2)}$$

Step 5
Check if the expression can be reduced further, and state the NPVs.

$$\frac{-3x+3}{3x(x+2)}$$
$$= \frac{\cancel{3}(-x+1)}{\cancel{3}x(x+2)}$$
$$= \frac{-x+1}{x(x+2)},\ x \neq 0, -2, -5$$

7. **Step 1**
Assign a variable to represent the unknown, and set up an equation using the given information.

Let *x* represent the unknown number. Set up an equation using the information that half of an unknown number is added to one quarter of the original unknown number, with the result equal to 10.

$$\frac{1}{2}\text{ of unknown}+\frac{1}{4}\text{ of unknown}=10$$
$$\frac{x}{2}+\frac{x}{4}=10$$

Step 2
Solve the rational equation.

Multiply the equation by the LCM to eliminate the denominators, and then solve for x.

$$\frac{x}{2}+\frac{x}{4}=10$$
$$(4)\left[\frac{x}{2}+\frac{x}{4}=10\right]$$
$$\left(\cancel{4}^{2}\right)\frac{x}{\cancel{2}}+\left(\cancel{4}\right)\frac{x}{\cancel{4}}=(4)10$$
$$2x+x=40$$
$$3x=40$$
$$x=\frac{40}{3}$$

Expressed as a fraction, the original number is $\frac{40}{3}$.

Step 3
Verify the solution.

One half of $\frac{40}{3}$ is $\frac{20}{3}$.

One quarter of $\frac{40}{3}$ is $\frac{10}{3}$.

Added together, these equal $\frac{30}{3}$, or 10, so the answer is verified.

8. **Step 1**
Assign a variable to represent the unknown, and set up an equation using the given information.

Let x represent the minimum mark that Molly must achieve on her sixth test.

To set up an equation, the test scores could each be written out as a series of fractions (representing the percentage marks) with the unknown included. For a desired passing grade, the overall average needs to be 50%, which can be represented by a mark of $\frac{50}{100}$ on each of the six tests. There are several different approaches that could be used to solve this type of question.

$$\frac{50}{100}+\frac{50}{100}+\frac{50}{100}+\frac{50}{100}+\frac{52}{100}+\frac{x}{100}=\frac{50(6)}{100}$$

Step 2
Solve the rational equation.

Simplify the left side, and then cross-multiply to solve for x.

$$\frac{50}{100}+\frac{50}{100}+\frac{50}{100}+\frac{50}{100}+\frac{52}{100}+\frac{x}{100}=\frac{50(6)}{100}$$
$$\frac{252+x}{100}=\frac{300}{100}$$
$$\left(\cancel{100}\right)(252+x)=\left(\cancel{100}\right)(300)$$
$$252+x=300$$
$$x=48$$

Therefore, Molly must achieve a minimum mark of 48% on the sixth test to receive an overall passing grade.

TRIGONOMETRY

Lesson 1—The Primary Trigonometric Ratios and Special Triangles

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Since the adjacent side to the given angle is known and the hypotenuse is unknown, use the cosine ratio.

$$\cos 38° = \frac{13}{m}$$
$$m\cos 38° = 13$$
$$m = \frac{13}{\cos 38}$$
$$m \doteq 16.5 \text{ cm}$$

2. Since the opposite and adjacent sides to θ are known, use the tangent ratio.

$$\tan\theta = \frac{21}{12}$$
$$\theta = \tan^{-1}\left(\frac{21}{12}\right)$$

$\theta \doteq$ [redacted]°

3. To find the exact value of $\sin 45°$, use the 45-45-90 triangle and the definition of the sine ratio.

$$\sin 45° = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin 45° = \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}$$

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

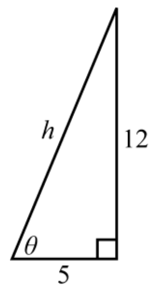
1. Since the adjacent side to θ and the hypotenuse are known, use the cosine ratio.

$$\cos\theta = \frac{24}{37}$$
$$\theta = \cos^{-1}\left(\frac{24}{37}\right)$$

$\theta \doteq$ [redacted]°

2. **Step 1**
Draw a diagram that represents the known information.

If $\tan\theta = \frac{12}{5}$, then the opposite and adjacent sides to θ are known.



Step 2
Use the Pythagorean theorem to find the missing side length, h.

$$h^2 = 12^2 + 5^2$$
$$h = \sqrt{12^2 + 5^2}$$
$$h = \sqrt{169}$$
$$h = 13$$

Step 3
Find $\cos\theta$.

Therefore, $\cos\theta = \frac{5}{13}$.

3. Since side x is opposite the given angle, and the hypotenuse is known, use the sine ratio.

$$\sin 52° = \frac{x}{15}$$
$$15\sin 52° = x$$

11.8 cm $\doteq$ [redacted]

4. Use the 45-45-90 and the 30-60-90 triangles and the primary trigonometric ratios:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

	$\sin\theta$	$\cos\theta$	$\tan\theta$
$\theta = 30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
$\theta = 45°$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	1
$\theta = 60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

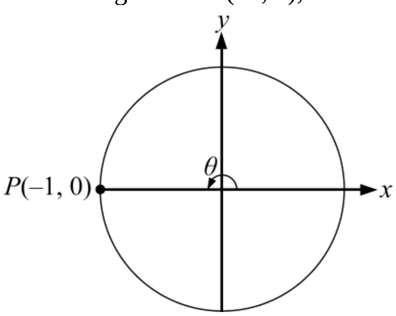
Lesson 2—Angles in Standard Position

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. A rotation angle of 260° will terminate in quadrant III. The reference angle is found by subtracting 180° from the rotation angle.

$$\begin{aligned}\text{reference angle} &= 260° - 180° \\ &= 80°\end{aligned}$$

2. Point P is given as $P(-1, 0)$, so $x = -1$ and $y = 0$.



Since the terminal arm happens to be a horizontal line, the radius and side x coincide and there is no reference angle (reference angles must be acute). However, since the radius is a length (not a coordinate), it remains positive, even though $x = -1$.

To find the value of $\sin\theta$, $\cos\theta$, and $\tan\theta$, substitute the known information.

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$
$$\sin\theta = \frac{0}{1} \qquad \cos\theta = \frac{-1}{1} \qquad \tan\theta = \frac{0}{-1}$$
$$\sin\theta = 0 \qquad \cos\theta = -1 \qquad \tan\theta = 0$$

3. **Step 1**
A rotation angle of 315° terminates in quadrant IV, so the reference angle will be 360° – 315° = 45°.

Step 2
In quadrant IV, the tangent ratio is negative.

Using the 45-45-90 special triangle, the exact value of tan 315° is –1.

4. It is known that $\sin\theta = \frac{y}{r}$ and $y = -3$ from $P(-4, -3)$.

Calculate the length of r using the Pythagorean theorem.

Substitute $x = -4$ and $y = -3$ into $r = \sqrt{x^2 + y^2}$.

$$\begin{aligned}r &= \sqrt{(-4)^2 + (-3)^2} \\ r &= \sqrt{16+9} \\ r &= \sqrt{25} \\ r &= 5\end{aligned}$$

Therefore, $\sin\theta = \frac{-3}{5}$ or $-\frac{3}{5}$.

5. The rotation angle of 210° terminates in quadrant III and has a reference angle of 210° – 180° = 30°. From $P(x, -2)$, it is known that $y = -2$.

Knowing that the reference angle is 30° and $y = -2$, use the sine ratio to find the exact value of r. To find the exact value, use the 30-60-90 triangle (since the reference angle is 30°).
In quadrant III, the sine ratio is negative.

$$\begin{aligned}\sin\theta &= \frac{y}{r} \\ -(\sin 30) &= \frac{-2}{r} \\ -\frac{1}{2} &= \frac{-2}{r} \\ -r &= -4 \\ r &= 4\end{aligned}$$

Therefore, the exact value of r is 4.

6. **Step 1**
Isolate $\cos\theta$.
$$2\cos\theta = \sqrt{3}$$
$$\cos\theta = \frac{\sqrt{3}}{2}$$

Step 2
Use the CAST rule to identify the quadrants in which the cosine ratio is positive.

The cosine ratio is positive in quadrants I and IV.

Step 3
Find the reference angle.
$$\begin{aligned}\theta_{ref} &= \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\\ &= 30°\end{aligned}$$

It is not necessary to use a calculator if you recognize the ratio $\frac{\sqrt{3}}{2}$ from the 30-60-90 special triangle.

Step 4
Find the rotation angles.

In quadrant I, the rotation angle is equal to the reference angle. In quadrant IV, the rotation angle is 360° – 30° = 330°.

Therefore, $\theta = 30°$ and $\theta = 330°$.

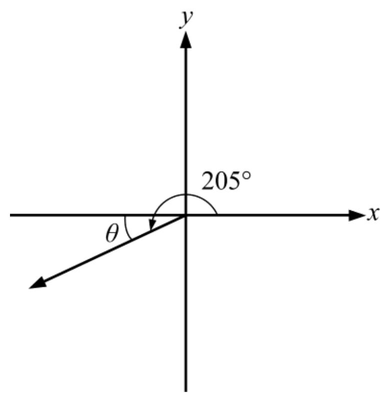
7. The CAST rule does not apply when solving $\sin\theta = 0$ since 0 is not considered a positive or negative number. However, $\sin\theta = \frac{y}{r}$.

If $\sin\theta = 0$, then $\frac{y}{r} = 0$, which means that $y = 0$. The value of y is 0 when the terminal arm of an angle θ in standard position lies on the x-axis.

The terminal arm of an angle lies on the x-axis when θ is 0°, 180°, and again at 360°, after a full rotation.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Draw the rotation angle in standard position, and identify the quadrant in which it terminates.



The rotation angle 205° terminates in quadrant III.

Step 2
Find the reference angle.

Let θ represent the reference angle. In quadrant III, the reference angle is equal to the rotation angle minus 180°.
$$\begin{aligned}\theta &= 205° - 180°\\ &= 25°\end{aligned}$$

The reference angle is 25°.

2. **Step 1**
Identify the quadrant in which the terminal arm of the rotation angle lies. Use the CAST rule to identify if the ratio is positive or negative.

The rotation angle 240° terminates in quadrant III, so the cosine ratio will be negative.

Step 2
Find the reference angle.

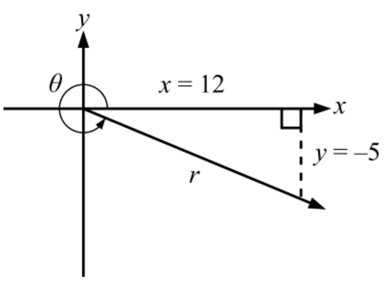
In quadrant III, the reference angle is equal to the rotation angle minus 180°.
240° – 180° = 60°

Step 3
Use the reference angle and related special triangle to find the exact value.

$$\begin{aligned}&\cos 240^\circ\\&=-\cos 60^\circ\\&=-\left(\frac{1}{2}\right)\\&=-\frac{1}{2}\end{aligned}$$

The exact value of cos 240° is $-\frac{1}{2}$.

3. **Step 1**
Since point $P(12, -5)$ is already sketched in quadrant IV, it is known that $x = 12$ and $y = -5$.



Step 2
Find the length of r using the Pythagorean theorem. The radius is always positive because it is a length (not a coordinate).

$$\begin{aligned}r&=\sqrt{12^2+(-5)^2}\\r&=\sqrt{169}\\r&=13\end{aligned}$$

Step 3
Find the primary trigonometric ratios.

Keep the CAST rule in mind to determine the sign of each.

$$\sin\theta=\frac{y}{r}=-\frac{5}{13}$$
$$\cos\theta=\frac{x}{r}=\frac{12}{13}$$
$$\tan\theta=\frac{y}{x}=-\frac{5}{12}$$

4. The angle 210º terminates in quadrant III, so it has a reference angle of 210° – 180° = 30º, and the cosine ratio will be negative.

Using a special triangle, the exact value of cos 210º in quadrant III is $-\frac{\sqrt{3}}{2}$. The cosine ratio is also negative in quadrant II. Therefore, the rotation angle with the same cosine ratio as 210º will be a quadrant II angle with a reference angle of 30º. The rotation angle in quadrant II will be 180º – 30º = 150º.

Therefore, 150º is the other angle that has the same cosine ratio as 210º, where $0^\circ \le \theta < 360^\circ$.

5. **Step 1**
Use the CAST rule to identify the two quadrants in which the sine ratio is positive.

The sine ratio is positive in quadrants I and II.

Step 2
Find the reference angle using the inverse sine function on a calculator.

$$\begin{aligned}\theta&=\sin^{-1}(0.7071)\\&\doteq 45^\circ\end{aligned}$$

Step 3
Find the two rotation angles.

In quadrant I, the rotation angle is equal to the reference angle. In quadrant II, the rotation angle is 180° – 45° = 135°.

Therefore, $\sin 45^\circ = 0.7071$ and $\sin 135^\circ = 0.7071$.

6. **Step 1**
Use the CAST rule to identify the two quadrants in which the tangent ratio is positive.

Since the tangent ratio is positive, the rotation angles will be in quadrants I and III.

Step 2
Find the reference angle using the 45-45-90 special triangle.

Using the 45-45-90 special triangle, the tangent ratio of 1 corresponds to a reference angle of 45°.

Step 3
Find the two rotation angles.

The rotation angles are $\theta = 45^\circ$ and $\theta = 180^\circ + 45^\circ$, which is $\theta = 225^\circ$.

Lesson 3—The Sine Law

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Find the missing angle.

The interior angles in any triangle have a sum of 180°.

$\angle X = 180° - 72° - 40°$
$\angle X = 68°$

Step 2
Solve for x.

The complete ratio $\frac{105}{\sin 72°}$ is known, and the angle opposite to the unknown side is 68°. Substitute these values into the sine law formula, and solve for x.

$$\frac{x}{\sin 68°} = \frac{105}{\sin 72°}$$
$$x = \frac{105\sin(68°)}{\sin 72°}$$
$$x \doteq 102 \text{ m}$$

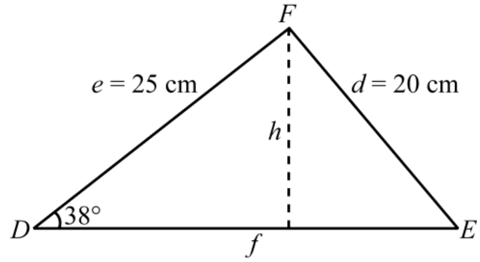
2. The complete ratio $\frac{34}{\sin 40°}$ is known, and the side opposite θ is 37 cm. Substitute these values into the sine law, and solve for θ.

$$\frac{\sin\theta}{37} = \frac{\sin 40°}{34}$$
$$\sin\theta = \frac{37\sin(40°)}{34}$$
$$\theta = \sin^{-1}\left(\frac{37\sin(40°)}{34}\right)$$

$\theta \doteq$ [redacted]°

3. **Step 1**
Draw and label a possible sketch of ΔDEF, where h represents the height of the triangle.

It is best to sketch the particular triangle such that the base of the triangle is the unknown side of the triangle and the given angle is the left vertex of the triangle.



Step 2
Find the value of h.

$$\sin D = \frac{h}{e}$$
$$\sin 38° = \frac{h}{25}$$
$$25\sin 38° = h$$

15.4 cm $\doteq$ [redacted]

Step 3
Find the number of possible triangles.

Since side d (20 cm) is less than side e (25 cm), but greater than h (15.4 cm), there are two triangles possible. In other words, since $d < e$ and $d > h = e\sin D$, there are two possible values for the length of side f.

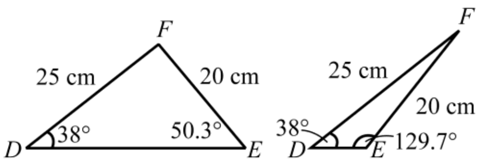
Step 4
Apply the sine law to calculate the measure of $\angle E$.

Substitute 20 for d, 25 for e, and 38° for $\angle D$.

$$\frac{\sin E}{e} = \frac{\sin D}{d}$$
$$\frac{\sin E}{25} = \frac{\sin 38°}{20}$$
$$\sin E = \frac{25\sin(38°)}{20}$$
$$E = \sin^{-1}\left(\frac{25\sin(38°)}{20}\right)$$
$$E \doteq 5 \quad 3$$

Therefore, if $\angle E$ is acute, then $\angle E \doteq 50.3°$; if $\angle E$ is obtuse, then $\angle E \doteq 180° - 50.3°$, which is $\angle E \doteq 129.7°$.

Step 5
Sketch the two possible triangles that will be used to find the two possible base lengths, *f*.



Step 6
Find the length of the base of each of the two possible triangles using the sine law.

In the first triangle, if $\angle E \doteq 50.3°$, then $\angle F \doteq 180° - (38° + 50.3°)$, which is $\angle F \doteq 91.7°$.

Substitute the known information into the sine law, and solve for *f*.

$$\frac{f}{\sin 91.7°} = \frac{20}{\sin 38°}$$
$$f = \frac{20\sin(91.7°)}{\sin 38°}$$
$$f \doteq 32.5 \text{ cm}$$

In the second triangle, if $\angle E \doteq 129.7°$, then $\angle F \doteq 180° - \ 38° + 129.7°)$, which is $\angle F \doteq 12.3°$.

Substitute the known information into the sine law, and solve for *f*.

$$\frac{f}{\sin 12.3°} = \frac{20}{\sin 38°}$$
$$f = \frac{20\sin(12.3°)}{\sin 38°}$$
$$f \doteq 6.9 \text{ cm}$$

Therefore, the two possible values for *f* are 32.5 cm and 6.9 cm.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. The complete ratio $\frac{25}{\sin 50°}$ is known, and the angle opposite side *x* is 83°. Substitute these values into the sine law, and solve for *x*.

$$\frac{x}{\sin 83°} = \frac{25}{\sin 50°}$$
$$x = \frac{25\sin(83°)}{\sin 50°}$$
$$x \doteq 32.4 \text{ m}$$

2. The complete ratio $\frac{\sin 81°}{55}$ is known, and the side opposite θ is 32 m. Substitute these values into the sine law, and solve for θ.

$$\frac{\sin\theta}{32} = \frac{\sin 81°}{55}$$
$$\sin\theta = \frac{32\sin(81°)}{55}$$
$$\theta = \sin^{-1}\left(\frac{32\sin(81°)}{55}\right)$$

$\theta \doteq$ [redacted]°

3. Do not use the primary trigonometric ratios since you cannot be sure that either triangle has a right angle.

Step 1
Solve for side *x*.

Find side *x* first since the complete ratio $\frac{52}{\sin 68°}$ is known and the angle opposite side *x* is 56°.

$$\frac{x}{\sin 56°} = \frac{52}{\sin 68°}$$
$$x = \frac{52\sin(56°)}{\sin 68°}$$
$$x \doteq 46.4956$$

Step 2
Solve for θ.

Now that the length of side *x* is known, solve for θ.

$$\frac{\sin\theta}{46.4956} \doteq \frac{\text{i} \quad 2°}{41}$$
$$\sin\theta \doteq \frac{46.4956\sin(42°)}{41}$$
$$\theta \doteq \sin^{-1}\left(\frac{46.4956\sin(42°)}{41}\right)$$
$$\theta \doteq 49.36°$$

Therefore, side *x* is 46 m and θ is 49°.

4. **Step 1**
Find the missing angle.
Find $\angle Z$ to create a complete ratio.

$$\angle Z = 180° - 100° - 50°$$
$$= 30°$$

The complete set is $\frac{25}{\sin 30°}$.

Step 2
Solve for the unknown.

The measure of $\angle Y = 50°$, which is opposite the unknown side *XZ*. Substitute the known values into the sine law, and solve for the length of side *XZ*. Side *XZ* could also be called side *y*.

$$\frac{XZ}{\sin 50°} = \frac{25}{\sin 30°}$$
$$XZ = \frac{25\sin(50°)}{\sin 30°}$$
$$XZ \doteq 38.3 \text{ m}$$

The length of side *XZ* is 38.3 m.

Lesson 4—The Cosine Law

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Decide which type of trigonometric law to use.

The given information is of the form SAS, so substitute the given values into the cosine law.

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$x^2 = 21^2 + 25^2 - 2(21)(25)\cos 71°$$

Step 2
Simplify the right side, and take the square root of both sides to solve for *x*.

$$x^2 = 21^2 + 25^2 - 2(21)(25)\cos 71°$$
$$x^2 \doteq \quad 4$$
$$x \doteq \sqrt{724.1534}$$
$$x \doteq 27 \text{ m}$$

2. **Step 1**
Decide which type of trigonometric law to use.

The given information is of the form SSS, so substitute the given values into the cosine law.

Find the measure of $\angle A$. The side measuring 8.7 cm becomes side *a*.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos A = \frac{9.6^2 + 6.4^2 - 8.7^2}{2(9.6)(6.4)}$$

Step 2
Simplify the right side, and then take the inverse cosine of both sides to solve for $\angle A$.

$$\cos A = \frac{9.6^2 + 6.4^2 - 8.7^2}{2(9.6)(6.4)}$$
$$\cos A = \frac{57.43}{122.88}$$
$$A = \cos^{-1}\left(\frac{57.43}{122.88}\right)$$

$A \doteq$ [redacted]°

Step 3
Solve for $\angle B$ and $\angle C$.

Now that $\angle A$ is known, you could use the sine or cosine of $\angle B$.

$$\cos B = \frac{8.7^2 + 6.4^2 - 9.6^2}{2(8.7)(6.4)}$$
$$\cos B = \frac{24.49}{111.36}$$
$$B = \cos^{-1}\left(\frac{24.49}{111.36}\right)$$

$B \doteq$ [redacted]°

To find $\angle C$, subtract the two known angles from 180°.

$$\angle C \doteq 180° - 62° - \quad °$$
$$\angle C \doteq 41°$$

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. To find the unknown side, use the cosine law since SAS is known.

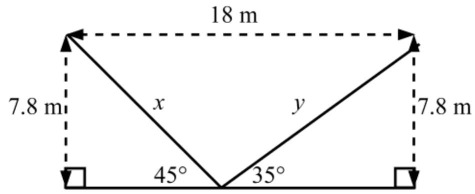
2. Since the side opposite the unknown angle is given and a complete side-angle ratio is known, use the sine law.

3. To find the unknown angle, use the cosine law since SSS is known.

4. Since the angle opposite the unknown side can be found by subtracting the two known angles from 180° and a complete side-angle ratio is known, use the sine law.

5. Step 1

Label the unknown sides as x and y.



The heights of the right triangles in the diagram are both 8.3 m – 0.5 m = 7.8 m.

Step 2

Find the length of the unknown sides.

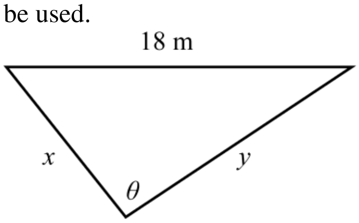
Since both sides are part of right triangles, a primary trigonometric ratio can be used to solve for x and y.

$$\sin 45° = \frac{7.8}{x}$$
$$x = \frac{7.8}{\sin 45°}$$
$$x \doteq 11.0309 \text{ m}$$
$$\sin 35° = \frac{7.8}{y}$$
$$y = \frac{7.8}{\sin 35°}$$
$$y \doteq 13.5989 \text{ m}$$

Step 3

Solve for the angle formed between the two kite strings.

Since all three sides are known, the cosine law can be used.



$$\cos\theta = \frac{x^2 + y^2 - 18^2}{2xy}$$
$$\theta = \cos^{-1}\left(\frac{\left(\frac{7.8}{\sin 45°}\right)^2 + \left(\frac{7.8}{\sin 35°}\right)^2 - 18^2}{2\left(\frac{7.8}{\sin 45°}\right)\left(\frac{7.8}{\sin 35°}\right)}\right)$$
$$\theta \doteq \quad 3°$$

Rather than using the decimal approximations for sides x and y, the exact ratios are used to avoid any rounding errors.

Therefore, the angle formed between the two kite strings is 93.3°.

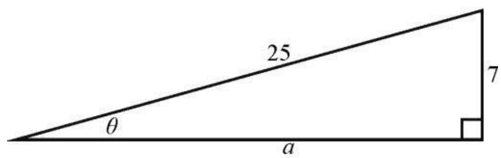
Practice Test

ANSWERS AND SOLUTIONS

1. Step 1

Draw a diagram that represents the information.

If $\sin\theta = \frac{7}{25}$, then the hypotenuse of the triangle could be 25 units, and the side opposite to θ could be 7 units (or any values that produce a ratio equivalent to $\frac{7}{25}$).



Step 2

Use the Pythagorean theorem to find a.

$$a^2 + 7^2 = 25^2$$
$$a^2 + 49 = 625$$
$$a^2 = 576$$
$$a = \sqrt{576}$$
$$a = 24$$

Step 3

Identify the required ratio now that all three sides are known.

$$\cos\theta = \frac{24}{25}$$

2. **Step 1**
Identify the reference angle.

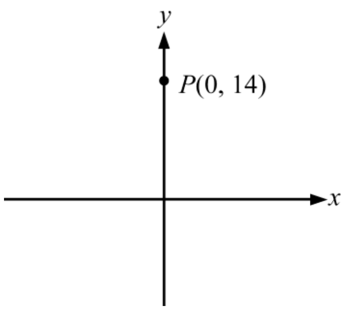
The rotation angle terminates in quadrant IV, so the reference angle is 360° – 315° = 45°.

Step 2
Use the 45-45-90 special triangle to find the exact value of the cosine ratio. Use the CAST rule to determine if the cosine ratio is positive or negative in quadrant IV.

The cosine ratio is positive in quadrant IV.

$$\begin{aligned}\cos 315^\circ &= \cos 45^\circ\\ &= \frac{1}{\sqrt{2}}\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

3. The coordinates of point P are given as $P(0, 14)$, and it is known that $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, and $\tan\theta = \frac{y}{x}$.



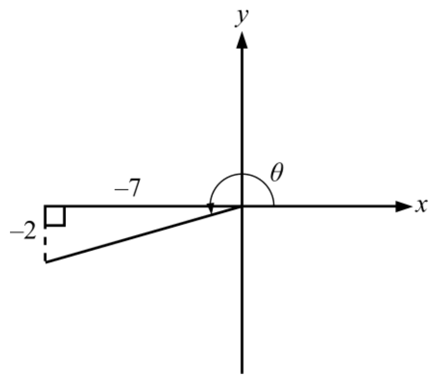
Since the terminal arm happens to be a vertical line, the radius and side y coincide (both are 14 units), and there is no reference angle. To find the value of $\sin\theta$, $\cos\theta$, and $\tan\theta$, substitute the known information into the definition of each primary trigonometric ratio.

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r}$$
$$\sin\theta = \frac{14}{14} \qquad \cos\theta = \frac{0}{14}$$
$$\sin\theta = 1 \qquad \cos\theta = 0$$

$$\tan\theta = \frac{y}{x}$$
$$\tan\theta = \frac{14}{0}$$
$$\tan\theta = \text{undefined}$$

4. **Step 1**
If necessary, make a sketch to identify the quadrant in which the rotation angle terminates.

If $\tan\theta = \frac{y}{x}$ and it is given that $\tan\theta = \frac{2}{7}$, where θ terminates in quadrant III, then possible values for x and y are $x = -7$ and $y = -2$. This is enough information to sketch angle θ in standard position in quadrant III.



Step 2
Find the unknown ratio.

Keep the CAST rule in mind to determine the sign.

If $\sin\theta = \frac{y}{r}$, it is necessary to determine the value of r using the Pythagorean theorem.

$$r^2 = x^2 + y^2$$
$$r = \sqrt{(-7)^2 + (-2)^2}$$
$$r = \sqrt{53}$$

Therefore, $\sin\theta = \frac{-2}{\sqrt{53}}$ or $-\frac{2\sqrt{53}}{53}$.

5. **D**
Step 1
Use the CAST rule to identify the two quadrants in which the cosine ratio is negative.

The cosine ratio is negative in quadrants II and III.

Step 2
Find the reference angle using the inverse cosine function on a calculator.

$$\theta = \cos^{-1}\left(\frac{3}{5}\right)$$
≐ [redacted]°

Step 3
Find the two rotation angles.

In quadrant II, the rotation angle is
180° – 53° = 127°.

In quadrant III, the rotation angle is
180° + 53° = 233°.

Therefore, $\cos 127° = -\frac{3}{5}$ and $\cos 233° = -\frac{3}{5}$.

6. Step 1
Use the CAST rule to identify the two quadrants in which the ratios are both positive or both negative.

Sine and cosine are positive in quadrant I and negative in quadrant III.

Step 2
Identify the reference angle.

For $\sin\theta = \cos\theta$, the values of x and y must be the same. If this is the case, then the special 45-45-90 triangle applies in which $x = 1$ and $y = 1$. Therefore, the reference angle is 45°.

Step 3
Find the two rotation angles.

In quadrant I, the rotation angle is equal to the reference angle (45°).

In quadrant III, the rotation angle is
180° + 45° = 225°.

Therefore, $\sin\theta = \cos\theta$ when θ = 45° and θ = 225°.

7. Step 1
Isolate $\tan\theta$.

In this case, add 1 to both sides and then divide both sides by 5.

$$5\tan\theta - 1 = 3$$
$$5\tan\theta = 4$$
$$\tan\theta = \frac{4}{5}$$

Step 2
Find the reference angle using the inverse tangent function on a calculator.

$$\theta = \tan^{-1}\left(\frac{4}{5}\right)$$
≐ [redacted]°

Step 3
Find the two rotation angles.

The tangent ratio is positive in quadrants I and III. In quadrant I, the rotation angle is equal to the reference angle (38.7°).

In quadrant III, the rotation angle is
180° + 38.7° = 218.7°.

Therefore, θ = 38.7° and θ = 218.7°.

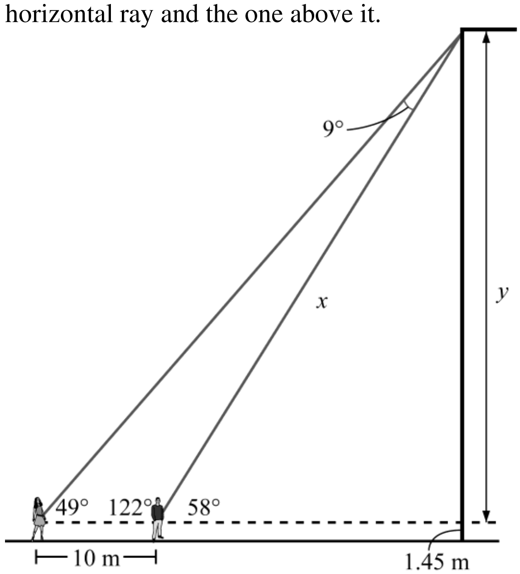
8. B
The primary trigonometric ratios (SOH CAH TOA) can only be used to find side or angle measures of right triangles. Therefore, the formula

$\tan A = \frac{\text{opposite}}{\text{adjacent}}$ works only with right triangles.

The sine law and cosine law can be used for any type of triangle.

9. Step 1
Draw a diagram representing the given information.

Assuming the ground is horizontal and the building is vertical, a right angle is formed. An angle of elevation is the measure of an angle between a horizontal ray and the one above it.



Since the angle measure of a straight line is 180°, the 122° angle can be determined by
180° – 58° = 122°.

Step 2
Solve for side x.

Find side x first since the complete ratio $\frac{10}{\sin 9°}$ is known and the angle opposite to side x is 49°.

$$\frac{x}{\sin 49°} = \frac{10}{\sin 9°}$$
$$x = \frac{10\sin(49°)}{\sin 9°}$$
$$x \doteq 48.2445 \text{ m}$$

Step 3
Solve for y.

In the right triangle, side y is opposite 58°, and now the length of x, the hypotenuse, is known. Therefore, use the sine ratio.

$$\sin 58° = \frac{y}{x}$$
$$\sin 58° = \frac{y}{\left(\frac{10\sin(49°)}{\sin 9°}\right)}$$
$$y = \frac{10\sin(49°)\sin(58°)}{\sin(9°)}$$

$y \doteq$ [redacted]

Step 4
Find the height of the tower.

The height of the student's eye level must be added to the value of y.
$40.91 + 1.45 = 42.36$

The height of the tower is 42.36 m.

10. Step 1
Decide which type of trigonometric law to use.

The given information is of the form SSS, so substitute the given values into the cosine law.

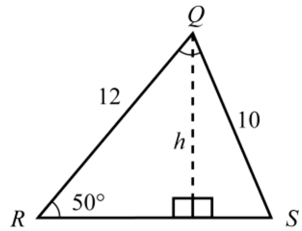
$$\cos\theta = \frac{18^2 + 12^2 - 19^2}{2(18)(12)}$$

Step 2
Simplify the right side, and then take the inverse cosine of both sides to solve for θ.

$$\cos\theta = \frac{107}{432}$$
$$\theta = \cos^{-1}\left(\frac{107}{432}\right)$$

$\theta \doteq$ [redacted]°

11. Step 1
Draw and label a possible sketch of ΔQRS, where h represents the height of the triangle.



Triangle QRS has known side lengths of r and s and a known angle measure, $\angle R$, which is a non-included angle (not between sides r and s).

As such, this may be an example of an ambiguous-case triangle (SSA).

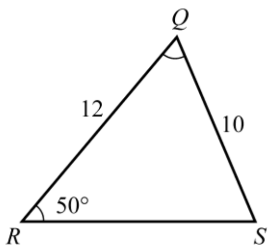
Step 2
Find the value of h.

$$\sin R = \frac{h}{s}$$
$$\sin 50° = \frac{h}{12}$$
$$12\sin 50° = h$$
$$9.2 \text{ cm} \doteq h$$

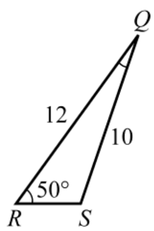
Step 3
Find the number of possible triangles.

Since side r (10 cm) is less than side s (12 cm), but greater than h (9.2 cm), there are two possible triangles and thus two possible angle measures for $\angle Q$.

Case 1



Case 2



Step 4
Apply the sine law to calculate the measure of $\angle S$.

Substitute 10 for *r*, 12 for *s*, and 50° for $\angle R$.

$$\frac{\sin S}{12} = \frac{\sin 50°}{10}$$
$$\sin S = \frac{12\sin(50°)}{10}$$
$$S = \sin^{-1}\left(\frac{12\sin(50°)}{10}\right)$$

$S \doteq$ [redacted]°

Since the value of $\sin S$ is positive in either quadrant I, $0° < S < 90°$, or quadrant II, $90° < S < 180°$, the measure of $\angle S$ could either be that of an acute or an obtuse angle.

Therefore, the possible measures of $\angle S$ are 66.8° or $\angle S \doteq 180° - 66.8°$, which is $\angle S \doteq 113.2°$.

Step 5
Determine the possible measures for $\angle Q$.

The sum of the angles in any triangle is 180°, so $\angle Q + \angle R + \angle S = 180°$. Since there are two possible measures for $\angle S$, there are also two possible measures for $\angle Q$.

If $\angle S \doteq 66.8°$, find the value of $\angle Q$.

$$\begin{aligned}\angle Q &\doteq 180° - 50° - 66.8° \\ &\doteq 63.2°\end{aligned}$$

If $\angle S \doteq 113.2°$, find the value of $\angle Q$.

$$\begin{aligned}\angle Q &\doteq 180° - 50° - 113.2° \\ &\doteq 16.8°\end{aligned}$$

To the nearest tenth of a degree, the measure of $\angle Q$ can be either 63.2° or 16.8°.

FACTORING POLYNOMIALS

Lesson 1—Factoring Using Decomposition

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. In $x^2 - 2x - 15$, the constant term is –15 and the middle term has a coefficient of –2.

Step 1
Find two integers with a product of –15 and a sum of –2.

The pairs of integers with a product of –15 and their respective sums are as follows:

Product	Sum
$(-1)\times(15) = -15$	$(-1)+(15) = 14$
$(-3)\times(5) = -15$	$(-3)+(5) = 2$
$1\times(-15) = -15$	$1+(-15) = -14$
$\boxed{3\times(-5) = -15}$	$\boxed{3+(-5) = -2}$

The integers required are 3 and –5.

Step 2
Express $-2x$ as $3x - 5x$.

$$\begin{aligned}&x^2 - 2x - 15 \\ &= x^2 + 3x - 5x - 15\end{aligned}$$

Step 3
Group the terms, and remove the greatest common factor from each group.

$$\begin{aligned}&x^2 + 3x - 5x - 15 \\ &= (x^2 + 3x) + (-5x - 15) \\ &= x(x+3) - 5(x+3)\end{aligned}$$

Step 4
Factor out the common binomial.

$$\begin{aligned}&x(x+3) - 5(x+3) \\ &= (x+3)(x-5)\end{aligned}$$

2. In $-6x^2+11x-3$, the product of the coefficient of x^2 and the constant term is $(-6)\times(-3)=18$.
The middle term has a coefficient of 11.

Step 1
Find two integers that have a product of 18 and a sum of 11.

The pairs of integers with a product of 18 and their respective sums are as follows:

Product	**Sum**
$(-1)\times(-18)=18$	$(-1)+(-18)=-19$
$(-2)\times(-9)=18$	$(-2)+(-9)=-11$
$(-3)\times(-6)=18$	$(-3)+(-6)=-9$
$1\times18=18$	$1+18=19$
$\boxed{2\times9=18}$	$\boxed{2+9=11}$
$3\times6=18$	$3+6=9$

The integers required are 2 and 9.

Step 2
Express $11x$ as $2x+9x$.
$-6x^2+11x-3$
$=-6x^2+2x+9x-3$

Step 3
Group the terms, and remove the greatest common factor from each group.
$-6x^2+2x+9x-3$
$=(-6x^2+2x)+(9x-3)$
$=-2x(3x-1)+3(3x-1)$

Step 4
Factor out the common binomial.
$-2x(3x-1)+3(3x-1)$
$=(3x-1)(-2x+3)$

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. In $x^2+3x-40$, the constant term is –40 and the middle term has a coefficient of 3.

Step 1
Find two integers with a product of –40 and a sum of 3.

The pairs of integers with a product of –40 and their respective sums are as follows:

Product	**Sum**
$(-1)\times40=-40$	$(-1)+40=39$
$(-2)\times20=-40$	$(-2)+20=18$
$(-4)\times10=-40$	$(-4)+10=6$
$\boxed{(-5)\times8=-40}$	$\boxed{(-5)+8=3}$
$1\times(-40)=-40$	$1+(-40)=-39$
$2\times(-20)=-40$	$2+(-20)=-18$
$4\times(-10)=-40$	$4+(-10)=-6$
$5\times(-8)=-40$	$5+(-8)=-3$

The integers required are –5 and 8.

Step 2
Express $3x$ as $-5x+8x$.
$x^2+3x-40$
$=x^2-5x+8x-40$

Step 3
Group the terms, and remove the greatest common factor from each group.
$x^2-5x+8x-40$
$=(x^2-5x)+(8x-40)$
$=x(x-5)+8(x-5)$

Step 4
Factor out the common binomial.
$x(x-5)+8(x-5)$
$=(x-5)(x+8)$

2. In $6x^2 - x - 15$, the product of the coefficient of x^2 and the constant term is $6 \times (-15) = -90$.
The middle term has a coefficient of –1.

Step 1
Find two integers that have a product of –90 and a sum of –1.

The pairs of integers with a product of –90 and their respective sums are as follows:

Product	Sum
$(-1) \times 90 = -90$	$(-1) + 90 = 89$
$(-2) \times 45 = -90$	$(-2) + 45 = 43$
$(-3) \times 30 = -90$	$(-3) + 30 = 27$
$(-5) \times 18 = -90$	$(-5) + 18 = 13$
$(-6) \times 15 = -90$	$(-6) + 15 = 9$
$(-9) \times 10 = -90$	$(-9) + 10 = 1$
$1 \times (-90) = -90$	$1 + (-90) = -89$
$2 \times (-45) = -90$	$2 + (-45) = -43$
$3 \times (-30) = -90$	$3 + (-30) = -27$
$5 \times (-18) = -90$	$5 + (-18) = -13$
$6 \times (-15) = -90$	$6 + (-15) = -9$
$\boxed{9 \times (-10) = -90}$	$\boxed{9 + (-10) = -1}$

The integers required are 9 and –10.

Step 2
Express $-x$ as $9x - 10x$.
$6x^2 - x - 15$
$= 6x^2 - 10x + 9x - 15$

Step 3
Group the terms, and remove the greatest common factor from each group.
$6x^2 - 10x + 9x - 15$
$= (6x^2 - 10x) + (9x - 15)$
$= 2x(3x - 5) + 3(3x - 5)$

Step 4
Factor out the common binomial.
$2x(3x - 5) + 3(3x - 5)$
$= (3x - 5)(2x + 3)$

3. **Step 1**
Factor out the greatest common factor from $9x^2 + 39x + 30$.

The greatest common factor is 3.
$9x^2 + 39x + 30$
$= 3(3x^2 + 13x + 10)$

In $3x^2 + 13 + 10$, the product of the coefficient of x^2 and the constant term is $3 \times 10 = 30$.
The middle term has a coefficient of 13.

Step 2
Find two integers that have a product of 30 and a sum of 13.

The pairs of integers with a product of 30 and their respective sums are as follows:

Product	Sum
$(-1) \times (-30) = 30$	$(-1) + (-30) = -31$
$(-2) \times (-15) = 30$	$(-2) + (-15) = -17$
$(-3) \times (-10) = 30$	$(-3) + (-10) = -13$
$(-5) \times (-6) = 30$	$(-6) + (-5) = -11$
$1 \times 30 = 30$	$1 + 30 = 31$
$2 \times 15 = 30$	$2 + 15 = 17$
$\boxed{3 \times 10 = 30}$	$\boxed{3 + 10 = 13}$
$5 \times 6 = 30$	$6 + 5 = 11$

The integers required are 3 and 10.

Step 3
Express $13x$ as $3x + 10x$.
$3(3x^2 + 13x + 10)$
$= 3(3x^2 + 3x + 10x + 10)$

Step 4
Group the terms, and remove the greatest common factor from each group.
$3\left(3x^2 + 3x + 10x + 10\right)$
$= 3\left[(3x^2 + 3x) + (10x + 10)\right]$
$= 3\left[3x(x + 1) + 10(x + 1)\right]$

Step 5
Factor out the common binomial.
$3\left[3x(x + 1) + 10(x + 1)\right]$
$= 3(x + 1)(3x + 10)$

4. In $-10x^2-19x-6$, the product of the coefficient of x^2 and the constant term is $(-10)\times(-6)=60$. The middle term has a coefficient of –19.

Step 1
Find two integers that have a product of 60 and a sum of –19.

The pairs of integers with a product of 60 and their respective sums are as follows:

Product	Sum
$(-1)\times(-60)=60$	$(-1)+(-60)=-61$
$(-2)\times(-30)=60$	$(-2)+(-30)=-32$
$(-3)\times(-20)=60$	$(-3)+(-20)=-23$
$\boxed{(-4)\times(-15)=60}$	$\boxed{(-4)+(-15)=-19}$
$(-5)\times(-12)=60$	$(-5)+(-12)=-17$
$(-6)\times(-10)=60$	$(-6)+(-10)=-16$
$1\times60=60$	$1+60=61$
$2\times30=60$	$2+30=32$
$3\times20=60$	$3+20=23$
$4\times15=60$	$4+15=19$
$5\times12=60$	$5+12=17$
$6\times10=60$	$6+10=16$

The integers required are –4 and –15.

Step 2
Express $-19x$ as $-4x-15x$.

$$-10x^2-19x-6$$
$$=-10x^2-4x-15x-6$$

Step 3
Group the terms, and remove the greatest common factor from each group.

$$-10x^2-4x-15x-6$$
$$=(-10x^2-4x)+(-15x-6)$$
$$=-2x(5x+2)-3(5x+2)$$

Step 4
Factor out the common binomial.

$$-2x(5x+2)-3(5x+2)$$
$$=(5x+2)(-2x-3)$$
$$=-(5x+2)(2x+3)$$

Alternatively, –1 could have been factored out as a first step, $-\left(10x^2+19x+6\right)$, followed by decomposition.

Lesson 2—Factoring a Perfect Square Trinomial and a Difference of Squares

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Because the first and last terms are perfect squares, this trinomial may be a perfect square trinomial.

Step 1
Comparing the trinomial $25x^2-60x+36$ to the form $a^2-2ab+b^2$, determine *a* and *b*.

$a^2=25x^2$ $\quad$ $b^2=36$
$a=5x$ $\quad$ $b=6$

Since $2ab=60x$, the trinomial $25x^2-60x+36$ is a perfect square trinomial.

Step 2
Factor the trinomial.

The trinomial $25x^2-60x+36$ is of the form $a^2-2ab+b^2$, which factors to $(a-b)^2$.

$$25x^2-60x+36$$
$$=(5x-6)^2$$

The trinomial $25x^2-60x+36$ factors to $(5x-6)(5x-6)$, or $(5x-6)^2$.

2. **Step 1**
Set up a product of two binomials, one with an addition operation and one with a subtraction operation.

$(\_\_+\_\_)(\_\_-\_\_)$

Step 2
Determine the square root of the first term in the difference-of-squares expression, and use the root as the first term in each of the bracketed binomials.

$\sqrt{4x^2}=2x\rightarrow(2x+\_\_)(2x-\_\_)$

Step 3
Determine the square root of the second term in the difference-of-squares expression, and use the root as the second term in each of the bracketed binomials.

$\sqrt{49y^2} = 7y \rightarrow (2x+7y)(2x-7y)$

Therefore, the factored form of the difference of squares $4x^2 - 49y^2$ is $(2x+7y)(2x-7y)$.

3. **Step 1**
Factor out the greatest common factor from each term of the trinomial. The greatest common factor of $12x^2$, $12x$, and 3 is 3.

$12x^2 + 12x + 3$
$= 3(4x^2 + 4x + 1)$

Because the first and last terms of the trinomial are perfect squares, the trinomial may be a perfect square trinomial.

Step 2
Comparing the trinomial $4x^2 + 4x + 1$ to the form $a^2 + 2ab + b^2$, determine *a* and *b*.

$a^2 = 4x^2$	$b^2 = 1$
$a = 2x$	$b = 1$

Since $2ab = 4x$, the trinomial $4x^2 + 4x + 1$ is a perfect square trinomial.

Step 3
Factor the trinomial.

The trinomial $4x^2 + 4x + 1$ is in the form $a^2 + 2ab + b^2$, which factors to $(a+b)^2$.

$3(4x^2 + 4x + 1)$
$= 3(2x+1)^2$

The trinomial $12x^2 + 12x + 3$ factors to $3(2x+1)(2x+1)$, or $3(2x+1)^2$.

4. **Step 1**
Factor out the greatest common factor from each term of the binomial. The greatest common factor of $-x^2$ and $64y^2$ is –1.

$-x^2 + 64y^2$
$= -(x^2 - 64y^2)$

Step 2
Set up a product of two binomials, one with an addition operation and one with a subtraction operation.
$(\_\_ + \_\_)(\_\_ - \_\_)$

Step 3
Determine the square root of the first term in the difference-of-squares expression, and use the root as the first term in each of the bracketed binomials.

$\sqrt{x^2} = x \rightarrow (x + \_\_)(x - \_\_)$

Step 4
Determine the square root of the second term in the difference-of-squares expression, and use the root as the second term in each of the bracketed binomials.

$\sqrt{64y^2} = 8y \rightarrow (x+8y)(x-8y)$

Therefore, the factored form of the binomial $-x^2 + 64y^2$ is $-(x+8y)(x-8y)$.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. Because the first and last terms are perfect squares, this trinomial may be a perfect square trinomial.

Step 1
Comparing the trinomial $64x^2 - 144x + 81$ to the form $a^2 - 2ab + b^2$, determine *a* and *b*.

$a^2 = 64x^2$	$b^2 = 81$
$a = 8x$	$b = 9$

Since $2ab = 144x$, the trinomial $64x^2 - 144x + 81$ is a perfect square trinomial.

Step 2
Factor the trinomial.

The trinomial $64x^2 - 144x + 81$ is in the form $a^2 - 2ab + b^2$, which factors to $(a-b)^2$.

$64x^2 - 144x + 81$
$= (8x-9)^2$

The trinomial $64x^2 - 144x + 81$ factors to $(8x-9)(8x-9)$, or $(8x-9)^2$.

2. Because the first and last terms are perfect squares, this trinomial may be a perfect square trinomial.

Step 1
Comparing the trinomial $4m^2+20m+25$ to the form $a^2+2ab+b^2$, determine *a* and *b*.

$a^2=4m^2$ $\quad$ $b^2=25$
$a=2m$ $\quad$ $b=5$

Since $2ab=20m$, the trinomial $4m^2+20m+25$ is a perfect square trinomial.

Step 2
Factor the trinomial.

The trinomial $4m^2+20m+25$ is in the form $a^2+2ab+b^2$, which factors to $(a+b)^2$.

$$4m^2+20m+25$$
$$=(2m+5)^2$$

The trinomial $4m^2+20m+25$ factors to $(2m+5)(2m+5)$, or $(2m+5)^2$.

3. **Step 1**
Factor out the greatest common factor from each term of the trinomial. The greatest common factor of $-4n^2$, $4n$, and –1 is –1.

$$-4n^2+4n-1$$
$$=-(4n^2-4n+1)$$

Because the first and last terms of the trinomial are perfect squares, this trinomial may be a perfect square trinomial.

Step 2
Comparing the trinomial $4n^2-4n+1$ to the form $a^2-2ab+b^2$, determine *a* and *b*.

$a^2=4n^2$ $\quad$ $b^2=1$
$a=2n$ $\quad$ $b=1$

Since $2ab=4n$, the trinomial $4n^2-4n+1$ is a perfect square trinomial.

Step 3
Factor the trinomial.

The trinomial $4n^2-4n+1$ is in the form $a^2-2ab+b^2$, which factors to $(a-b)^2$.

$$-(4n^2-4n+1)$$
$$=-(2n-1)^2$$

The trinomial $-4n^2+4n-1$ factors to $-(2n-1)(2n-1)$, or $-(2n-1)^2$.

4. **Step 1**
Set up a product of two binomials, one with an addition operation and one with a subtraction operation.

$(\_\_+\_\_)(\_\_-\_\_)$

Step 2
Determine the square root of the first term in the difference-of-squares expression, and use the root as the first term in each of the bracketed binomials.

$\sqrt{x^2}=x\to(x+\_\_)(x-\_\_)$

Step 3
Determine the square root of the second term in the difference-of-squares expression, and use the root as the second term in each of the bracketed binomials.

$\sqrt{9y^2}=3y\to(x+3y)(x-3y)$

Therefore, the factored form of the difference of squares x^2-9y^2 is $(x+3y)(x-3y)$.

5. **Step 1**
Set up a product of two binomials, one with an addition operation and one with a subtraction operation.
$(\_\_+\_\_)(\_\_-\_\_)$

Step 2
Determine the square root of the first term in the difference-of-squares expression, and use the root as the first term in each of the bracketed binomials.
$\sqrt{9m^2}=3m \to (3m+\_\_)(3m-\_\_)$

Step 3
Determine the square root of the second term in the difference-of-squares expression, and use the root as the second term in each of the bracketed binomials.
$\sqrt{4n^2}=2n \to (3m+2n)(3m-2n)$

Therefore, the factored form of the difference of squares $9m^2-4n^2$ is $(3m+2n)(3m-2n)$.

6. **Step 1**
Factor out the greatest common factor from each term of the binomial. The greatest common factor of –81 and $121y^2$ is –1.

$$-81+121y^2$$
$$=-(81-121y^2)$$

Step 2
Set up a product of two binomials, one with an addition operation and one with a subtraction operation.
$(\_\_+\_\_)(\_\_-\_\_)$

Step 3
Determine the square root of the first term in the difference-of-squares expression, and use the root as the first term in each of the bracketed binomials.
$\sqrt{81}=9 \to (9+\_\_)(9-\_\_)$

Step 4
Determine the square root of the second term in the difference-of-squares expression, and use the root as the second term in each of the bracketed binomials.
$\sqrt{121y^2}=11y \to (9+11y)(9-11y)$

Therefore, the factored form of the binomial $-81+121y^2$ is $-(9+11y)(9-11y)$.

Lesson 3—Factoring Polynomial Expressions with Quadratic Patterns

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. In $-(x^2+4)^2+4(x^2+4)+21$, the product of the coefficient of $\left(x^2+4\right)^2$ and the constant term is $(-1)\times 21=-21$. The middle term has a coefficient of 4.

Step 1
Find two integers that have a product of –21 and a sum of 4.

The pairs of integers with a product of –21 and their respective sums are as follows:

Product	Sum
$(-1)\times 21=-21$	$(-1)+21=20$
$\boxed{(-3)\times 7=-21}$	$\boxed{(-3)+7=4}$
$1\times(-21)=-21$	$1+(-21)=-20$
$3\times(-7)=-21$	$3+(-7)=-4$

The integers required are –3 and 7.

Step 2
Express $4(x^2+4)$ as $-3(x^2+4)+7(x^2+4)$.

$$-(x^2+4)^2+4(x^2+4)+21$$
$$=-(x^2+4)^2-3(x^2+4)+7(x^2+4)+21$$

Step 3
Group the terms, and remove the greatest common factor from each group.

$$[-(x^2+4)^2-3(x^2+4)]+[7(x^2+4)+21]$$
$$=-(x^2+4)[(x^2+4)+3]+7[(x^2+4)+3]$$

Step 4
Factor out the common binomial.

$$-(x^2+4)[(x^2+4)+3]+7[(x^2+4)+3]$$
$$=[(x^2+4)+3][-(x^2+4)+7]$$
$$=-[(x^2+4)+3][(x^2+4)-7]$$

Step 5
Collect like terms in each binomial.
$-[(x^2+4)+3][(x^2+4)-7]$
$=-(x^2+7)(x^2-3)$

The trinomial $-(x^2+4)^2+4(x^2+4)+21$ factors to $-(x^2+7)(x^2-3)$.

2. **Step 1**
Set up a product of two binomials, one with an addition operation and one with a subtraction operation.
$(\_\_+\_\_)(\_\_-\_\_)$

Step 2
Determine the square root of the first term in the difference-of-squares expression, and use the root as the first term in each of the bracketed binomials.
$\sqrt{25(x-3)^2}=5(x-3)$
$\rightarrow[5(x-3)+\_\_][5(x-3)-\_\_]$

Step 3
Determine the square root of the second term in the difference-of-squares expression, and use the root as the second term in each of the bracketed binomials.
$\sqrt{(y+8)^2}=y+8$
$\rightarrow[5(x-3)+(y+8)][5(x-3)-(y+8)]$

Step 4
Expand and collect like terms in each binomial.
$[5(x-3)+(y+8)][5(x-3)-(y+8)]$
$=(5x-15+y+8)(5x-15-y-8)$
$=(5x+y-7)(5x-y-23)$

Therefore, the factored form of the difference of squares $25(x-3)^2-(y+8)^2$ is $(5x+y-7)(5x-y-23)$.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. In $(x^2+1)^2+(x^2+1)-6$, the constant term is –6, and the middle term has a coefficient of 1.

Step 1
Find two integers with a product of –6 and a sum of 1.

The pairs of integers with a product of –6 and their respective sums are as follows:

Product	Sum
$(-1)\times6=-6$	$(-1)+6=5$
$\boxed{(-2)\times3=-6}$	$\boxed{(-2)+3=1}$
$(-3)\times2=-6$	$(-3)+2=-1$
$1\times(-6)=-6$	$1+(-6)=-5$
$2\times(-3)=-6$	$2+(-3)=-1$
$3\times(-2)=-6$	$3+(-2)=1$

The integers required are –2 and 3.

Step 2
Express (x^2+1) as $-2(x^2+1)+3(x^2+1)$.
$(x^2+1)^2+(x^2+1)-6$
$=(x^2+1)^2-2(x^2+1)+3(x^2+1)-6$

Step 3
Group the terms, and remove the greatest common factor from each group.
$(x^2+1)^2-2(x^2+1)+3(x^2+1)-6$
$=[(x^2+1)^2-2(x^2+1)]+[3(x^2+1)-6]$
$=(x^2+1)[(x^2+1)-2]+3[(x^2+1)-2]$

Step 4
Factor out the common binomial.
$(x^2+1)[(x^2+1)-2]+3[(x^2+1)-2]$
$=[(x^2+1)-2][(x^2+1)+3]$

Step 5
Collect like terms in each binomial.
$[(x^2+1)-2][(x^2+1)+3]$
$=(x^2-1)(x^2+4)$

Step 6
Factor the difference of squares.
$(x^2-1)(x^2+4)$
$=(x-1)\left(x+1\right)(x^2+4)$

The trinomial $(x^2+1)^2+(x^2+1)-6$ factors to $(x-1)\left(x+1\right)(x^2+4)$.

2. In $2(x-9)^2-5(x-9)-3$, the product of the coefficient of $\left(x^2-9\right)^2$ and the constant term is $2\times(-3)=-6$. The middle term has a coefficient of –5.

Step 1
Find two integers that have a product of –6 and a sum of –5.

The pairs of integers with a product of –6 and their respective sums are as follows:

Product	Sum
$(-1)\times 6=-6$	$(-1)+6=5$
$(-2)\times 3=-6$	$(-2)+3=1$
$(-3)\times 2=-6$	$(-3)+2=-1$
$\boxed{1\times(-6)=-6}$	$\boxed{1+(-6)=-5}$
$2\times(-3)=-6$	$2+(-3)=-1$
$3\times(-2)=-6$	$3+(-2)=1$

The integers required are 1 and –6.

Step 2
Express $-5(x-9)$ as $(x-9)-6(x-9)$.
$2(x-9)^2-5(x-9)-3$
$=2(x-9)^2+(x-9)-6(x-9)-3$

Step 3
Group the terms, and remove the greatest common factor from each group.
$[2(x-9)^2+(x-9)]+[-6(x-9)-3]$
$=(x-9)[2(x-9)+1]-3[2(x-9)+1]$

Step 4
Factor out the common binomial.
$(x-9)[2(x-9)+1]-3[2(x-9)+1]$
$=[2(x-9)+1][(x-9)-3]$

Step 5
Expand and collect like terms in each binomial.
$[2(x-9)+1][(x-9)-3]$
$=(2x-18+1)(x-9-3)$
$=(2x-17)(x-12)$

The trinomial $2(x-9)^2-5(x-9)-3$ factors to $(2x-17)(x-12)$.

3. **Step 1**
Set up a product of two binomials, one with an addition operation and one with a subtraction operation.
$(\_\_+\_\_)(\_\_-\_\_)$

Step 2
Determine the square root of the first term in the difference-of-squares expression, and use the root as the first term in each of the bracketed binomials.
$\sqrt{4(x-7)^2}=2(x-7)$
$\rightarrow[2(x-7)+\_\_][2(x-7)-\_\_]$

Step 3
Determine the square root of the second term in the difference-of-squares expression, and use the root as the second term in each of the bracketed binomials.
$\sqrt{144(y+6)^2}=12(y+6)$
$\rightarrow[2(x-7)+12(y+6)][2(x-7)-12(y+6)]$

Step 4
Expand and collect like terms in each binomial.
$[2(x-7)+12(y+6)][2(x-7)-12(y+6)]$
$=(2x-14+12y+72)(2x-14-12y-72)$
$=(2x+12y+58)(2x-12y-86)$

Therefore, the factored form of the difference of squares $4(x-7)^2-144(y+6)^2$ is $(2x+12y+58)(2x-12y-86)$.

4. **Step 1**
Set up a product of two binomials, one with an addition operation and one with a subtraction operation.
$(\_\_+\_\_)(\_\_-\_\_)$

Step 2
Determine the square root of the first term in the difference-of-squares expression, and use the root as the first term in each of the bracketed binomials.
$\sqrt{16x^2}=4x \rightarrow [4x+\_\_][4x-\_\_]$

Step 3
Determine the square root of the second term in the difference-of-squares expression, and use the root as the second term in each of the bracketed binomials.
$\sqrt{49(y-3)^2}=7(y-3)$
$\rightarrow [4x+7(y-3)][4x-7(y-3)]$

Step 4
Expand and collect like terms in each binomial.
$[4x+7(y-3)][4x-7(y-3)]$
$=(4x+7y-21)(4x-7y+21)$

Therefore, the factored form of the difference of squares $16x^2-49(y-3)^2$ is $(4x+7y-21)(4x-7y+21)$.

Practice Test

ANSWERS AND SOLUTIONS

1. In $x^2-4x-21$, the constant term is –21, and the middle term has a coefficient of –4.

Step 1
Find two integers with a product of –21 and a sum of –4.

The pairs of integers with a product of –21 and their respective sums are as follows:

Product	Sum
$(-1)\times 21=-21$	$(-1)+21=20$
$(-3)\times 7=-21$	$(-3)+7=4$
$1\times(-21)=-21$	$1+(-21)=-20$
$\boxed{3\times(-7)=-21}$	$\boxed{3+(-7)=-4}$

The integers required are 3 and –7.

Step 2
Express $-4x$ as $3x-7x$.
$x^2-4x-21$
$=x^2+3x-7x-21$

Step 3
Group the terms, and remove the greatest common factor from each group.
$x^2+3x-7x-21$
$=(x^2+3x)+(-7x-21)$
$=x(x+3)-7(x+3)$

Step 4
Factor out the common binomial.
$x(x+3)-7(x+3)$
$=(x+3)(x-7)$

2. In $2x^2+13x+15$, the product of the coefficient of x^2 and the constant term is $2\times 15=30$. The middle term has a coefficient of 13.

Step 1
Find two integers that have a product of 30 and a sum of 13.

The pairs of integers with a product of 30 and their respective sums are as follows:

Product	Sum
$(-1)\times(-30)=30$	$(-1)+(-30)=-31$
$(-2)\times(-15)=30$	$(-2)+(-15)=-17$
$(-3)\times(-10)=30$	$(-3)+(-10)=-13$
$(-5)\times(-6)=30$	$(-5)+(-6)=-11$
$1\times 30=30$	$1+30=31$
$2\times 15=30$	$2+15=17$
$\boxed{3\times 10=30}$	$\boxed{3+10=13}$
$5\times 6=30$	$5+6=11$

The integers required are 3 and 10.

Step 2
Express $13x$ as $3x+10x$.
$2x^2+13x+15$
$=2x^2+3x+10x+15$

Step 3
Group the terms, and remove the greatest common factor from each group.
$2x^2+3x+10x+15$
$=(2x^2+3x)+(10x+15)$
$=x(2x+3)+5(2x+3)$

Step 4
Factor out the common binomial.
$x(2x+3)+5(2x+3)$
$=(2x+3)(x+5)$

3. Because the first and last terms are perfect squares, this trinomial may be a perfect square trinomial.

Step 1
Comparing the trinomial $225x^2-30x+1$ to the form $a^2-2ab+b^2$, determine *a* and *b*.

$a^2=225x^2$ $\quad b^2=1$
$a=15x$ $\quad b=1$

Since $2ab=30x$, the trinomial $225x^2-30x+1$ is a perfect square trinomial.

Step 2
Factor the trinomial.

The trinomial $225x^2-30x+1$ is in the form $a^2-2ab+b^2$, which factors to $(a-b)^2$.
$225x^2-30x+1$
$=(15x-1)^2$

The trinomial $225x^2-30x+1$ factors to $(15x-1)(15x-1)$, or $(15x-1)^2$.

4. Step 1
Factor out the greatest common factor from each term of the trinomial. The greatest common factor of $24x^2$, $120x$, and 150 is 6.
$24x^2+120x+150$
$=6(4x^2+20x+25)$

Because the first and last terms of the trinomial are perfect squares, this trinomial may be a perfect square trinomial.

Step 2
Comparing the trinomial $4x^2+20x+25$ to the form $a^2+2ab+b^2$, determine *a* and *b*.

$a^2=4x^2$ $\quad b^2=25$
$a=2x$ $\quad b=5$

Since $2ab=20x$, the trinomial $4x^2+20x+25$ is a perfect square trinomial.

Step 3
Factor the trinomial.

The trinomial $4x^2+20x+25$ is in the form $a^2+2ab+b^2$, which factors to $(a+b)^2$.
$6(4x^2+20x+25)$
$=6(2x+5)^2$

The trinomial $24x^2+120x+150$ factors to $6(2x+5)(2x+5)$, or $6(2x+5)^2$.

5. Step 1
Set up a product of two binomials, one with an addition operation and one with a subtraction operation.
$(\_\_+\_\_)(\_\_-\_\_)$

Step 2
Determine the square root of the first term in the difference-of-squares expression, and use the root as the first term in each of the bracketed binomials.
$\sqrt{36x^2}=6x\rightarrow(6x+\_\_)(6x-\_\_)$

Step 3
Determine the square root of the second term in the difference-of-squares expression, and use the root as the second term in each of the bracketed binomials.
$\sqrt{49y^2}=7y\rightarrow(6x+7y)(6x-7y)$

Therefore, the factored form of the difference of squares $36x^2-49y^2$ is $(6x+7y)(6x-7y)$.

6. Step 1
Factor out the greatest common factor from each term of the binomial. The greatest common factor of $45x^2$ and $125y^2$ is 5.
$45x^2-125y^2$
$=5(9x^2-25y^2)$

Step 2
Set up a product of two binomials, one with an addition operation and one with a subtraction operation.
$(\_\_+\_\_)(\_\_-\_\_)$

Step 3
Determine the square root of the first term in the difference-of-squares expression, and use the root as the first term in each of the bracketed binomials.

$\sqrt{9x^2} = 3x \rightarrow (3x + \_\_)(3x - \_\_)$

Step 4
Determine the square root of the second term in the difference-of-squares expression, and use the root as the second term in each of the bracketed binomials.

$\sqrt{25y^2} = 5y \rightarrow (3x + 5y)(3x - 5y)$

Therefore, the factored form of the binomial $45x^2 - 125y^2$ is $5(3x + 5y)(3x - 5y)$.

7. In $(x+6)^2 - 8(x+6) + 15$, the constant term is 15, and the middle term has a coefficient of –8.

Step 1
Find two integers with a product of 15 and a sum of –8.

The pairs of integers with a product of 15 and their respective sums are as follows:

Product	**Sum**
$(-1)\times(-15) = 15$	$(-1)+(-15) = -16$
$\boxed{(-3)\times(-5) = 15}$	$\boxed{(-3)+(-5) = -8}$
$1\times 15 = 15$	$1+15 = 16$
$3\times 5 = 15$	$3+5 = 8$

The integers required are –3 and –5.

Step 2
Express $-8(x+6)$ as $-3(x+6) - 5(x+6)$.

$$\begin{aligned}&(x+6)^2 - 8(x+6) + 15\\ &= (x+6)^2 - 3(x+6) - 5(x+6) + 15\end{aligned}$$

Step 3
Group the terms, and remove the greatest common factor from each group.

$$\begin{aligned}&(x+6)^2 - 3(x+6) - 5(x+6) + 15\\ &= [(x+6)^2 - 3(x+6)] + [-5(x+6) + 15]\\ &= (x+6)[(x+6) - 3] - 5[(x+6) - 3]\end{aligned}$$

Step 4
Factor out the common binomial.

$$\begin{aligned}&(x+6)[(x+6) - 3] - 5[(x+6) - 3]\\ &= [(x+6) - 3][(x+6) - 5]\end{aligned}$$

Step 5
Collect like terms in each binomial.

$$\begin{aligned}&[(x+6) - 3][(x+6) - 5]\\ &= (x+3)(x+1)\end{aligned}$$

The trinomial $(x+6)^2 - 8(x+6) + 15$ factors to $(x+3)(x+1)$.

8. **Step 1**
Set up a product of two binomials, one with an addition operation and one with a subtraction operation.

$(\_\_ + \_\_)(\_\_ - \_\_)$

Step 2
Determine the square root of the first term in the difference-of-squares expression, and use the root as the first term in each of the bracketed binomials.

$$\begin{aligned}&\sqrt{81(x-7)^2} = 9(x-7)\\ &\rightarrow [9(x-7) + \_\_][9(x-7) - \_\_]\end{aligned}$$

Step 3
Determine the square root of the second term in the difference-of-squares expression, and use the root as the second term in each of the bracketed binomials.

$$\begin{aligned}&\sqrt{16y^2} = 4y\\ &\rightarrow [9(x-7) + 4y][9(x-7) - 4y]\end{aligned}$$

Step 4
Expand and collect like terms in each binomial.

$$\begin{aligned}&[9(x-7) + 4y][9(x-7) - 4y]\\ &= (9x - 63 + 4y)(9x - 63 - 4y)\\ &= (9x + 4y - 63)(9x - 4y - 63)\end{aligned}$$

Therefore, the factored form of the difference of squares $81(x-7)^2 - 16y^2$ is $(9x + 4y - 63)(9x - 4y - 63)$.

ABSOLUTE VALUE FUNCTIONS

Lesson 1—The Absolute Value of Real Numbers

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Since the number is positive, its absolute value equals the number.
$|17.5| = 17.5$

2. **Step 1**
Replace each absolute value with its equivalent positive number.
$|12| - |-8| \div 4$
$= (12) - (8) \div (4)$

Step 2
Perform the calculations following proper order of operations.
$(12) - (8) \div (4)$
$= 12 - 2$
$= 10$

The value of the expression $|12| - |-8| \div 4$ is 10.

3. **Step 1**
Perform the operation inside the last absolute value.
$|-6+4| = |-2|$

The expression simplifies to $|3| \times |-7| - |-2|$.

Step 2
Replace each absolute value with its equivalent positive number.
$|3| \times |-7| - |-2|$
$= (3) \times (7) - (2)$

Step 3
Perform the calculations following proper order of operations.
$(3) \times (7) - (2)$
$= 21 - 2$
$= 19$

The value of the expression $|3| \times |-7| - |-6+4|$ is 19.

4. **Step 1**
Perform all operations inside the absolute values.

There are operations inside all four of the absolute values.
$|5-7| = |-2|$ $\quad$ $|-8-7+3| = |-12|$

$|10-6| = |4|$ $\quad$ $|-5-2| = |-7|$

The expression simplifies to $|-2| - |-12| \div |4| + |-7|$.

Step 2
Replace each absolute value with its equivalent positive value.
$(2) - (12) \div (4) + (7)$

Step 3
Evaluate the expression following proper order of operations.
$(2) - (12) \div (4) + (7)$
$= 2 - 3 + 7$
$= 6$

The value of $|5-7| - |-8-7+3| \div |10-6| + |-5-2|$ is 6.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Perform the operation inside the absolute value.
$|24-9| = |15|$

The expression simplifies to $|15|$.

Step 2
Replace the absolute value with its equivalent positive number.
$|15| = 15$

2. **Step 1**
Perform the operation inside the absolute value.
$|7-9|=|-2|$

The expression simplifies to $|-2|$.

Step 2
Replace the absolute value with its equivalent positive number.
$|-2|=2$

3. **Step 1**
Replace each absolute value with its equivalent positive number.
$|-7.2|+|16.3|$
$=(7.2)+(16.3)$

Step 2
Perform the calculation.
$(7.2)+(16.3)$
$=23.5$

The value of the expression $|-7.2|+|16.3|$ is 23.5.

4. **Step 1**
Replace each absolute value with its equivalent positive number.
$|-9|\times|6|\div|-3|$
$=(9)\times(6)\div(3)$

Step 2
Perform the calculations following proper order of operations.
$(9)\times(6)\div(3)$
$=9\times 2$
$=18$

The value of the expression $|-9|\times|6|\div|-3|$ is 18.

5. **Step 1**
Perform the operation inside the first absolute value.
$|-8+3|=|-5|$

The expression now simplifies to $|-5|+|9|\times|-5|$.

Step 2
Replace each absolute value with its equivalent positive number.
$|-5|+|9|\times|-5|$
$=(5)+(9)\times(5)$

Step 3
Perform the calculations following proper order of operations.
$(5)+(9)\times(5)$
$=5+45$
$=50$

The value of the expression $|-8+3|+|9|\times|-5|$ is 50.

6. **Step 1**
Perform the operation inside the second absolute value.
$|12-2|=|10|$

The expression now simplifies to $|-4|-|10|\div|-5|$.

Step 2
Replace each absolute value with its equivalent positive number.
$|-4|-|10|\div|-5|$
$=(4)-(10)\div(5)$

Step 3
Perform the calculations following proper order of operations.
$(4)-(10)\div(5)$
$=4-2$
$=2$

The value of the expression $|-4|-|12-2|\div|-5|$ is 2.

7. **Step 1**
Perform all operations inside the absolute values.
$|-36\div 2-10|$
$=|-18-10|$
$=|-28|$

$|-7\times -2|$
$=|14|$

The expression simplifies to $|-28|\div|14|-|9|$.

Step 2
Replace each absolute value with its equivalent positive value.
$(28) \div (14) - (9)$

Step 3
Evaluate the expression following proper order of operations.
$(28) \div (14) - (9)$
$= 2 - 9$
$= -7$

The value of $|-36 \div 2 - 10| \div |-7 \times -2| - |9|$ is –7.

8. **Step 1**
Perform all operations inside the absolute values.
$|100 \div 5| = |20| \qquad |-16 - 8| = |-24|$

The expression simplifies to $|20| \div |-4| + |-24| + |7|$.

Step 2
Replace each absolute value with its equivalent positive value.
$(20) \div (4) + (24) + (7)$

Step 3
Evaluate the expression following proper order of operations.
$(20) \div (4) + (24) + (7)$
$= 5 + 24 + 7$
$= 29 + 7$
$= 36$

The value of $|100 \div 5| \div |-4| + |-16 - 8| + |7|$ is 36.

Lesson 2—Analyzing Absolute Value Functions

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Apply the definition of an absolute value.

If $(x+2) \geq 0$, which means $x \geq -2$,
then $|x+2| = x+2$.
If $(x+2) < 0$, which means $x < -2$,
then $|x+2| = -(x+2)$.

Step 2
Express $y = |x+2|$ as a piecewise function.

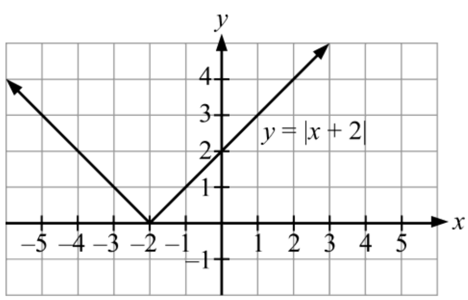
$$y = \begin{cases} x+2, & x \geq -2 \\ -(x+2), & x < -2 \end{cases}$$

Step 3
Construct a table of values for the piecewise function.

$y = x + 2, x \geq -2$		$y = -(x+2), x < -2$	
x	y	x	y
–2	0		
–1	1	–3	1
0	2	–4	2
1	3	–5	3
2	4	–6	4

Step 4
Sketch the graph.

Plot the ordered pairs from the table of values on a Cartesian plane, and draw lines through the points to give a graph of the function.



Step 5
State the intercepts and the domain and range.

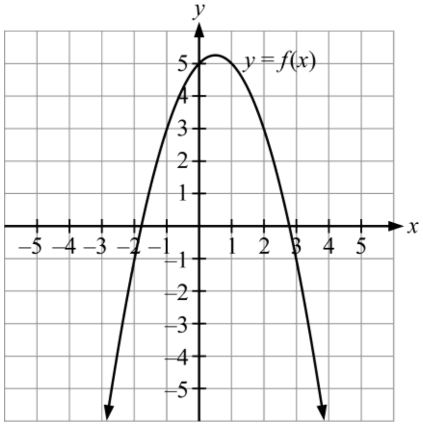
The graph of $y = |x+2|$ intersects the x-axis at (–2, 0)and the y-axis at (0, 2).

The domain is $x \in \mathrm{R}$, and the range is $y \geq 0$.

2. Step 1
Let $f(x) = -x^2 + x + 5$.

Sketch the graph of $f(x) = -x^2 + x + 5$.

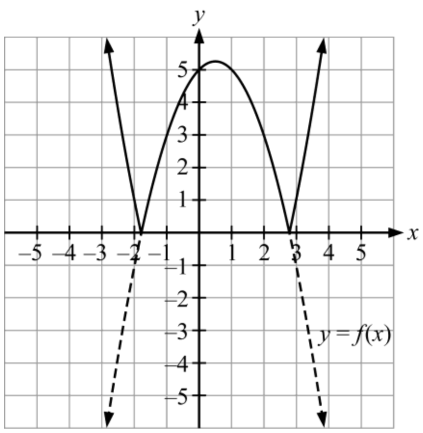
The graph of $f(x) = -x^2 + x + 5$ can be sketched by completing the square, $f(x) = -\left(x - \frac{1}{2}\right)^2 + \frac{21}{4}$, and plotting known points (such as the vertex and intercepts) or using a TI-83 or similar graphing calculator.



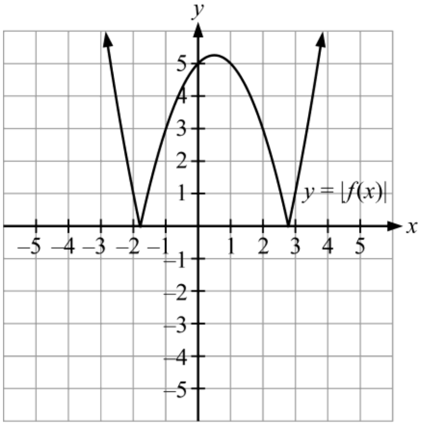
Step 2
Identify points where the graph of $y = f(x)$ is zero or positive (above the x-axis) and where the graph of $y = f(x)$ is negative (below the x-axis).

Using a TI-83 or similar graphing calculator, press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 2:zero to determine that the function $f(x)$ is zero or positive when $-1.8 \le x \le 2.8$. The function is negative when $x < -1.8$ and $x > 2.8$.

Step 3
Reflect the negative parts of $y = f(x)$ about the x-axis.



Step 4
Show the sketch of $y = |f(x)|$.



Step 5
State the intercepts and the domain and range of the function.

The graph of $y = |-x^2 + x + 5|$ intersects the x-axis at (–1.8, 0) and (2.8, 0). The y-intercept is (0, 5). The domain is $x \in \text{R}$, and the range of the function $y = |f(x)|$ is $y \ge 0$.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. Step 1
Apply the definition of an absolute value.

If $(x+3) \ge 0$, which means $x \ge -3$, then $|x+3| = x+3$.
If $(x+3) < 0$, which means $x < -3$, then $|x+3| = -(x+3)$.

Step 2
Express $y = |x+3|$ as a piecewise function.

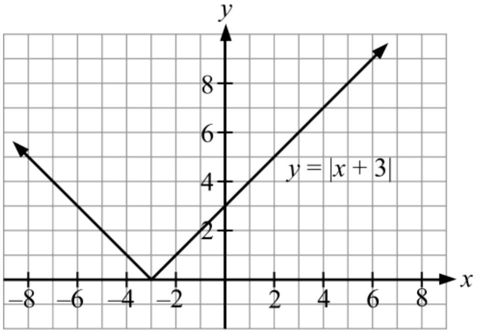
$$y = \begin{cases} x+3, & x \ge -3 \\ -(x+3), & x < -3 \end{cases}$$

Step 3
Construct a table of values for the piecewise function.

$y = x + 3, x \geq -3$		$y = -(x + 3), x < -3$	
x	y	x	y
–3	0		
–2	1	–4	1
–1	2	–5	2
0	3	–6	3
1	4	–7	4

Step 4
Sketch the graph.

Plot the ordered pairs from the table of values on a Cartesian plane, and draw lines through the points to give a graph of the function.



Step 5
State the intercepts and the domain and range.

The graph of $y = |x + 3|$ intersects the x-axis at (–3, 0) and the y-axis at (0, 3).

The domain is $x \in \mathrm{R}$, and the range is $y \geq 0$.

2. Step 1
Apply the definition of an absolute value.

If $(x - 7) \geq 0$, which means $x \geq 7$,
then $|x - 7| = x - 7$.
If $(x - 7) < 0$, which means $x < 7$,
then $|x - 7| = -(x - 7)$.

Step 2
Express $y = |x - 7|$ as a piecewise function.

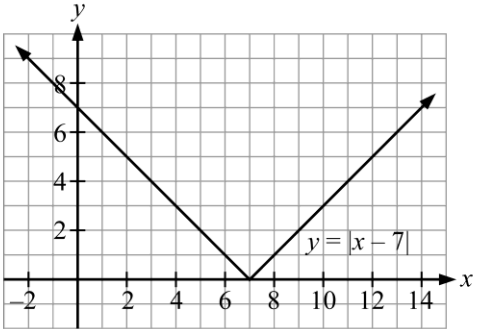
$$y = \begin{cases} x - 7, & x \geq 7 \\ -(x - 7), & x < 7 \end{cases}$$

Step 3
Construct a table of values for the piecewise function.

$y = x - 7, x \geq 7$		$y = -(x - 7), x < 7$	
x	y	x	y
7	0		
8	1	6	1
9	2	5	2
10	3	4	3
11	4	3	4

Step 4
Sketch the graph.

Plot the ordered pairs from the table of values on a Cartesian plane, and draw lines through the points to give a graph of the function.



Step 5
State the intercepts and the domain and range.

The graph of $y = |x - 7|$ intersects the x-axis at (7, 0) and the y-axis at (7, 0).

The domain is $x \in \mathrm{R}$, and the range is $y \geq 0$.

3. Step 1
Apply the definition of an absolute value.

If $(-x - 4) \geq 0$, which means $x \leq -4$,
then $|-x - 4| = (-x - 4)$.
If $(-x - 4) < 0$, which means $x > -4$,
then $|-x - 4| = -(-x - 4)$.

Step 2
Express $y = |-x-4|$ as a piecewise function.

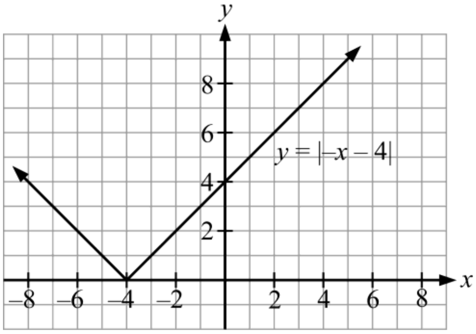
$$y = \begin{cases} -x-4, & x \le -4 \\ x+4, & x > -4 \end{cases}$$

Step 3
Construct a table of values for the piecewise function.

$y = -x-4, x \le -4$		$y = x+4, x > -4$	
x	y	x	y
–4	0		
–5	1	–3	1
–6	2	–2	2
–7	3	–1	3
–8	4	0	4

Step 4
Sketch the graph.

Plot the ordered pairs from the table of values on a Cartesian plane, and draw lines through the points to give a graph of the function.



Step 5
State the intercepts and the domain and range.

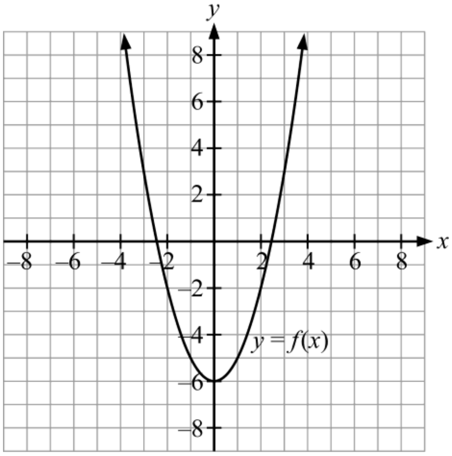
The graph of $y = |-x-4|$ intersects the x-axis at (–4, 0) and the y-axis at (0, 4).

The domain is $x \in \mathrm{R}$, and the range is $y \ge 0$.

4. Step 1
Let $f(x) = x^2 - 6$.

Sketch the graph of $f(x) = x^2 - 6$.

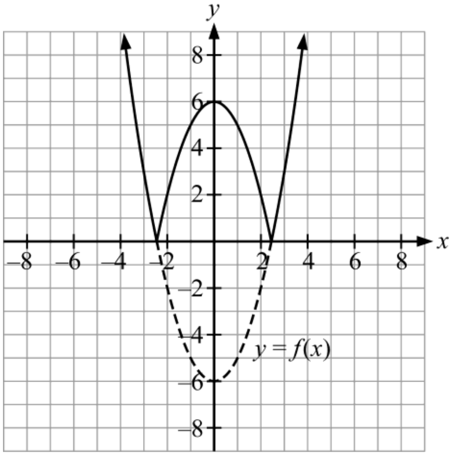
The graph of $f(x) = x^2 - 6$ can be sketched by transforming the graph of the parabola $y = x^2$ 6 units down or by using a TI-83 or similar graphing calculator.



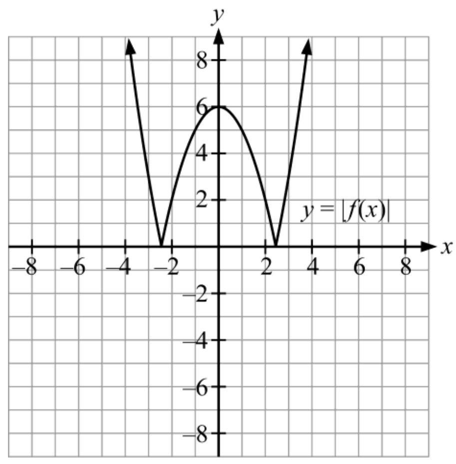
Step 2
Identify points where the graph of $y = f(x)$ is zero or positive (above the x-axis) and where the graph of $y = f(x)$ is negative (below the x-axis).

Using a TI-83 or similar graphing calculator, press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 2:zero to determine that the function $f(x)$ is zero or positive when $x \le -2.4$ and $x \ge 2.4$. The function is negative when $-2.4 < x < 2.4$.

Step 3
Reflect the negative parts of $y = f(x)$ about the x-axis.



Step 4
Show the sketch of $y = |f(x)|$.



Step 5
State the intercepts and the domain and range of the function.

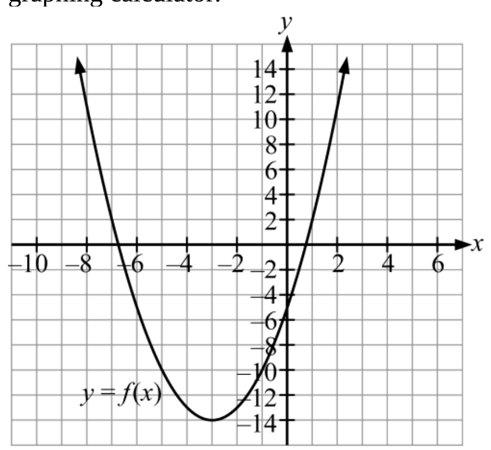
The graph of $y = |x^2 - 6|$ intersects the x-axis at (–2.4, 0) and (2.4, 0). The y-intercept is (0, 6).

The domain is $x \in R$. The range of the function $y = |f(x)|$ is $y \geq 0$.

5. Step 1
Let $f(x) = x^2 + 6x - 5$.

Sketch the graph of $f(x) = x^2 + 6x - 5$.

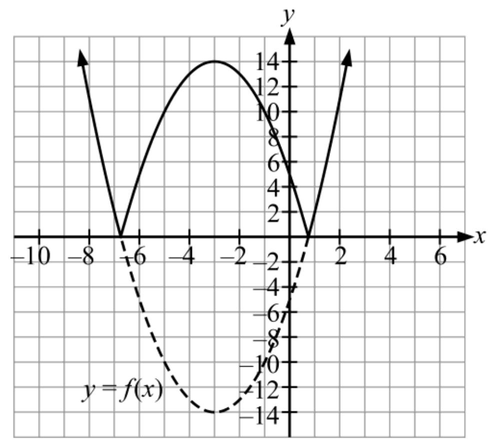
The graph of $f(x) = x^2 + 6x - 5$ can be sketched by completing the square, $f(x) = (x+3)^2 - 14$, and plotting known points (such as the vertex and intercepts) or using a TI-83 or similar graphing calculator.



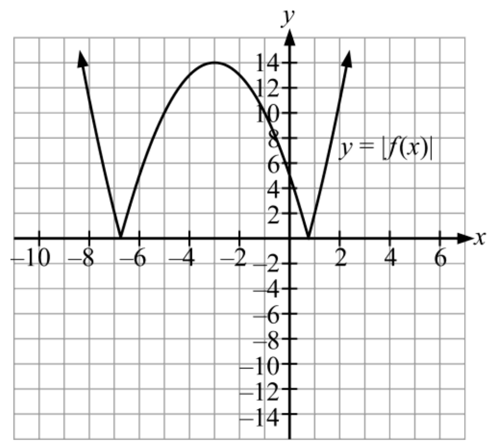
Step 2
Identify points where the graph of $y = f(x)$ is zero or positive (above the x-axis) and where the graph of $y = f(x)$ is negative (below the x-axis).

Using a TI-83 or similar graphing calculator, press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 2:zero to determine that the function $f(x)$ is zero or positive when $x \leq -6.7$ and $x \geq 0.7$. The function is negative when $-6.7 < x < 0.7$.

Step 3
Reflect the negative parts of $y = f(x)$ about the x-axis.



Step 4
Show the sketch of $y = |f(x)|$.



Step 5
State the intercepts and the domain and range of the function.

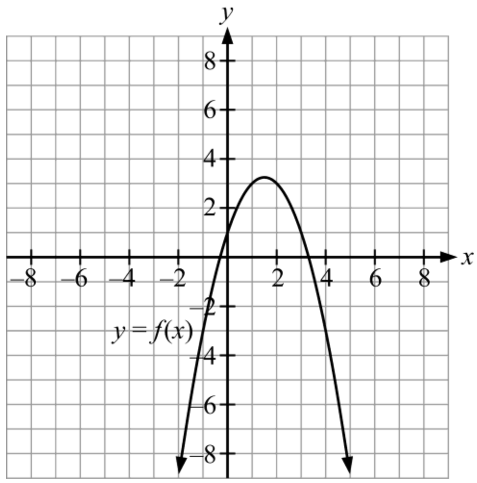
The graph of $y = \left|x^2 + 6x - 5\right|$ intersects the x-axis at (–6.7, 0) and (0.7, 0). The y-intercept is (0, 5).

The domain is $x \in \mathrm{R}$, and the range of the function $y = \left|f\left(x\right)\right|$ is $y \geq 0$.

6. Step 1
Let $f(x) = -x^2 + 3x + 1$.

Sketch the graph of $f(x) = -x^2 + 3x + 1$.

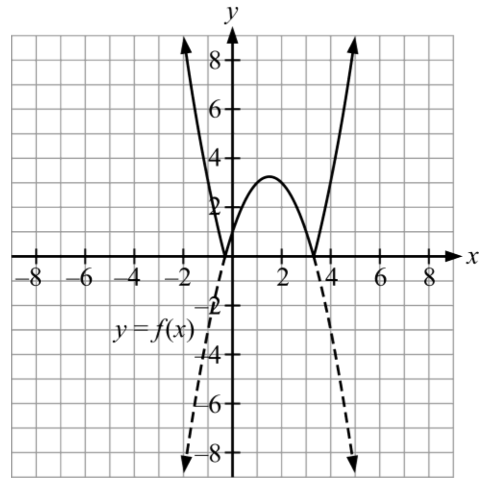
The graph of $f(x) = -x^2 + 3x + 1$ can be sketched by completing the square, $f\left(x\right) = -\left(x - \frac{3}{2}\right)^2 + \frac{13}{4}$, and plotting known points (such as the vertex and intercepts) or using a TI-83 or similar graphing calculator.



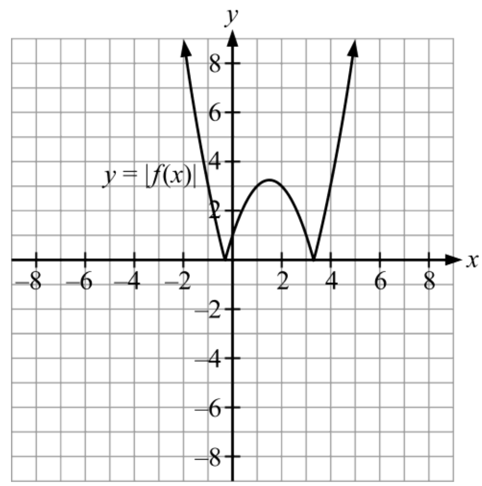
Step 2
Identify points where the graph of $y = f(x)$ is zero or positive (above the x-axis) and where the graph of $y = f(x)$ is negative (below the x-axis).

Using a TI-83 or similar graphing calculator, press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 2:zero to determine that the function $f(x)$ is zero or positive when $-0.3 \leq x \leq 3.3$. The function is negative when $x < -0.3$ and $x > 3.3$.

Step 3
Reflect the negative parts of $y = f(x)$ about the x-axis.



Step 4
Show the sketch of $y = \left|f\left(x\right)\right|$.



Step 5
State the intercepts and the domain and range of the function.

The graph of $y = \left|-x^2 + 3x + 1\right|$ intersects the x-axis at (–0.3, 0) and (3.3, 0). The y-intercept is (0, 1).

The domain is $x \in \mathrm{R}$, and the range of the function $y = \left|f\left(x\right)\right|$ is $y \geq 0$.

Lesson 3—Solving Absolute Value Equations

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Apply the definition of absolute value.

If $(2x-6)\geq 0$, which means $x\geq 3$,
then $|2x-6|=2x-6$.
If $(2x-6)<0$, which means $x<3$,
then $|2x-6|=-(2x-6)$.

The outcome is two equations that are solved separately.

Step 2
Solve for x if $(2x-6)\geq 0$.

$$\begin{aligned}|2x-6|&=x+4\\2x-6&=x+4\\x&=10\end{aligned}$$

Step 3
Solve for x if $(2x-6)<0$.

$$\begin{aligned}|2x-6|&=x+4\\-(2x-6)&=x+4\\-2x+6&=x+4\\-3x&=-2\\x&=\frac{2}{3}\end{aligned}$$

Step 4
Verify the solutions.

Both obtained values appear to be valid because $x=10$ satisfies $x\geq 3$ and $x=\frac{2}{3}$ satisfies $x<3$.
Substitute these values into the original equation to verify.

$\boldsymbol{x=10}$		$\boldsymbol{x=\frac{2}{3}}$	
LHS	**RHS**	**LHS**	**RHS**
$\lvert 2(10)-6\rvert$ $\lvert 14\rvert$ 14	$10+4$ 14	$\left\lvert 2\left(\frac{2}{3}\right)-6\right\rvert$ $\left\lvert -\frac{14}{3}\right\rvert$ $\frac{14}{3}$	$\frac{2}{3}+4$ $\frac{14}{3}$

The solution set is $\left\{\frac{2}{3},\ 10\right\}$.

2. **Step 1**
Apply the definition of absolute value.

If $(2x+1)\geq 0$, which means $x\geq -\frac{1}{2}$,
then $|2x+1|=2x+1$.

If $(2x+1)<0$, which means $x<-\frac{1}{2}$,
then $|2x+1|=-(2x+1)$.

The outcome is two equations that are solved separately.

Step 2
Solve for x if $(2x+1)\geq 0$.

$$\begin{aligned}|2x+1|&=4x-3\\2x+1&=4x-3\\-2x&=-4\\x&=2\end{aligned}$$

Step 3
Solve for x if $(2x+1)<0$.

$$\begin{aligned}|2x+1|&=4x-3\\-(2x+1)&=4x-3\\-2x-1&=4x-3\\-6x&=-2\\x&=\frac{1}{3}\end{aligned}$$

Step 4
Verify the solutions.

It appears that $x=\frac{1}{3}$ is extraneous because it does not satisfy $x<-\frac{1}{2}$, whereas the value x = 2 does satisfy $x\geq-\frac{1}{2}$.

Substitute these values into the original equation to verify.

***x* = 2**		$\boldsymbol{x=\frac{1}{3}}$	
LHS	**RHS**	**LHS**	**RHS**
$\lvert 2(2)+1\rvert$ $\lvert 5\rvert$ 5	$4(2)-3$ $8-3$ 5	$\left\lvert 2\left(\frac{1}{3}\right)+1\right\rvert$ $\left\lvert\frac{5}{3}\right\rvert$ $\frac{5}{3}$	$4\left(\frac{1}{3}\right)-3$ $-\frac{5}{3}$

The value $\frac{1}{3}$ is extraneous. The solution set is {2}.

3. **Step 1**
Apply the definition of absolute value.

If $\left((x+3)^2-2\right)\geq0$, then

$\left|(x+3)^2-2\right|=(x+3)^2-2$.

If $\left((x+3)^2-2\right)<0$, then

$\left|(x+3)^2-2\right|=-\left((x+3)^2-2\right)$.

The outcome is two equations that are solved separately.

Step 2
Solve for x if $\left((x+3)^2-2\right)\geq0$.

$$\begin{aligned}\left|(x+3)^2-2\right|&=2\\(x+3)^2-2&=2\\(x+3)^2&=4\\\sqrt{(x+3)^2}&=\pm\sqrt{4}\\x+3&=\pm2\\x&=-3\pm2\end{aligned}$$

$$\begin{aligned}x&=-3+2\\x&=-1\end{aligned}\quad\text{or}\quad\begin{aligned}x&=-3-2\\x&=-5\end{aligned}$$

Step 3
Solve for x if $\left((x+3)^2-2\right)<0$.

$$\begin{aligned}\left|(x+3)^2-2\right|&=2\\-\left((x+3)^2-2\right)&=2\\-(x+3)^2+2&=2\\-(x+3)^2&=0\\(x+3)(x+3)&=0\\x&=-3\end{aligned}$$

Step 4
Verify the solutions.

The possible solutions are $x=-5$, $x=-3$, and $x=-1$.

Substitute these values into the original equation to verify.

***x* = –5**	
LHS	**RHS**
$\left\lvert\left((-5)+3\right)^2-2\right\rvert$ $\left\lvert(-2)^2-2\right\rvert$ $\lvert 4-2\rvert$ $\lvert 2\rvert$ 2	2

Since the LHS = RHS, x = –5 is a valid solution.

$x = -3$	
LHS	**RHS**
$\left\|\left((-3)+3\right)^2-2\right\|$ $\left\|(0)^2-2\right\|$ $\|-2\|$ 2	2

Since the LHS = RHS, $x = -3$ is a valid solution.

$x = -1$	
LHS	**RHS**
$\left\|\left((-1)+3\right)^2-2\right\|$ $\left\|(2)^2-2\right\|$ $\|4-2\|$ $\|2\|$ 2	2

Since the LHS = RHS, $x = -1$ is a valid solution.

The solution set is $\{-5, -3, -1\}$.

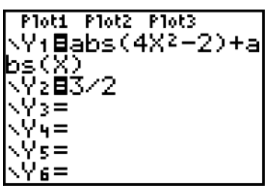
4. **Step 1**
Break the equation into two separate functions.

Write the left side of the equation as Y_1 and the right side as Y_2.

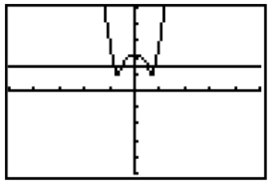
$Y_1 = \left|4x^2-2\right| + |x|$ and $Y_2 = \dfrac{3}{2}$.

Step 2
Graph the two functions using a TI-83 or similar graphing calculator.

Press $\boxed{Y =}$, and input the functions $Y_1 = \left|4x^2-2\right| + |x|$ and $Y_2 = \dfrac{3}{2}$ as shown in this window.



Use the window settings of $x:[-5,5,1]$ and $y:[-5,5,1]$, and then press GRAPH to obtain this window.



Step 3
Determine the points of intersection of the two graphs.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 5:intersect to determine the points of intersection.

The *x*-coordinates of the points of intersection are approximately $x = -0.8$, $x = -0.5$, $x = 0.5$, and $x = 0.8$.

The solution set is $\{-0.8, -0.5, 0.5, 0.8\}$.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Multiply both sides of the equation by –1.

$$-3 = -|3x-1|$$
$$3 = |3x-1|$$

Step 2
Apply the definition of absolute value.

If $3x-1 \geq 0$, which means $x \geq \dfrac{1}{3}$,

then $|3x-1| = 3x-1$.

If $3x-1 < 0$, which means $x < \dfrac{1}{3}$,

then $|3x-1| = -(3x-1)$.

The outcome is two equations that are solved separately.

Step 3
Solve for x if $3x-1\geq 0$.

$$3=|3x-1|$$
$$3=3x-1$$
$$4=3x$$
$$\frac{4}{3}=x$$

Step 4
Solve for x if $3x-1<0$.

$$3=|3x-1|$$
$$3=-(3x-1)$$
$$3=-3x+1$$
$$3x=-2$$
$$x=-\frac{2}{3}$$

Step 5
Verify the solutions.

Both values appear to be valid because $x=\frac{4}{3}$ satisfies $x\geq\frac{1}{3}$ and $x=-\frac{2}{3}$ satisfies $x<\frac{1}{3}$.

Substitute these solutions into the original equation to verify.

$\boldsymbol{x=\frac{4}{3}}$		$\boldsymbol{x=-\frac{2}{3}}$	
LHS	**RHS**	**LHS**	**RHS**
-3	$-\left\|3\left(\frac{4}{3}\right)-1\right\|$ $-\|4-1\|$ $-\|3\|$ -3	-3	$-\left\|3\left(-\frac{2}{3}\right)-1\right\|$ $-\|-2-1\|$ $-\|-3\|$ -3

The solution set is $\left\{-\frac{2}{3},\frac{4}{3}\right\}$.

2. **Step 1**
Apply the definition of absolute value.

If $(3x-4)\geq 0$, which means $x\geq\frac{4}{3}$,

then $|3x-4|=3x-4$.

If $(3x-4)<0$, which means $x<\frac{4}{3}$,

then $|3x-4|=-(3x-4)$.

The outcome is two equations that are solved separately.

Step 2
Solve for x if $(3x-4)\geq 0$.

$$|3x-4|=2x+1$$
$$3x-4=2x+1$$
$$x=5$$

Step 3
Solve for x if $(3x-4)<0$.

$$|3x-4|=2x+1$$
$$-(3x-4)=2x+1$$
$$-3x+4=2x+1$$
$$-5x=-3$$
$$x=\frac{3}{5}$$

Step 4
Verify the solutions.

Both values appear to be valid because $x=5$ satisfies $x\geq\frac{4}{3}$ and $x=\frac{3}{5}$ satisfies $x<\frac{4}{3}$.

Substitute these solutions into the original equation to verify.

$\boldsymbol{x=\frac{3}{5}}$		$\boldsymbol{x=5}$	
LHS	**RHS**	**LHS**	**RHS**
$\left\|3\left(\frac{3}{5}\right)-4\right\|$ $\left\|\frac{9}{5}-\frac{20}{5}\right\|$ $\left\|-\frac{11}{5}\right\|$ $\frac{11}{5}$	$2\left(\frac{3}{5}\right)+1$ $\frac{6}{5}+\frac{5}{5}$ $\frac{11}{5}$	$\|3(5)-4\|$ $\|15-4\|$ $\|11\|$ 11	$2(5)+1$ $10+1$ 11

The solution set is $\left\{\frac{3}{5},5\right\}$.

3. **Step 1**
Apply the definition of absolute value.

If $(3x+4)\geq 0$, which means $x\geq -\frac{4}{3}$,

then $|3x+4|=3x+4$.

If $(3x+4)<0$, which means $x<-\frac{4}{3}$,

then $|3x+4|=-(3x+4)$.

The outcome is two equations that are solved separately.

Step 2
Solve for x if $(3x+4)\geq 0$.

$$\begin{aligned}|3x+4|&=13\\3x+4&=13\\3x&=9\\x&=3\end{aligned}$$

Step 3
Solve for x if $(3x+4)<0$.

$$\begin{aligned}|3x+4|&=13\\-(3x+4)&=13\\3x+4&=-13\\3x&=-17\\x&=-\frac{17}{3}\end{aligned}$$

Step 4
Verify the solutions.

Both values appear to be valid because $x=3$ satisfies $x\geq -\frac{4}{3}$ and $x=-\frac{17}{3}$ satisfies $x<-\frac{4}{3}$.
Substitute these solutions into the original equation to verify.

$x=3$		**$x=-\frac{17}{3}$**	
LHS	**RHS**	**LHS**	**RHS**
$\lvert 3(3)+4\rvert$ $\lvert 9+4\rvert$ $\lvert 13\rvert$ 13	13	$\left\lvert 3\left(-\frac{17}{3}\right)+4\right\rvert$ $\lvert -17+4\rvert$ $\lvert -13\rvert$ 13	13

The solution set is $\left\{-\frac{17}{3}, 3\right\}$.

4. **Step 1**
Apply the definition of absolute value.

If $\left(x^2+6x+5\right)\geq 0$, then
$\left|x^2+6x+5\right|=x^2+6x+5$.
If $\left(x^2+6x+5\right)<0$, then
$\left|x^2+6x+5\right|=-\left(x^2+6x+5\right)$.

The outcome is two equations that are solved separately.

Step 2
Solve for x if $\left(x^2+6x+5\right)\geq 0$.

$$\begin{aligned}\left|x^2+6x+5\right|&=x+1\\x^2+6x+5&=x+1\\x^2+5x+4&=0\\x^2+x+4x+4&=0\\x(x+1)+4(x+1)&=0\\(x+4)(x+1)&=0\end{aligned}$$

$x+4=0$ or $x+1=0$
$x=-4$ $\quad$ $x=-1$

Step 3
Solve for x if $\left(x^2+6x+5\right)<0$.

$$\begin{aligned}\left|x^2+6x+5\right|&=x+1\\-\left(x^2+6x+5\right)&=x+1\\-x^2-6x-5&=x+1\\-x^2-7x-6&=0\\x^2+7x+6&=0\\x^2+x+6x+6&=0\\x(x+1)+6(x+1)&=0\\(x+6)(x+1)&=0\end{aligned}$$

$x+6=0$ or $x+1=0$
$x=-6$ $\quad$ $x=-1$

Step 4
Verify the solutions.

The possible solutions are $x = -6$, $x = -4$, and $x = -1$.

Substitute these solutions into the original equation and verify.

$x = -6$	
LHS	**RHS**
$\lvert(-6)^2 + 6(-6) + 5\rvert$ $\lvert 36 - 36 + 5\rvert$ $\lvert 5\rvert$ 5	$(-6) + 1$ $-6 + 1$ -5

The $\text{LHS} \neq \text{RHS}$, so $x = -6$ is an extraneous solution.

$x = -4$	
LHS	**RHS**
$\lvert(-4)^2 + 6(-4) + 5\rvert$ $\lvert 16 - 24 + 5\rvert$ $\lvert -3\rvert$ 3	$(-4) + 1$ $-4 + 1$ -3

The $\text{LHS} \neq \text{RHS}$, so $x = -4$ is an extraneous solution.

$x = -1$	
LHS	**RHS**
$\lvert(-1)^2 + 6(-1) + 5\rvert$ $\lvert 1 - 6 + 5\rvert$ $\lvert 0\rvert$ 0	$(-1) + 1$ $-1 + 1$ 0

The $\text{LHS} = \text{RHS}$, so $x = -1$ is a valid solution.

The solution set is {–1}.

5. **Step 1**
Apply the definition of absolute value.

If $(2x^2 + x - 6) \geq 0$, then
$\lvert 2x^2 + x - 6\rvert = 2x^2 + x - 6$.

If $(2x^2 + x - 6) < 0$, then
$\lvert 2x^2 + x - 6\rvert = -(2x^2 + x - 6)$.

The outcome is two equations that are solved separately.

Step 2
Solve for x if $(2x^2 + x - 6) \geq 0$.

$$\begin{aligned}
\lvert 2x^2 + x - 6\rvert &= 4x + 3 \\
2x^2 + x - 6 &= 4x + 3 \\
2x^2 - 3x - 9 &= 0 \\
2x^2 - 6x + 3x - 9 &= 0 \\
2x(x - 3) + 3(x - 3) &= 0 \\
(2x + 3)(x - 3) &= 0
\end{aligned}$$

$$2x + 3 = 0 \quad \text{or} \quad x - 3 = 0$$
$$x = -\frac{3}{2} \qquad\qquad x = 3$$

Step 3
Solve for x if $(2x^2 + x - 6) < 0$.

$$\begin{aligned}
\lvert 2x^2 + x - 6\rvert &= 4x + 3 \\
-(2x^2 + x - 6) &= 4x + 3 \\
-2x^2 - x + 6 &= 4x + 3 \\
-2x^2 - 5x + 3 &= 0 \\
2x^2 + 5x - 3 &= 0 \\
2x^2 + 6x - x - 3 &= 0 \\
2x(x + 3) - 1(x + 3) &= 0 \\
(2x - 1)(x + 3) &= 0
\end{aligned}$$

$$2x - 1 = 0 \quad \text{or} \quad x + 3 = 0$$
$$x = \frac{1}{2} \qquad\qquad x = -3$$

Step 4
Verify the solutions.

The possible solutions are $x=-3,\ x=-\frac{3}{2},\ x=\frac{1}{2}$, and $x=3$.

Substitute these solutions into the original equation and verify.

$x = -3$	
LHS	**RHS**
$\lvert 2(-3)^2+(-3)-6\rvert$ $\lvert 2(9)-3-6\rvert$ $\lvert 18-9\rvert$ $\lvert 9\rvert$ 9	$4(-3)+3$ $-12+3$ -9

The $\text{LHS}\neq\text{RHS}$, so $x=-3$ is an extraneous solution.

$x=-\frac{3}{2}$	
LHS	**RHS**
$\left\lvert 2\left(-\frac{3}{2}\right)^2+\left(-\frac{3}{2}\right)-6\right\rvert$ $\left\lvert 2\left(\frac{9}{4}\right)-\frac{3}{2}-6\right\rvert$ $\left\lvert \frac{9}{2}-\frac{3}{2}-6\right\rvert$ $\left\lvert \frac{6}{2}-6\right\rvert$ $\lvert 3-6\rvert$ $\lvert -3\rvert$ 3	$4\left(-\frac{3}{2}\right)+3$ $-6+3$ -3

The $\text{LHS}\neq\text{RHS}$, so $x=-\frac{3}{2}$ is an extraneous solution.

$x=\frac{1}{2}$	
LHS	**RHS**
$\left\lvert 2\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)-6\right\rvert$ $\left\lvert 2\left(\frac{1}{4}\right)+\frac{1}{2}-6\right\rvert$ $\left\lvert \frac{1}{2}+\frac{1}{2}-6\right\rvert$ $\lvert -5\rvert$ 5	$4\left(\frac{1}{2}\right)+3$ $2+3$ 5

The $\text{LHS}=\text{RHS}$, so $x=\frac{1}{2}$ is a valid solution.

$x = 3$	
LHS	**RHS**
$\lvert 2(3)^2+(3)-6\rvert$ $\lvert 2(9)+3-6\rvert$ $\lvert 18-3\rvert$ $\lvert 15\rvert$ 15	$4(3)+3$ $12+3$ 15

The $\text{LHS}=\text{RHS}$, so $x=3$ is a valid solution.

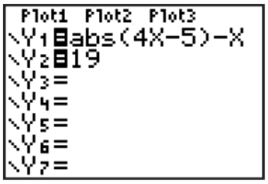
The solution set is $\left\{\frac{1}{2},\ 3\right\}$.

6. **Step 1**
Break the equation into two separate functions.

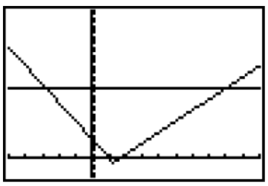
Write the left side of the equation as Y_1 and the right side as Y_2.
$Y_1 = |4x - 5| - x$ and $Y_2 = 19$.

Step 2
Graph the two functions using a TI-83 or similar graphing calculator.

Press $\boxed{\text{Y =}}$, and input the functions $Y_1 = |4x - 5| - x$ and $Y_2 = 19$ as shown in the given window.



Use the window settings of $x:[-5,10,1]$ and $y:[-5,40,1]$, and then press $\boxed{\text{GRAPH}}$ to obtain this window.



Step 3
Determine the points of intersection of the two graphs.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 5:intersect to determine the points of intersection.

The *x*-coordinates of the points of intersection are $x = -2.8$ and $x = 8$.

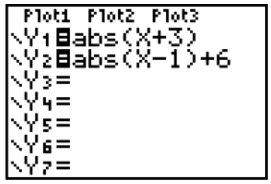
The solution set is $\{-2.8, 8\}$.

7. **Step 1**
Break the equation into two separate functions.

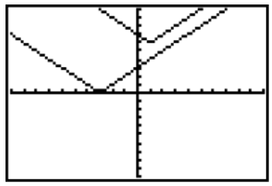
Write the left side of the equation as Y_1 and the right side as Y_2.
$Y_1 = |x + 3|$ and $Y_2 = |x - 1| + 6$.

Step 2
Graph the two functions using a TI-83 or similar graphing calculator.

Press $\boxed{\text{Y =}}$, and input the functions $Y_1 = |x + 3|$ and $Y_2 = |x - 1| + 6$ as shown in the given window.



Use the window settings of $x:[-10,10,1]$ and $y:[-10,10,1]$ then press $\boxed{\text{GRAPH}}$ to obtain this window.



Step 3
Determine the points of intersection of the two graphs.

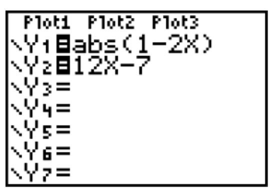
Since there are no intersection points for the graphs, there is no solution to the equation $|x + 3| = |x - 1| + 6$.

8. **Step 1**
Break the equation into two separate functions.

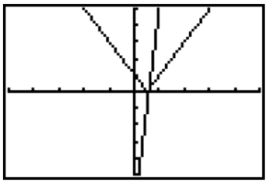
Write the left side of the equation as Y_1 and the right side as Y_2.
$Y_1 = |1 - 2x|$ and $Y_2 = 12x - 7$.

Step 2
Graph the two functions using a TI-83 or similar graphing calculator.

Press $\boxed{\text{Y =}}$, and input the functions $Y_1 = |1 - 2x|$ and $Y_2 = 12x - 7$ as shown in the given window.



Use the window settings of $x:[-5,5,1]$ and $y:[-5,5,1]$, and then press $\boxed{\text{GRAPH}}$ to obtain this window.



Step 3
Determine the points of intersection of the two graphs.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 5:intersect to determine the points of intersection. At the point of intersection, the *x*-coordinate is $x = 0.6$.

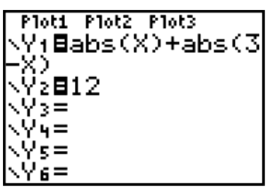
The solution set is {0.6}.

9. **Step 1**
Break the equation into two separate functions.

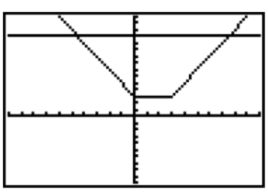
Write the left side of the equation as Y_1 and the right side as Y_2.
$\mathrm{Y}_1 = |x| + |3 - x|$ and $\mathrm{Y}_2 = 12$.

Step 2
Graph the two functions using a TI-83 or similar graphing calculator.

Press $\boxed{\text{Y =}}$, and input the functions $\mathrm{Y}_1 = |x| + |3 - x|$ and $\mathrm{Y}_2 = 12$ as shown in the given window.



Use the window settings of $x:[-10,10,1]$ and $y:[-10,15,1]$, and then press $\boxed{\text{GRAPH}}$ to obtain this window.



Step 3
Determine the points of intersection of the two graphs.

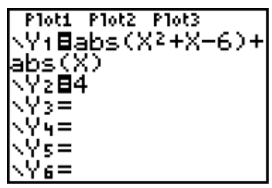
Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 5:intersect to determine the points of intersection. At the points of intersection, the *x*-coordinates are $x = –4.5$ and $x = 7.5$.

The solution set is {–4.5, 7.5}.

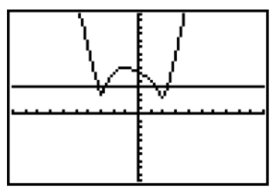
10. **Step 1**
Break the equation into two separate functions.
$\mathrm{Y}_1 = |x^2 + x - 6| + |x|$ and $\mathrm{Y}_2 = 4$.

Step 2
Graph the two functions using a TI-83 or similar graphing calculator.

Press $\boxed{\text{Y =}}$, and input the functions $\mathrm{Y}_1 = |x^2 + x - 6| + |x|$ and $\mathrm{Y}_2 = 4$ as shown in the given window.



Use the window settings of $x:[-10,10,1]$ and $y:[-10,10,1]$, and then press $\boxed{\text{GRAPH}}$ to obtain this window.



Step 3
Determine the points of intersection of the two graphs.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 5:intersect to determine the points of intersection. At the points of intersection, the *x*-coordinates are approximately $x = –3.2$, $x = –2.7$, $x = 1.4$, and $x = 2.3$.

The solution set is {–3.2, –2.7, 1.4, 2.3}.

Practice Test

ANSWERS AND SOLUTIONS

1. **Step 1**
Perform the operation inside the last absolute value.
$|-2-1| = |-3|$

The expression simplifies to $|-6| + |9| \div |-3|$.

Step 2
Replace each absolute value with its equivalent positive number.
$|-6| + |9| \div |-3|$
$= (6) + (9) \div (3)$

Step 3
Perform the calculations following proper order of operations.
$(6) + (9) \div (3)$
$= 6 + 3$
$= 9$

The value of the expression $|-6| + |9| \div |-2-1|$ is 9.

2. **Step 1**
Perform all operations inside the absolute values.
$|6 \div 3| = |2|$ $\quad |-2 \times 5| = |-10|$

The expression simplifies to $|2| + |-4| - |-10|$.

Step 2
Replace each absolute value with its equivalent positive value.
$(2) + (4) - (10)$

Step 3
Evaluate the expression following proper order of operations.
$(2) + (4) - (10)$
$= 6 - 10$
$= -4$

The value of $|6 \div 3| + |-4| - |-2 \times 5|$ is –4.

3. **Step 1**
Perform all operations inside the absolute values.
$|9 \div 3 \times 2|$
$= |3 \times 2|$
$= |6|$

$|-6-7| = |-13|$

The expression simplifies to $120 + |6| - |-13|$.

Step 2
Replace each absolute value with its equivalent positive value.
$120 + (6) - (13)$

Step 3
Evaluate the expression following proper order of operations.
$120 + (6) - (13)$
$= 126 - 13$
$= 113$

The value of $120 + |9 \div 3 \times 2| - |-6-7|$ is 113.

4. **Step 1**
Perform all operations inside the absolute values.
$|8 \times (-3)| = |-24|$ $\quad |5-3| = |2|$

The expression simplifies to
$|-24| - |2| + |64| \div |-16|$.

Step 2
Replace each absolute value with its equivalent positive value.
$(24) - (2) + (64) \div (16)$

Step 3
Evaluate the expression following proper order of operations.
$(24) - (2) + (64) \div (16)$
$= 24 - 2 + 4$
$= 24 + 2$
$= 26$

The value of $|8 \times (-3)| - |5-3| + |64| \div |-16|$ is 26.

5. **Step 1**
Apply the definition of an absolute value.

If $(-2x+9) \geq 0$, which means $x \leq \frac{9}{2}$,
then $|-2x+9| = -2x+9$.

If $(-2x+9) < 0$, which means $x > \frac{9}{2}$,
then $|-2x+9| = -(-2x+9)$.

Step 2
Express $y = |-2x+9|$ as a piecewise function.

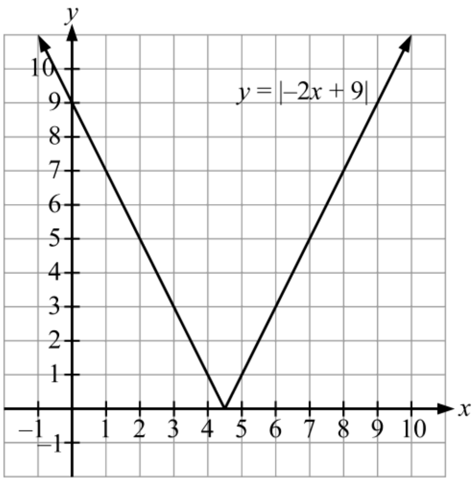
$$y = \begin{cases} -2x+9, & x \leq \frac{9}{2} \\ 2x-9, & x > \frac{9}{2} \end{cases}$$

Step 3
Construct a table of values for the piecewise function.

$\boldsymbol{y = -2x+9, x \leq \frac{9}{2}}$		$\boldsymbol{y = 2x-9, x > \frac{9}{2}}$,	
x	***y***	***x***	***y***
$\frac{9}{2}$	0		
4	1	5	1
3	3	6	3
2	5	7	5
1	7	8	7
0	9	9	9

Step 4
Sketch the graph.

Plot the ordered pairs from the table of values on a Cartesian plane, and draw lines through the points to give a graph of the function.



Step 5
State the intercepts and the domain and range of the function.

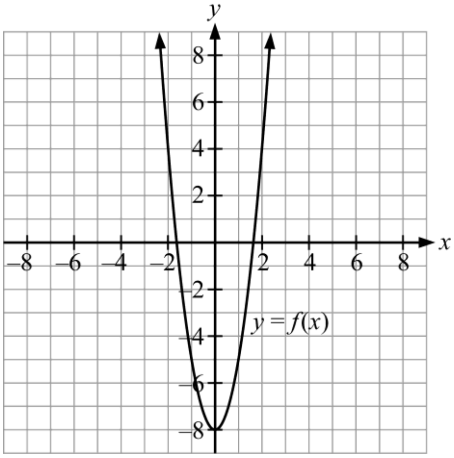
The graph of $y = |-2x+9|$ intersects the x-axis at $\left(\frac{9}{2}, 0\right)$. The y-intercept is (0, 9).

The domain is $x \in \text{R}$, and the range is $y \geq 0$.

6. Step 1
Let $f(x) = 3x^2 - 8$.

Sketch the graph of $f(x) = 3x^2 - 8$.

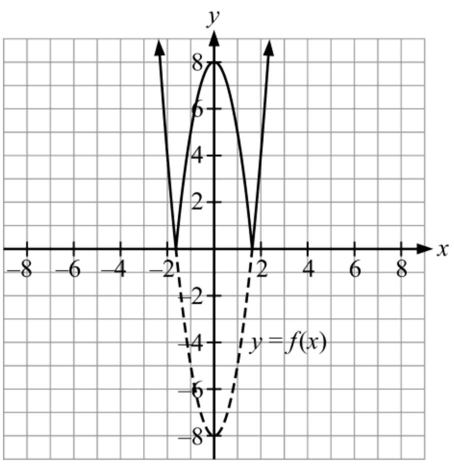
The graph of $f(x) = 3x^2 - 8$ can be sketched by stretching the graph of the parabola $y = x^2$ by a factor of 3 followed by a vertical translation 8 units down. Alternatively, a TI-83 or similar graphing calculator can be used.



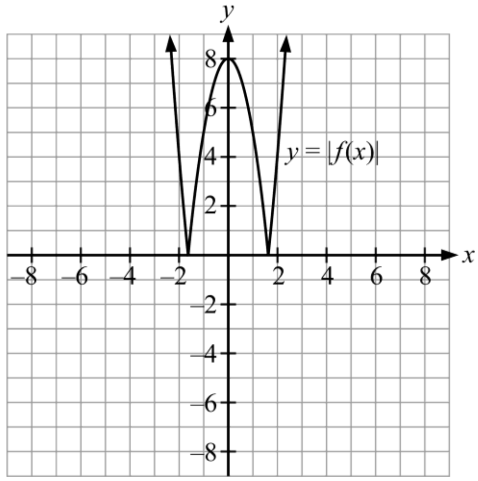
Step 2
Identify points where the graph of $y = f(x)$ is zero or positive (above the x-axis) and where the graph of $y = f(x)$ is negative (below the x-axis).

The function $f(x)$ is zero or positive when $x \le -1.6$ and $x \ge 1.6$. The function is negative when $-1.6 < x < 1.6$.

Step 3
Reflect the negative parts of $y = f(x)$ about the x-axis.



Step 4
Show the sketch of $y = \left|f(x)\right|$.



Step 5
State the intercepts and the domain and range of the function.

The graph of $y = \left|3x^2 - 8\right|$ intersects the x-axis at (–1.6, 0) and (1.6, 0). The y-intercept is (0, 8).

The domain is $x \in \mathrm{R}$. The range of the function $y = \left|f(x)\right|$ is $y \ge 0$.

7. Step 1
Apply the definition of an absolute value.

If $(4x - 3) \ge 0$, which means $x \ge \dfrac{3}{4}$,
then $|4x - 3| = 4x - 3$.

If $(4x - 3) < 0$, which means $x < \dfrac{3}{4}$,
then $|4x - 3| = -(4x - 3)$.

The outcome is two equations that are solved separately.

Step 2
Solve for x if $(4x - 3) \ge 0$.

$$
\begin{aligned}
|4x - 3| &= 77 \\
4x - 3 &= 77 \\
4x &= 80 \\
x &= 20
\end{aligned}
$$

Step 3
Solve for x if $(4x-3)<0$.

$$\begin{aligned}|4x-3|&=77\\-(4x-3)&=77\\4x-3&=-77\\4x&=-74\\x&=-\frac{37}{2}\end{aligned}$$

Step 4
Verify the solutions.

Both values appear to be valid because $x=20$ satisfies $x\geq\frac{3}{4}$ and $x=-\frac{37}{2}$ satisfies $x<\frac{3}{4}$.

Substitute the solutions into the original equation to verify.

$\boldsymbol{x=20}$		$\boldsymbol{x=-\frac{37}{2}}$	
LHS	**RHS**	**LHS**	**RHS**
$\lvert 4(20)-3\rvert$ $\lvert 80-3\rvert$ $\lvert 77\rvert$ 77	77	$\left\lvert 4\left(-\frac{37}{2}\right)-3\right\rvert$ $\lvert -74-3\rvert$ $\lvert -77\rvert$ 77	77

The solution set is $\left\{20,-\frac{37}{2}\right\}$.

8. **Step 1**
Apply the definition of absolute value.

If $(x^2-6)\geq 0$, then $|x^2-6|=x^2-6$.
If $(x^2-6)<0$, then $|x^2-6|=-(x^2-6)$.

The outcome is two equations that are solved separately.

Step 2
Solve for x if $(x^2-6)\geq 0$.

$$\begin{aligned}|x^2-6|&=10\\x^2-6&=10\\x^2&=16\\x&=\pm4\end{aligned}$$

Step 3
Solve for x if $(x^2-6)<0$.

$$\begin{aligned}|x^2-6|&=10\\-(x^2-6)&=10\\x^2-6&=-10\\x^2&=-4\end{aligned}$$

Since the square root of a negative number is undefined, there is no solution when $(x^2-6)<0$.

Step 4
Verify the solutions.

The possible solutions are $x=-4$ and $x=4$.

Substitute the solutions into the original equation to verify.

$\boldsymbol{x=-4}$		$\boldsymbol{x=4}$	
LHS	**RHS**	**LHS**	**RHS**
$\lvert(-4)^2-6\rvert$ $\lvert 16-6\rvert$ 10	10	$\lvert(4)^2-6\rvert$ $\lvert 16-6\rvert$ 10	10

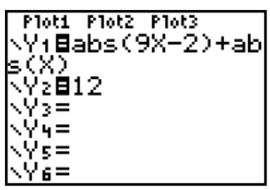
The solution set is {–4, 4}.

9. **Step 1**
Break the equation into two separate functions.

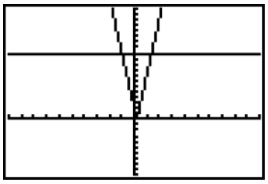
Write the left side of the equation as Y_1 and the right side as Y_2.
$\mathrm{Y}_1=|9x-2|+|x|$ and $\mathrm{Y}_2=12$.

Step 2
Graph the two functions using a TI-83 or similar graphing calculator.

Press $\boxed{\text{Y =}}$, and input the functions $\mathrm{Y}_1=|9x-2|+|x|$ and $\mathrm{Y}_2=12$ as shown in the given window.



Use the window settings of $x:[-10,10,1]$ and $y:[-10,20,1]$, and then press $\boxed{\text{GRAPH}}$ to obtain this window.



Step 3
Determine the points of intersection of the two graphs.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 5:intersect to determine the points of intersection. At the points of intersection, the *x*-coordinates are $x = -1$ and $x = 1.4$.

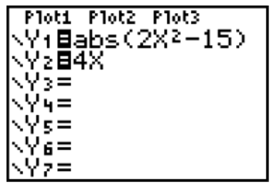
Therefore, the solution set is {–1, 1.4}.

10. Step 1
Break the equation into two separate functions.

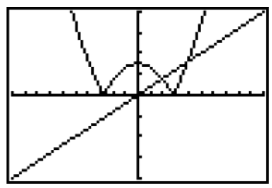
Write the left side of the equation as Y_1 and the right side as Y_2.
$Y_1 = \left|2x^2 - 15\right|$ and $Y_2 = 4x$.

Step 2
Graph the two functions using a TI-83 or similar graphing calculator.

Press $\boxed{\text{Y =}}$, and input the functions $Y_1 = \left|2x^2 - 15\right|$ and $Y_2 = 4x$ as shown in the given window.



Use the window settings of $x:[-10,10,1]$ and $y:[-40,40,10]$, and then press $\boxed{\text{GRAPH}}$ to obtain this window.



Step 3
Determine the points of intersection of the two graphs.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 5:intersect to determine the points of intersection.

At the points of intersection, the *x*-coordinates are approximately $x = 1.9$ and $x = 3.9$.

Therefore, the solution set is {1.9, 3.9}.

QUADRATIC FUNCTIONS AND EQUATIONS

Lesson 1—Quadratic Functions in the Form $y=a(x-p)^2+q$

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. The equation $\frac{1}{2}y = x^2$ becomes $y = 2x^2$ by multiplying both sides by 2. The graph of $y = x^2$ is vertically stretched about the *x*-axis by a factor of 2 to produce the graph of $\frac{1}{2}y = x^2$.

2. To find the equation of function *g*, *x* must be replaced with *x* + 7 using the equation of function *f*.

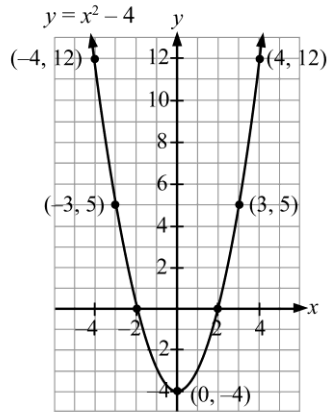
$$g(x) = f(x+7)$$
$$g(x) = 2((x+7)-3)^2$$
$$g(x) = 2(x+4)^2$$

The value of *p* in function *g* is –4. Therefore, the vertex of the graph of function *g* is (–4, 0).

Alternate solution:
Since the vertex can be defined as (*p*, 0), the graph of function *f* has its vertex at (3, 0). Function *g* is equal to function *f* after *f* undergoes a horizontal translation of 7 units to the left. Therefore, the vertex of the graph of function *f* at (3, 0) will move 7 units to the left and become the vertex at (–4, 0) for the graph of function *g*.

3. The transformation is a vertical translation 4 units down. Choose several points from the graph of $y = x^2$, and subtract 4 from the *y*-coordinates. Plot the new points, and join them with a smooth curve.

(–4, 16) → (–4, 12)	(2, 4) → (2, 0)
(–3, 9) → (–3, 5)	(3, 9) → (3, 5)
(–2, 4) → (–2, 0)	(4, 16)→ (4, 12)
(0, 0) → (0, –4)	



4. The function $y = x^2$ is transformed to the function $y = a(x - p)^2 + q$. A vertical stretch by a factor of 5 means $a = 5$. A translation 4 units left means $p = –4$, and a translation 6 units up means $q = 6$. The equation of the transformed function is $y = 5(x+4)^2 + 6$.

5. **Step 1**
State the vertex, axis of symmetry, range, and maximum value.

The values of *a*, *p*, and *q* are $a = -\frac{1}{2}$, $p = -2$, and $q = 5$.

The vertex (*p*, *q*) is at (–2, 5). The equation of the axis of symmetry is *x* = –2, and it is a vertical line passing through the vertex.

The range is $y \le 5$ (the graph opens down).
The maximum value is *y* = 5.

Step 2
Find the *y*-intercept.
Let *x* = 0.

$$y = -\frac{1}{2}(0+2)^2 + 5$$
$$y = 3$$

The *y*-intercept is the point (0, 3).

Step 3
Find the *x*-intercepts.
Substitute *y* = 0 in the equation.

$$y = -\frac{1}{2}(x+2)^2 + 5$$
$$0 = -\frac{1}{2}(x+2)^2 + 5$$

Use a TI-83 or similar graphing calculator to find the zeros of the function (the *x*-intercepts of the graph).

Press $\boxed{\text{Y=}}$, and enter the equation of the function.
$Y_1 = -1/2(x+2)\wedge 2+5$

Press $\boxed{\text{WINDOW}}$ to define the window settings as $x:[-10,10,1]$ and $y:[-10,10,1]$, and then press $\boxed{\text{GRAPH}}$ to display the graph.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$ to access the calculate menu. Select 2:zero, and press $\boxed{\text{ENTER}}$.

For "Left Bound?", position the cursor just left of the zero that is farthest to the left, and press $\boxed{\text{ENTER}}$.

For "Right Bound?", position the cursor just right of the zero that is farthest to the left, and press $\boxed{\text{ENTER}}$ $\boxed{\text{ENTER}}$

For "Guess?", press $\boxed{\text{ENTER}}$. The result is one of the zeros of the function.

The value of the leftmost *x*-intercept is approximately –5.2.

Repeat the procedure to find the other zero (*x*-intercept of the graph). The value of the second *x*-intercept is approximately 1.2.

The coordinates of the *x*-intercepts are approximately (–5.2, 0) and (1.2, 0).

6. **Step 1**
Substitute the known values into the function $y = a(x-p)^2 + q$.
The values of *p* and *q* are –3 and 17, respectively.
A *y*-intercept of –8 means the parabola contains the point (0, –8).

$$\begin{aligned} y &= a(x-p)^2+q \\ y &= a(x+3)^2+17 \\ -8 &= a(0+3)^2+17 \end{aligned}$$

Step 2
Solve for *a*.

$$\begin{aligned} -8 &= a(0+3)^2+17 \\ -25 &= 9a \\ -\frac{25}{9} &= a \end{aligned}$$

The equation of the parabola is
$y = -\frac{25}{9}(x+3)^2+17$

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. The equation $y=\sqrt{x}+5$ is equivalent to $y = x^{\frac{1}{2}}+5$. A quadratic function can be written in the form $y = ax^2+bx+c$, where *a*, *b*, and *c* are real numbers and $a \neq 0$. Therefore, $y=\sqrt{x}+5$ is not a quadratic function since there is no degree two term.

2. Since $p(x)=(x+3)(x+4)$ becomes $p(x)=x^2+7x+12$ when expanded, it is a quadratic function written in the form $p(x)=ax^2+bx+c$, where *a* = 1, *b* = 7, and *c* = 12.

3. The function $y=x^2$ becomes $y=3x^2$.

4. The function $y=x^2$ becomes $y=(x-9)^2-8$.

5. The function $y=x^2$ becomes $y=-x^2$.

6. The equation $y+9=(x+9)^2$ becomes $y=(x+9)^2-9$. There is a horizontal translation 9 units left and a vertical translation 9 units down.

7. The equation $2y=-6(x-5)^2+18$ becomes $y=-3(x-5)^2+9$. There is a vertical stretch about the *x*-axis by a factor of 3, a reflection in the *x*-axis, a horizontal translation 5 units right, and vertical translation 9 units up.

8. **Step 1**
Find 4*f*(*x* – 1).

$$\begin{aligned} 4f(x-1) &= 4\left[\frac{1}{4}((x-1)+4)^2-11\right] \\ 4f(x-1) &= (x+3)^2-44 \end{aligned}$$

Step 2
Find $g(x)$.
$$g(x) = 4f(x-1)+2$$
$$g(x) = \left((x+3)^2 - 44\right)+2$$
$$g(x) = (x+3)^2 - 42$$

Step 3
Find the y-intercept for $g(x)$.
Find $g(0)$.
$$g(0) = (0+3)^2 - 42$$
$$g(0) = 9 - 42$$
$$g(0) = -33$$

Therefore, the y-intercept for $g(x)$ is the point (0, –33).

9. The vertex is (–3, 5).
The axis of symmetry is $x = -3$.
The range is $y \le 5$.
The maximum value is $y = 5$.
The y-intercept is at $y = 2$.
The x-intercepts are at $x = -7$ and $x = 1$.

10. Step 1
Apply the transformations in the order shown.
i) $y = -x^2$

ii) $y = -\frac{2}{3}x^2$

iii) $y = -\frac{2}{3}(x+2)^2$

iv) $y = -\frac{2}{3}(x+2)^2 + 10$

Step 2
Find k.
Substitute 1 for x and k for y.
$$k = -\frac{2}{3}(1+2)^2 + 10$$
$$k = -\frac{2}{3}(3)^2 + 10$$
$$k = -\frac{2}{3}(9) + 10$$
$$k = -6 + 10$$
$$k = 4$$

The value of k is 4.

11. D
Step 1
Identify the transformations in the order of stretches and reflections followed by translations.
If the function $f(x) = x^2$ is transformed to

$g(x) = \frac{1}{2}(x+3)^2 + 4$, the following transformations were applied to $f(x)$ in this order:

1) A vertical stretch about the x-axis by a factor of $\frac{1}{2}$
2) A horizontal translation 3 units left
3) A vertical translation 4 units up

Step 2
Find the new point on function g.
The original point (2, 4) on function f would transform as follows:

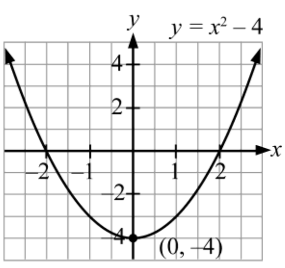
1) $(2,4) \to (2,2)$ Multiply y by $\frac{1}{2}$.

2) $(2,2) \to (-1,2)$ Subtract 3 from x.

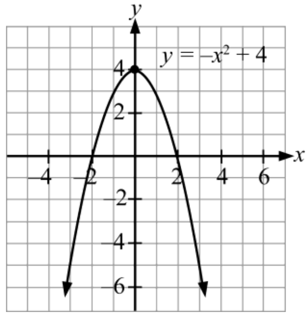
3) $(-1,2) \to (-1,6)$ Add 4 to y.

The point (2, 4) on the graph of f would be transformed to the point (–1, 6) on the graph of g.

12. Compared to the graph of $y = x^2$, the graph of $y = x^2 - 4$ has been vertically translated 4 units down. The vertex (0, 0) of the graph of $y = x^2$ will now be at (0, –4). The x-intercepts are at $x = -2$ and $x = 2$ (using technology).



The graph of $-y = x^2 - 4$ is a reflection of the graph of $y = x^2 - 4$ about the x-axis ($y \to -y$). The y-intercept (0, –4) on the graph of $y = x^2 - 4$ will be transformed to (0, 4) on the graph of $-y = x^2 - 4$. The x-intercepts have not changed because at the x-axis the y-coordinate is 0, a value that remains unchanged with a reflection in the x-axis.



Lesson 2—Quadratic Functions in the Form y = ax² +bx + c

CLASS EXERCISES
ANSWERS AND SOLUTIONS

1. Since $a < 0$, the parabola opens down.
Since $c = 7$, the y-intercept is the point (0, 7).

2. **Step 1**
Factor the coefficient of the x^2-term out of the x^2- and x-terms only, and place those terms together in brackets.

$$y = x^2 - 12x + 11$$
$$y = 1\left(x^2 - 12x + \_\_\_\right) + 11$$

Step 2
Find the c-value of the perfect square trinomial using $\left[\frac{1}{2}(b)\right]^2$. Add and subtract this value inside the brackets.

$$\left[\frac{1}{2}(-12)\right]^2 = 36$$
$$y = 1\left(x^2 - 12x + \_\_\_\right) + 11$$
$$y = 1\left(x^2 - 12x + 36 - 36\right) + 11$$

Step 3
Multiply the subtracted term inside the brackets by a (the coefficient that was factored out), and move the result outside the brackets.

$$y = 1\left(x^2 - 12x + 36 - 36\right) + 11$$
$$y = 1\left(x^2 - 12x + 36\right) - 36(1) + 11$$

Step 4
Factor the perfect square trinomial in the brackets, and combine like terms outside the brackets.

$$y = 1\left(x^2 - 12x + 36\right) - 36(1) + 11$$
$$y = (x - 6)^2 - 25$$

Step 5
Identify the characteristics of the function.
Vertex: (6, –25)
Range: $y \geq -25$
Minimum y-value: –25
Axis of symmetry: $x = 6$
y-intercept: 11 (from c in the original equation $y = x^2 - 12x + 11$)

3. **Step 1**
Factor the coefficient of the x^2-term out of the x^2- and x-terms only, and place those terms together in brackets.

$$y = -\frac{1}{3}x^2 - 4x + 11$$
$$y = -\frac{1}{3}\left(x^2 + 12x + \_\_\_\right) + 11$$

Step 2
Find the c-value of the perfect square trinomial using $\left[\frac{1}{2}(b)\right]^2$. Add and subtract this value inside the brackets.

$$\left[\frac{1}{2}(12)\right]^2 = 36$$
$$y = -\frac{1}{3}\left(x^2 + 12x + \_\_\_\right) + 11$$
$$y = -\frac{1}{3}\left(x^2 + 12x + 36 - 36\right) + 11$$

Step 3
Multiply the subtracted term inside the brackets by a (the coefficient that was factored out), and move the result outside the brackets.

$$y = -\frac{1}{3}\left(x^2 + 12x + 36 - 36\right) + 11$$
$$y = -\frac{1}{3}\left(x^2 + 12x + 36\right) - 36\left(-\frac{1}{3}\right) + 11$$

Step 4
Factor the perfect square trinomial in the brackets, and combine like terms outside the brackets.

$$y = -\frac{1}{3}\left(x^2 + 12x + 36\right) - 36\left(-\frac{1}{3}\right) + 11$$
$$y = -\frac{1}{3}(x + 6)^2 + 23$$

Step 5
Identify the characteristics of the function.
Vertex: (–6, 23)
Range: $y \leq 23$

Maximum y-value: 23
Axis of symmetry: $x = -6$
y-intercept: 11 (from c in the original equation $y = -\frac{1}{3}x^2 - 4x + 11$)

4. **Step 1**
Use $x = \frac{-b}{2a}$ to find the x-coordinate of the vertex.

$$x = \frac{-3}{2\left(\frac{1}{4}\right)}$$
$$x = \frac{-3}{\frac{1}{2}}$$
$$x = -6$$

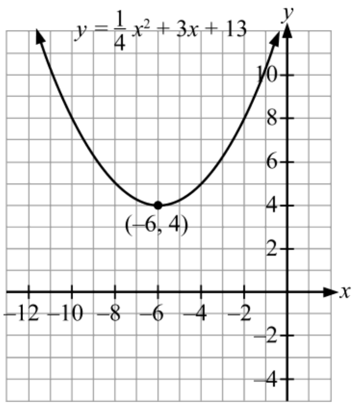
Step 2
Solve for y by substituting the x-coordinate of the vertex into the equation of the function.

$$y = \frac{1}{4}(-6)^2 + 3(-6) + 13$$
$$y = 9 - 18 + 13$$
$$y = 4$$

The vertex is (–6, 4).

The y-intercept is (0, 13). The 13 corresponds to the constant term c when the equation is in the form $y = ax^2 + bx + c$.

Step 3
Sketch the graph of the function.



PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. Expand and gather like terms to write the equation in the form $y = ax^2 + bx + c$.

$$y = 2(x+3)^2 + 5$$
$$y = 2(x+3)(x+3) + 5$$
$$y = 2(x^2 + 6x + 9) + 5$$
$$y = 2x^2 + 12x + 18 + 5$$
$$y = 2x^2 + 12x + 23$$

2. Expand and gather like terms to write the equation in the form $y = ax^2 + bx + c$.

$$3y = -\frac{3}{5}(x-5)^2 - 9$$
$$3y = -\frac{3}{5}(x-5)(x-5) - 9$$
$$3y = -\frac{3}{5}(x^2 - 10x + 25) - 9$$
$$3y = -\frac{3}{5}x^2 + \frac{30}{5}x - \frac{75}{5} - 9$$
$$3y = -\frac{3}{5}x^2 + 6x - 24$$
$$y = -\frac{1}{5}x^2 + 2x - 8$$

3. Step 1
Factor the coefficient of the x^2-term out of the x^2- and x-terms only, and place those terms together in brackets.

$$y = x^2 - 12x - 5$$
$$y = (x^2 - 12x + \underline{\quad\quad}) - 5$$

Step 2
Find the c-value of the perfect square trinomial using $\left[\frac{1}{2}(b)\right]^2$. Add and subtract this value inside the brackets.

$$y = (x^2 - 12x + \underline{\quad\quad}) - 5$$
$$y = (x^2 - 12x + 36 - 36) - 5$$

Step 3
Multiply the subtracted term inside the brackets by a (the coefficient that was factored out), and move the result outside the brackets.

$$y = (x^2 - 12x + 36 - 36) - 5$$
$$y = (x^2 - 12x + 36) - 36(1) - 5$$

Step 4
Factor the perfect square trinomial in the brackets, and combine like terms outside the brackets.

$y=\left(x^2-12x+36\right)-36(1)-5$

$y=(x-6)^2-41$

4. **Step 1**
Factor the coefficient of the x^2-term out of the x^2- and x-terms only, and place those terms together in brackets.

$y=-3x^2+12x-1$

$y=-3\left(x^2-4x+\underline{\quad}\right)-1$

Step 2
Find the c-value of the perfect square trinomial using $\left[\frac{1}{2}(b)\right]^2$. Add and subtract this value inside the brackets.

$y=-3\left(x^2-4x+\underline{\quad}\right)-1$

$y=-3\left(x^2-4x+4-4\right)-1$

Step 3
Multiply the subtracted term inside the brackets by a (the coefficient that was factored out), and move the result outside the brackets.

$y=-3\left(x^2-4x+4-4\right)-1$

$y=-3\left(x^2-4x+4\right)-4(-3)-1$

Step 4
Factor the perfect square trinomial in the brackets, and combine like terms outside the brackets.

$y=-3\left(x^2-4x+4\right)-4(-3)-1$

$y=-3(x-2)^2+11$

5. **Step 1**
Factor the coefficient of the x^2-term out of the x^2- and x-terms only, and place those terms together in brackets.

$y=\frac{1}{2}x^2-12x+5$

$y=\frac{1}{2}\left(x^2-24x+\underline{\quad}\right)+5$

Step 2
Find the c-value of the perfect square trinomial using $\left[\frac{1}{2}(b)\right]^2$. Add and subtract this value inside the brackets.

$y=\frac{1}{2}\left(x^2-24x+\underline{\quad}\right)+5$

$y=\frac{1}{2}\left(x^2-24x+144-144\right)+5$

Step 3
Multiply the subtracted term inside the brackets by a (the coefficient that was factored out), and move the result outside the brackets.

$y=\frac{1}{2}\left(x^2-24x+144-144\right)+5$

$y=\frac{1}{2}\left(x^2-24x+144\right)-144\left(\frac{1}{2}\right)+5$

Step 4
Factor the perfect square trinomial in the brackets, and combine like terms outside the brackets.

$y=\frac{1}{2}\left(x^2-24x+144\right)-144\left(\frac{1}{2}\right)+5$

$y=\frac{1}{2}(x-12)^2-67$

6. Step 1
Expand and then factor the coefficient of the x^2-term out of the x^2- and x-terms only, and place those terms together in brackets.

$y=(x-8)(x+30)$

$y=x^2+22x-240$

$y=\left(x^2+22x+\underline{\quad}\right)-240$

Step 2
Find the c-value of the perfect square trinomial using $\left[\frac{1}{2}(b)\right]^2$. Add and subtract this value inside the brackets.

$y=\left(x^2+22x+\underline{\quad}\right)-240$

$y=\left(x^2+22x+121-121\right)-240$

Step 3
Multiply the subtracted term inside the brackets by a (the coefficient that was factored out), and move the result outside the brackets.

$y=\left(x^2+22x+121-121\right)-240$

$y=\left(x^2+22x+121\right)-121(1)-240$

Step 4
Factor the perfect square trinomial in the brackets, and combine like terms outside the brackets.

$y=\left(x^2+22x+121\right)-121(1)-240$

$y=\left(x+11\right)^2-361$

7. **Step 1**

Use $x=\dfrac{-b}{2a}$ to find the x-coordinate of the vertex.

$x=\dfrac{-12}{2(-3)}$

$x=2$

Step 2
Solve for the y-coordinate of the vertex by substituting the x-coordinate into the equation of the original function.

$y=-3(2)^2+12(2)-1$

$y=-12+24-1$

$y=11$

The vertex is (2, 11).

8. **A**

Ronald's first mistake is in step 1 because $\dfrac{1}{4}$ should be correctly factored out of the x2- and x-terms. The correct step should be

$y=\dfrac{1}{4}\left(x^2-320x \qquad \right)+11.$

Lesson 3—Applications of Quadratic Functions

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **a)** An equation that relates the income, I, produced by the sale of the fishing rods and n can be written in dollars or cents.
In dollars, $I=\left(64+4n\right)\left(12-0.50n\right)$.
In cents, $I=\left(64+4n\right)\left(1\,200-50n\right)$.

b) **Step 1**
Find the maximum by expanding and then completing the square to find the vertex. (Avoiding decimals is often simpler, so this solution is shown in cents.)

$I=\left(64+4n\right)\left(1\,200-50n\right)$

$I=-200n^2+1\,600n+76\,800$

$I=-200\left(n^2-8n+\_\_\_\right)+76\,800$

$I=-200\left(n^2-8n+16-16\right)+76\,800$

$I=-200\left(n^2-8n+16\right)-16(-200)+76\,800$

$I=-200\left(n-4\right)^2+80\,000$

The maximum income is 80 000¢ or $800. Since the vertex of this quadratic function is (4, 80 000), the maximum income occurs when n = 4.

Step 2
Determine the number of fishing rods that must be sold to attain the maximum income. Replace n with 4 in (64 + 4n).
64 + 4(4) = 80

Therefore, 80 fishing rods must be sold in order to attain the maximum income of $800.

Step 3
Determine the price of each fishing rod that yields the maximum income.
Replace n with 4 in (1 200 – 50n).
1 200 – 50(4) = 1 000

1 000¢ ÷ 100 = $10

Therefore, the price of each fishing rod must be $10 to attain the maximum income of $800.

2. **a)** Determine the equation when l is a function of w.

$4w+2l=800$

$2l=-4w+800$

$l=-2w+400$

b) Determine the equation when A is a function of w.

$A=(l)(w)$

$A=(-2w+400)(w)$

$A=-2w^2+400w$

c) Write the equation in completed square form.

$A=-2w^2+400w$

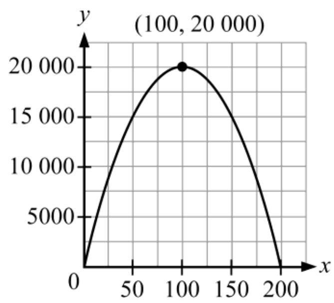
$A=-2\left(w^2-200w\right)$

$A=-2\left(w^2-200w+10\,000-10\,000\right)$

$A=-2\left(w^2-200w+10\,000\right)-10\,000(-2)$

$A=-2\left(w-100\right)^2+20\,000$

d)



The vertex of the parabola is (100, 20 000).

e) The *y*-coordinate of the vertex represents the maximum value of the function.

The maximum total enclosed area is 20 000 m^2.

f) The *x*-coordinate of the vertex represents the value of the width, *w*, required to produce the maximum total enclosed area.
w = 100 m

The length, *l*, can be calculated using either of two methods.

Method 1
From the perimeter equation:
$l = -2w + 400$, *w* = 100 m

$l = -2(100) + 400$
$l = 200$

Therefore, *l* = 200 m.

Method 2
From the area equation:
$(w)(l) = A$

$l = \frac{A}{w}$

$l = \frac{20\,000}{100}$

$l = 200$

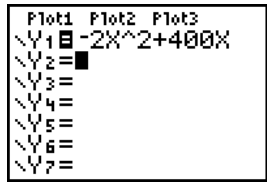
Therefore, *l* = 200 m.

The width must be 100 m and the length must be 200 m in order to produce a maximum enclosed area.

g) Find the maximum possible value of the total enclosed area, *A*, using a TI-83 or similar graphing calculator.

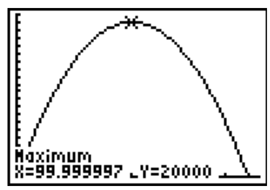
In this case, the function is $A = -2w^2 + 400w$. The graph of the area function is a parabola opening down. Find the coordinates of the maximum of the parabola.

Step 1
Press $\boxed{\text{Y=}}$, and enter the equation of the function, $y = -2x^2 + 400x$.



Press $\boxed{\text{WINDOW}}$ to define the window settings as $x:[-2, 210, 20]$ and $y:[-20, 21000, 1000]$.

Press $\boxed{\text{GRAPH}}$ to display the graph of the function. Then, determine the coordinates of the maximum by pressing $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$ and selecting 4:maximum.



This confirms that the vertex of (100, 20 000) is correct, which corresponds to a maximum enclosed area of 20 000 m^2 when the width is 100 m.

3. a) The initial height is equal to the value of *h*(0).
$h(0) = -5(0)^2 + 60(0) + 11$
$h(0) = 11$

The object's initial height is 11 m.

b) Complete the square, or use the fact that the maximum occurs when $t=\frac{-b}{2a}$.

$$t=\frac{-b}{2a}$$
$$t=\frac{-60}{2(-5)}$$
$$t=6$$

The maximum occurs at t = 6 s.

Substitute t = 6 into the equation of the height function to determine the maximum value of the function.

$$h(6)=-5(6)^2+60(6)+11$$
$$h(6)=-180+360+11$$
$$h(6)=191$$

The graph of this function is a parabola that opens down with a vertex at (6, 191).

The maximum height reached by the object is 191 m, and it occurred 6 s after being launched.

c) The height of the object above the ground 3 s after being launched is the value of $h(3)$.

$$h(3)=-5(3)^2+60(3)+11$$
$$h(3)=146\text{ m}$$

At t = 3, the object is still rising. As such, the distance that the object travelled in 3 s is its height above the ground at t = 3 minus its initial starting height.
146 m – 11 m = 135 m

The object travelled a distance of 135 m in its first 3 s of motion.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Write an equation of the function that relates the income, I, to n, the number of weeks to wait until harvest.

In n weeks, the farmer will have $(1\,200+100n)$ bushels.

In n weeks, the price will be $\$(18-0.5n)$ per bushel.

$$I=(\text{number of bushels})(\text{price per bushel})$$
$$I=(1\,200+100n)(18-0.5n)$$

Step 2
Expand and complete the square for this function.

$$I=(1\,200+100n)(18-0.5n)$$
$$I=-50n^2+1\,200n+21\,600$$
$$I=-50\left(n^2-24n+\_\_\_\right)+21\,600$$
$$I=-50\left(n^2-24n+144-144\right)+21\,600$$
$$I=-50\left(n^2-24n+144\right)-144(-50)+21\,600$$
$$I=-50(n-12)^2+28\,800$$

The vertex of the graph of the function is at (12, 28 800).

Therefore, the harvest should occur in 12 weeks when the maximum income is \$28 800.

2. **a)** **Step 1**
Write the equation of the function that relates the income, I, to n, the number of students in excess of 75.

If n students over 75 are taken, there will be $(75+n)$ students.

For n students over 75, the price per student will be $\$(6-0.05n)$ or $(600-5n)¢$.

$$I=(\text{number of students})(\text{price per student})$$
$$I=(75+n)(600-5n)$$

Step 2
Expand and complete the square for this function.

$$I=(75+n)(600-5n)$$
$$I=-5n^2+225n+45\ 000$$
$$I=-5\left(n^2-45n+\_\_\_\right)+45\ 000$$
$$I=-5\left(n^2-45n+506.25-506.25\right)+45\ 000$$
$$I=-5\left(n^2-45n+506.25\right)-506.25(-5)+45\ 000$$
$$I=-5(n-22.5)^2+47\ 531.25$$

Step 3
Determine the number of students required to generate the maximum income.

The vertex of the graph of the function is at (22.5, 47 531.25). The maximum income would occur if the boat company took 22.5 extra students (over the 75). This is impossible because you cannot take half of a student.

Therefore, the company must transport 22 or 23 extra students to maximize their income (since the *x*-coordinate of the vertex is 22.5 and the parabola is symmetrical about its vertex, the value of the function when $n = 22$ will be the same as the value of the function when $n = 23$).

If $n = 22$, the number of students taken is $(75+n)=97$.
If $n = 23$, the number of students taken is $(75+n)=98$.

b) Determine the price per ticket if $n = 22$.

$$\begin{aligned}&(600-5n)\\&=600-5(22)\\&=490¢\\&=\$4.90\end{aligned}$$

The income would be $(97)(\$4.90)=\$475.30$.

Determine the price per ticket if $n = 23$.
The price per ticket is $(600-5n)$.

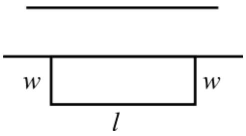
$$\begin{aligned}&(600-5n)\\&=600-5(23)\\&=485¢\\&=\$4.85\end{aligned}$$

The income would be $(98)(\$4.85)=\$475.30$.

The students should be charged either \$4.85 or \$4.90, depending on the number transported.

c) If either 97 or 98 students are taken, the maximum income would be \$475.30.

3. Step 1
Draw a diagram.
The diagram shows the street and the rectangular fence on three sides. It also shows the width, *w*, and the length, *l*.



Step 2
Write an equation for the area, *A*, of the rectangle as a function of *w*.

Write the equation relating *l* and *w*.

$$\begin{aligned}2w+l&=2\,500\\l&=2\,500-2w\end{aligned}$$

Write the equation for the area, A.

$$\begin{aligned}A&=(l)(w)\\A&=(2\,500-2w)(w)\end{aligned}$$

Step 3
Expand and complete the square for this function.

$$\begin{aligned}A&=(2\,500-2w)(w)\\A&=-2w^2+2\,500w\\A&=-2\left(w^2-1\,250w+\_\_\_\right)\\A&=-2\left(w^2-1\,250w+390\,625-390\,625\right)\\A&=-2\left(w^2-1\,250w+390\,625\right)-390\,625(-2)\\A&=-2(w-625)^2+781\,250\end{aligned}$$

The vertex of the graph of this function is at (625, 781 250).

Step 4
Determine the maximum area and the dimensions that produce the maximum.

The maximum area is 781 250 m^2, which occurs when $w = 625$ m.

$$\begin{aligned}l&=(2\,500-2w)\\l&=(2\,500-2(625))\\l&=1\,250\text{ m}\end{aligned}$$

The maximum area of 781 250 m^2 occurs when the width is 625 m and the length is 1 250 m.

4. **Step 1**
Write the equation of a function that relates the sum, S, to one of the parts, x.

If x is one part, then $(20 – x)$ will be the other part. Let S be the sum of the squares of the two parts.
$S = x^2 + (20 - x)^2$

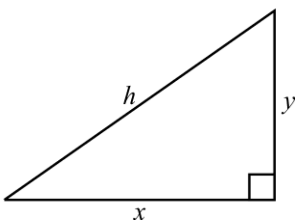
Step 2
Expand and complete the square for this function.
$S = x^2 + (20 - x)^2$
$S = x^2 + 400 - 40x + x^2$

$S = 2x^2 - 40x + 400$
$S = 2(x^2 - 20x + \_\_\_) + 400$
$S = 2(x^2 - 20x + 100 - 100) + 400$
$S = 2(x^2 - 20x + 100) - 100(2) + 400$
$S = 2(x - 10)^2 + 200$

Step 3
Determine the two parts.
The vertex of the graph of this function is at (10, 200). The minimum value of 200 occurs when x =10. The second part is $(20 - x) = 10.$
The two parts are 10 and 10.

You can confirm that the sum of the squares of these two parts is the minimum of 200.
$10^2 + 10^2 = 200$

5. **Step 1**
Draw a diagram.
The diagram shows a right triangle with hypotenuse h.



Step 2
Write the equation of a function that relates the length of the hypotenuse, h, to the length of side x.
Write an equation relating x and y.
$x + y = 40$
$y = 40 - x$

Write an equation for the hypotenuse.
$h^2 = x^2 + y^2$
$h^2 = x^2 + (40 - x)^2$

Step 2
Expand and complete the square for this function.
$h^2 = x^2 + (40 - x)^2$
$h^2 = x^2 + (40 - x)(40 - x)$
$h^2 = x^2 + 1\,600 - 80x + x^2$
$h^2 = 2x^2 - 80x + 1\,600$
$h^2 = 2(x^2 - 40x + \_\_\_) + 1\,600$
$h^2 = 2(x^2 - 40x + 400 - 400) + 1\,600$
$h^2 = 2(x^2 - 40x + 400) - 400(2) + 1600$
$h^2 = 2(x - 20)^2 + 800$

The vertex is at (20, 800).

The minimum value for h^2 is 800. If h^2 is a minimum, then h will also be a minimum.
This minimum occurs when x = 20.
$y = (40 - x)$
$y = (40 - 20)$
$y = 20$

Therefore, each part of the string is 20 cm long.

6. Complete the square for the given function.
$h(t) = 30t - 5t^2$
$h(t) = -5(t^2 - 6t + \_\_\_)$
$h(t) = -5(t^2 - 6t + 9 - 9)$
$h(t) = -5(t^2 - 6t + 9) - 9(-5)$
$h(t) = -5(t - 3)^2 + 45$

The vertex is at (3, 45).

It takes 3 s for the baseball to reach its maximum height of 45 m.

Lesson 4—Solving Quadratic Equations

CLASS EXERCISES
ANSWERS AND SOLUTIONS

1. **Step 1**
Eliminate any denominators.
Multiply every term on each side of the equation by the lowest common denominator, which is 6.

$$\frac{1}{2}x = \frac{2}{3}(5x-7)^2 - 10$$
$$(6)\frac{1}{2}x = (6)\frac{2}{3}(5x-7)^2 - (6)10$$
$$3x = 4(5x-7)^2 - 60$$

Step 2
Write the equation in the form $ax^2 + bx + c = 0$ by expanding and gathering like terms.

$$3x = 4(5x-7)^2 - 60$$
$$3x = 4\left(25x^2 - 70x + 49\right) - 60$$
$$0 = 100x^2 - 280x - 3x + 196 - 60$$
$$0 = 100x^2 - 283x + 136$$

2. Given the nature of the equation, factoring a difference of squares would be complicated, so solve using method 1.

Method 1
Isolate x^2, and find the square root of both sides of the equation.

Be sure to include both the positive and negative roots.

$$2x^2 - 7 = 0$$
$$2x^2 = 7$$
$$x^2 = \frac{7}{2}$$
$$x = \pm\sqrt{\frac{7}{2}}$$
$$x = \pm\frac{\sqrt{7}}{\sqrt{2}}$$

Rationalize the denominator.

$$x = \pm\frac{\left(\sqrt{7}\right)\left(\sqrt{2}\right)}{\left(\sqrt{2}\right)\left(\sqrt{2}\right)}$$
$$x = \pm\frac{\sqrt{14}}{2}$$

3. **Step 1**
Eliminate any denominators.
Multiply each term on both sides of the equation by 2.

$$x(x-2) = \frac{x}{2}$$
$$(2)\left[x(x-2)\right] = (2)\frac{x}{2}$$
$$2x(x-2) = x$$

Step 2
Write the equation in the form $ax^2 + bx + c = 0$ by expanding and gathering like terms.

$$2x(x-2) = x$$
$$2x^2 - 4x = x$$
$$2x^2 - 5x = 0$$

Step 3
Solve the equation by factoring.

$$2x^2 - 5x = 0$$
$$x(2x-5) = 0$$
$$x = 0 \quad \text{or} \quad 2x - 5 = 0$$
$$x = 0 \qquad\qquad x = \frac{5}{2}$$

4. **Step 1**
Write the equation in the form $ax^2 + bx + c = 0$ by expanding and gathering like terms.

$$15x(x+1) + 1 = (x+1)^2 - x(x+1) + 8$$
$$15x^2 + 15x + 1 = (x+1)(x+1) - x^2 - x + 8$$
$$15x^2 + 15x + 1 = x^2 + 2x + 1 - x^2 - x + 8$$
$$15x^2 + 15x + 1 = x + 9$$
$$15x^2 + 14x - 8 = 0$$

Step 2
Solve the equation by factoring.

$$15x^2 + 14x - 8 = 0$$
$$(5x-2)(3x+4) = 0$$
$$x = \frac{2}{5} \quad \text{or} \quad x = -\frac{4}{3}$$

5. Step 1

Isolate the x^2- and x-terms on one side of the equation by moving the constant to the other side.

Divide all terms by the coefficient a.

$$2x^2+10x-7=0$$
$$2x^2+10x=7$$
$$x^2+5x=\frac{7}{2}$$

Step 2

Find the c-value of the perfect square trinomial using $\left[\frac{1}{2}(b)\right]^2$.

$$\left[\frac{1}{2}(5)\right]^2=\frac{25}{4}$$

Add this value to both sides of the equation.

$$x^2+5x=\frac{7}{2}$$
$$x^2+5x+\frac{25}{4}=\frac{7}{2}+\frac{25}{4}$$

Step 3

Factor the perfect square trinomial, and simplify the other side of the equation.

$$x^2+5x+\frac{25}{4}=\frac{7}{2}+\frac{25}{4}$$
$$\left(x+\frac{5}{2}\right)^2=\frac{14}{4}+\frac{25}{4}$$
$$\left(x+\frac{5}{2}\right)^2=\frac{39}{4}$$

Step 4

Solve for x by taking the square root of both sides of the equation.

$$\left(x+\frac{5}{2}\right)^2=\frac{39}{4}$$
$$x+\frac{5}{2}=\pm\sqrt{\frac{39}{4}}$$
$$x=-\frac{5}{2}\pm\frac{\sqrt{39}}{2}$$
$$x=\frac{-5\pm\sqrt{39}}{2}$$

Therefore, the two roots of the quadratic equation $2x^2+10x-7=0$ are $\frac{-5+\sqrt{39}}{2}$ and $\frac{-5-\sqrt{39}}{2}$.

Approximating these roots to the nearest hundredth gives the following values:

$\frac{-5+\sqrt{39}}{2}\doteq$ [redacted]

$$\frac{-5-\sqrt{39}}{2}\doteq-5.62$$

6. Step 1

Write the equation in the form $ax^2+bx+c=0$, and identify the values of a, b, and c.

$$\frac{3}{4}x^2+2x=5$$
$$(\cancel{4})\frac{3}{\cancel{4}}x^2+(4)2x=(4)5$$
$$3x^2+8x=20$$
$$3x^2+8x-20=0$$

Therefore, $a=3$, $b=8$, and $c=-20$.

The trinomial cannot be factored.

Step 2

Substitute the values of a, b, and c into the quadratic formula, and simplify.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-8\pm\sqrt{8^2-4(3)(-20)}}{2(3)}$$
$$x=\frac{-8\pm\sqrt{304}}{6}$$
$$x=\frac{-8\pm\sqrt{(16)(19)}}{6}$$
$$x=\frac{-8\pm4\sqrt{19}}{6}$$
$$x=\frac{-4\pm2\sqrt{19}}{3}$$

7. The equation will have no real roots if the discriminant is less than 0.

$$b^2-4ac<0$$
$$(4)^2-4(a)(-3)<0$$
$$16+12a<0$$
$$12a<-16$$
$$a<-\frac{16}{12}$$
$$a<-\frac{4}{3}$$

The equation $0 = ax^2 + 4x - 3$ will have no real roots if $a < -\frac{4}{3}$.

8. Set $y = 0$, and rearrange the resulting quadratic equation in the form $-\frac{q}{a} = \left(x - p\right)^2$.

$$0 = -5\left(x-6\right)^2 + 0$$
$$\frac{0}{5} = \left(x-6\right)^2$$
$$0 = \left(x-6\right)^2$$

Since the ratio $-\frac{q}{a}$ is 0, there is one real solution to the equation.

The graph of the given function has its vertex on the x-axis (one x-intercept).

9. **Step 1**
Find the vertex.

Since the function $y = \frac{1}{3}\left(x-3\right)^2 - 2$ is in completed square form, the vertex is (3, –2).

Step 2
Find the y-intercept by substituting $x = 0$ into the equation of the function and solving for y.

$$y = \frac{1}{3}\left(x-3\right)^2 - 2$$
$$y = \frac{1}{3}\left(0-3\right)^2 - 2$$
$$y = \frac{1}{3}\left(9\right) - 2$$
$$y = 3 - 2$$
$$y = 1$$

The y-intercept is (0, 1).

Step 3
Find the x-intercepts by substituting $y = 0$ into the equation of the function and using the square root method to solve the resulting quadratic equation for x.

$$0 = \frac{1}{3}\left(x-3\right)^2 - 2$$
$$2 = \frac{1}{3}\left(x-3\right)^2$$
$$6 = \left(x-3\right)^2$$
$$\pm\sqrt{6} = x - 3$$
$$3 \pm \sqrt{6} = x$$

The x-intercepts are $(3+\sqrt{6}, 0)$ and $(3-\sqrt{6}, 0)$.

10. **Step 1**
Use the given zeros to write the function in the form $y = a\left(x-s\right)\left(x-t\right)$, and expand.

$s = 2\sqrt{3}$ and $t = -2\sqrt{3}$
$$y = a\left(x-2\sqrt{3}\right)\left(x-\left(-2\sqrt{3}\right)\right)$$
$$y = a\left(x-2\sqrt{3}\right)\left(x+2\sqrt{3}\right)$$
$$y = a\left(x^2 - 4\sqrt{9}\right)$$
$$y = a\left(x^2 - 12\right)$$

Step 2
Find the value of a by substituting the coordinates of the given point for x and y. Substitute $x = 1$ and $y = 2$.

$$y = a\left(x^2 - 12\right)$$
$$2 = a\left(\left(1\right)^2 - 12\right)$$
$$2 = a\left(-11\right)$$
$$-\frac{2}{11} = a$$

Step 3
Write the equation in the form $y = ax^2 + bx + c$.

Substitute the value for a into $y = a\left(x^2 - 12\right)$, and expand.

$$y = -\frac{2}{11}\left(x^2 - 12\right)$$
$$y = -\frac{2}{11}x^2 + \frac{24}{11}$$

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. Solve the equation by factoring.

$$2x^2+5x-12=0$$
$$2x^2-3x+8x-12=0$$
$$x(2x-3)+4(2x-3)=0$$
$$(2x-3)(x+4)=0$$
$$2x-3=0 \text{ or } x+4=0$$
$$2x=3 \qquad x=-4$$
$$x=\frac{3}{2}$$

2. **Step 1**

Write the equation in the form $ax^2+bx+c=0$. Even though the variable used in the equation $\frac{2}{3}y^2=54$ is *y*, it can be solved exactly the same as if it involved *x*.

$$\frac{2}{3}y^2=54$$
$$\cancel{3}\left(\frac{2}{\cancel{3}}y^2\right)=3(54)$$
$$2y^2=162$$
$$2y^2-162=0$$

Step 2

Solve by factoring.

$$2y^2-162=0$$
$$2(y^2-81)=0$$
$$(y-9)(y+9)=0$$

$$y-9=0 \qquad y+9=0$$
$$y=9 \qquad y=-9$$

3. Solve by taking the square root of both sides of the equation.

$$\left(y+\frac{2}{3}\right)^2=\frac{25}{9}$$
$$y+\frac{2}{3}=\pm\sqrt{\frac{25}{9}}$$
$$y+\frac{2}{3}=\pm\frac{5}{3}$$
$$y=-\frac{2}{3}\pm\frac{5}{3}$$
$$y=-\frac{7}{3} \text{ or } y=1$$

4. **Step 1**

Write the equation in the form $ax^2+bx+c=0$.

$$b^2=7b+11$$
$$b^2-7b-11=0$$

Step 2

Complete the square.

$$b^2-7b=11$$
$$b^2-7b+\frac{49}{4}=11+\frac{49}{4}$$
$$\left(b-\frac{7}{2}\right)^2=\frac{93}{4}$$

Step 3

Solve the equation by taking the square root of both sides of the equation.

$$\left(b-\frac{7}{2}\right)^2=\frac{93}{4}$$
$$b-\frac{7}{2}=\pm\sqrt{\frac{93}{4}}$$
$$b-\frac{7}{2}=\pm\frac{\sqrt{93}}{2}$$
$$b=\frac{7}{2}\pm\frac{\sqrt{93}}{2}$$
$$b=\frac{7\pm\sqrt{93}}{2}$$

5. For an equation to have two real and equal roots, $b^2-4ac=0$.

Substitute the values of *a*, *b*, and *c* from the equation $3x^2-2x+c=0$ into the discriminant, and solve for *c*.

$$(-2)^2-4(3)(c)=0$$
$$4-12c=0$$
$$-12c=-4$$
$$c=\frac{-4}{-12}$$
$$c=\frac{1}{3}$$

6. Write the equation in the form $-\frac{q}{a} = (x-p)^2$.

$$0 = \frac{1}{2}(x-7)^2 + 5$$
$$-5 = \frac{1}{2}(x-7)^2$$
$$-10 = (x-7)^2$$

There is no real solution since the ratio $-\frac{q}{a}$ is negative. Taking the square root of both sides is not possible.

7. Step 1
Write the equation in the form $ax^2 + bx + c = 0$.

$$2y^2 + 11y = -15$$
$$2y^2 + 11y + 15 = 0$$

Step 2
Solve the equation using the quadratic formula.

$$y = \frac{-11 \pm \sqrt{11^2 - 4(2)(15)}}{2(2)}$$
$$y = \frac{-11 \pm \sqrt{121 - 120}}{4}$$
$$y = \frac{-11 \pm \sqrt{1}}{4}$$
$$y = \frac{-11+1}{4} \text{ or } y = \frac{-11-1}{4}$$
$$y = -\frac{10}{4} \text{ or } y = -\frac{12}{4}$$
$$y = -\frac{5}{2} \text{ or } y = -3$$

8. Step 1
Write the equation in the form $ax^2 + bx + c = 0$.

$$\frac{2x-1}{3} = \frac{x^2 + 2x}{5}$$
$$5(2x-1) = 3(x^2 + 2x)$$
$$10x - 5 = 3x^2 + 6x$$
$$0 = 3x^2 - 4x + 5$$

Step 2
Solve the equation using the quadratic formula.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(5)}}{2(3)}$$
$$x = \frac{4 \pm \sqrt{16 - 60}}{6}$$
$$x = \frac{4 \pm \sqrt{-44}}{6}$$

This equation has no real roots since $b^2 - 4ac < 0$.

9. Step 1
Write a quadratic equation that represents the problem.

The points at which $y = 2x^2 + 4x + 2$ and $y = 6$ intersect will satisfy both equations.

Since the y-coordinate of these intersection points must be 6, substitute $y = 6$ into the equation $y = 2x^2 + 4x + 2$.

$$6 = 2x^2 + 4x + 2$$
$$0 = 2x^2 + 4x - 4$$
$$0 = 2(x^2 + 2x - 2)$$
$$0 = x^2 + 2x - 2$$

Step 2
Solve for the x-coordinates of the points of intersection using the quadratic formula.

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-2)}}{2(1)}$$
$$x = \frac{-2 \pm \sqrt{4+8}}{2}$$
$$x = \frac{-2 \pm \sqrt{12}}{2}$$
$$x = \frac{-2 \pm 2\sqrt{3}}{2}$$
$$x = -1 \pm \sqrt{3}$$
$$x = -1 + \sqrt{3} \text{ or } x = -1 - \sqrt{3}$$

10. Step 1
Write the function in the form $y = ax^2 + bx + c$.

$$y = (-3x-2)(2x-3)$$
$$y = -6x^2 + 5x + 6$$

Step 2
Find the vertex by completing the square.

$$y=-6\left(x^2-\frac{5}{6}x+\_\_\_\right)+6$$

$$y=-6\left(x^2-\frac{5}{6}x+\frac{25}{144}-\frac{25}{144}\right)+6$$

$$y=-6\left(x^2-\frac{5}{6}x+\frac{25}{144}\right)-\frac{25}{144}(-6)+6$$

$$y=-6\left(x-\frac{5}{12}\right)^2+\frac{25}{24}+6$$

$$y=-6\left(x-\frac{5}{12}\right)^2+\frac{169}{24}$$

The vertex is at $\left(\frac{5}{12},\frac{169}{24}\right)$.

Step 3
Find the y-intercept by substituting $x = 0$ into the equation $y=-6x^2+5x+6$ and solving for y.

$$y=-6x^2+5x+6$$
$$y=-6(0)^2+5(0)+6$$
$$y=6$$

The y-intercept is (0, 6).

Step 4
Find the x-intercepts by substituting $y = 0$ into the original equation, and solve each factor for x.

$$y=(-3x-2)(2x-3)$$
$$0=(-3x-2)(2x-3)$$
$$-3x-2=0 \quad \text{or} \quad 2x-3=0$$
$$-3x=2 \quad \text{or} \quad 2x=3$$
$$x=-\frac{2}{3} \quad \text{or} \quad x=\frac{3}{2}$$

The x-intercepts are $\left(-\frac{2}{3},0\right)$ and $\left(\frac{3}{2},0\right)$.

11. **Step 1**
Use the given zeros to write the function in the form $y=a(x-s)(x-t)$, and expand.

Let $s = -3$ and $t = 4$.

$$y=a(x-(-3))(x-4)$$
$$y=a(x+3)(x-4)$$
$$y=a(x^2-x-12)$$

Step 2
Find the value of a by substituting the coordinates of the given point for x and y. Substitute $x = 5$ and $y = 8$.

$$y=a(x^2-x-12)$$
$$8=a((5)^2-(5)-12)$$
$$8=8a$$
$$1=a$$

Step 3
Write the equation of the function in the form $y=ax^2+bx+c$.

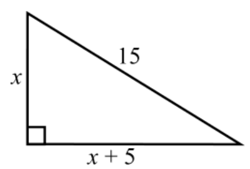
Substitute the value for a into $y=a(x^2-x-12)$, and expand.

$$y=1(x^2-x-12)$$
$$y=x^2-x-12$$

Lesson 5—Applications of Quadratic Equations

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Draw a diagram.



Let x be the length of the shorter leg and $x + 5$ be the length of the other leg.

Step 2
Write a quadratic equation in the form $ax^2+bx+c=0$ using the Pythagorean theorem to relate the lengths of all three sides of the right triangle.

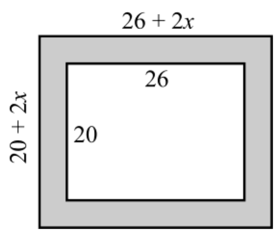
$$x^2+(x+5)^2=15^2$$
$$x^2+x^2+10x+25=225$$
$$2x^2+10x-200=0$$
$$x^2+5x-100=0$$

Step 3
Solve the equation using the quadratic formula.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-5\pm\sqrt{(5)^2-4(1)(-100)}}{2(1)}$$
$$x=\frac{-5\pm\sqrt{425}}{2(1)}$$
$$x=\frac{-5\pm5\sqrt{17}}{2}$$

Since the length of a leg must be a positive number, the lengths of the legs are $\frac{-5+5\sqrt{17}}{2}$ cm and $\frac{-5+5\sqrt{17}}{2}+5=\frac{5+5\sqrt{17}}{2}$ cm .

2. **Step 1**
Draw a diagram.
Let x be the width of the picture frame.



Step 2
Write a quadratic equation in the form $ax^2+bx+c=0$ using the area formula for a rectangle.

$$(20+2x)(26+2x)=952$$
$$2(10+x)2(13+x)=952$$
$$4(10+x)(13+x)=952$$
$$(10+x)(13+x)=238$$
$$130+23x+x^2=238$$
$$x^2+23x-108=0$$

Step 3
Solve the equation by factoring.

$$x^2+23x-108=0$$
$$(x-4)(x+27)=0$$
$$x=4 \text{ or } x=-27$$

Since the width of the frame must be a positive number, the width is 4 cm.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Substitute the given information into the quadratic equation, and write it in the form $ax^2+bx+c=0$.

$$h(t)=-5t^2+30t+1$$
$$41=-5t^2+30t+1$$
$$0=-5t^2+30t-40$$

Step 2
Solve the equation by factoring.

$$0=-5t^2+30t-40$$
$$0=-5(t^2-6t+8)$$
$$0=(t-4)(t-2)$$
$$t-4=0 \quad \text{or} \quad t-2=0$$
$$t=4 \qquad\qquad t=2$$

The ball reaches a height of 41 m at t = 2 s and at t = 4 s.

2. **Step 1**
Write a quadratic equation in the form $ax^2+bx+c=0$.

If x is one number, then $x+2\sqrt{2}$ is the other number. Their product is 2.

$$x\left(x+2\sqrt{2}\right)=2$$
$$x^2+2\sqrt{2}(x)=2$$
$$x^2+2\sqrt{2}(x)-2=0$$

(Note: It is also possible to define the two numbers as x and $x-2\sqrt{2}$).

Step 2
Solve the equation using the quadratic formula.

$$x=\frac{-2\sqrt{2}\pm\sqrt{\left(2\sqrt{2}\right)^2-4(1)(-2)}}{2(1)}$$
$$x=\frac{-2\sqrt{2}\pm\sqrt{8+8}}{2}$$
$$x=\frac{-2\sqrt{2}\pm\sqrt{16}}{2}$$
$$x=\frac{-2\sqrt{2}\pm4}{2}$$
$$x=-\sqrt{2}\pm2$$

Since x must be positive, $x=-\sqrt{2}+2$ only.

Determine the other number if $x=-\sqrt{2}+2$.

$x+2\sqrt{2}$
$=-\sqrt{2}+2+2\sqrt{2}$
$=2+\sqrt{2}$

The two numbers are $-\sqrt{2}+2$ and $2+\sqrt{2}$.

3. **Step 1**
Write a quadratic equation in the form $ax^2+bx+c=0$ using the area formula for a rectangle.

If w is the width and l is the length of the rectangle, then the sum of the length and width can be written as $l+w=19$.

The area is 60 m^2.

$$A=lw$$
$$60=(19-w)w$$
$$60=19w-w^2$$
$$w^2-19w+60=0$$

Step 2
Solve the equation by factoring.

$$w^2-19w+60=0$$
$$(w-15)(w-4)=0$$
$$w-15=0 \text{ or } w-4=0$$
$$w=15 \qquad w=4$$

Determine l if w = 4.

$l=19-w$
$l=19-4$
$l=15$

The rectangle has a width of 4 m and a length of 15 m. (Alternatively, if w = 15, l = 4.)

Practice Test

ANSWERS AND SOLUTIONS

1. The function $y=x^2$ becomes $y=\frac{1}{2}x^2$, which then becomes $y=-\frac{1}{2}x^2$. Then, $y=-\frac{1}{2}x^2$ becomes $y=-\frac{1}{2}(x-3)^2-1$.

2. **Step 1**
Write the function in $y=a(x-p)^2+q$ form.

Solve for y by dividing both sides by 3.

$$3y=(x-5)^2-6$$
$$\frac{3}{3}y=\frac{(x-5)^2}{3}-\frac{6}{3}$$
$$y=\frac{1}{3}(x-5)^2-2$$

Step 2
Apply the transformations in the order of stretches and reflections, followed by translations.
The graph of $y=x^2$ is transformed to the graph of $y=\frac{1}{3}x^2$ by applying a vertical stretch about the x-axis by a factor of $\frac{1}{3}$.

Then, the graph of $y=\frac{1}{3}x^2$ is transformed to the graph of $y=\frac{1}{3}(x-5)^2-2$ by translating the graph 5 units to the right and 2 units down.

3. **Step 1**
Find the vertex of the graph of the function $y=(x-0.5)^2-6.25$ using transformations.
Compared to $y=x^2$, the parabola is horizontally translated 0.5 units to the right and vertically translated 6.25 units down.

Therefore, vertex (0, 0) on the graph of $y=x^2$ becomes (0.5, –6.25) on the graph of $y=(x-0.5)^2-6.25$.

Step 2
Identify the characteristics of the transformed parabola.

If the vertex of the graph of $y=(x-0.5)^2-6.25$ is (0.5, –6.25), the axis of symmetry is x = 0.5.
The domain is $x\in\mathbb{R}$. Since $a>0$, the parabola opens up. The range is $y\geq-6.25$. The function has a minimum y-value of –6.25.

Step 3
Find the intercepts.

Substitute $x = 0$, and solve for y to find the y-intercept.

$$\begin{aligned} y &= (x-0.5)^2 - 6.25 \\ y &= (0-0.5)^2 - 6.25 \\ y &= 0.25 - 6.25 \\ y &= -6 \end{aligned}$$

The y-intercept is (0, –6).

Substitute $y = 0$, and solve for x to find the x-intercepts.

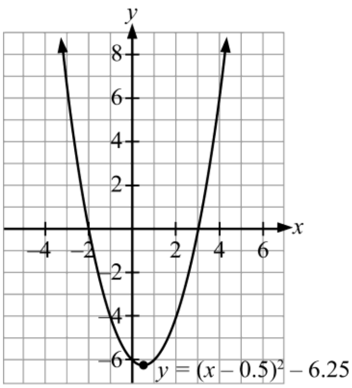
$$\begin{aligned} y &= (x-0.5)^2 - 6.25 \\ 0 &= (x-0.5)^2 - 6.25 \\ 6.25 &= (x-0.5)^2 \\ \pm 2.5 &= x - 0.5 \\ 0.5 \pm 2.5 &= x \\ 3 &= x \\ -2 &= x \text{ and } 3 = x \end{aligned}$$

The x-intercepts are (3, 0) and (–2, 0).

Step 4
Sketch the graph of the function $y = (x-0.5)^2 - 6.25$.

Plot the vertex and intercepts.

The graph of $y = (x-0.5)^2 - 6.25$ is shown.



4. **Step 1**
Substitute the known values into the equation $y = a(x - p)^2 + q$.

Since the vertex is (–2, –5), $p = -2$ and $q = -5$. If one of the x-intercepts is –3, this means the parabola contains the point (–3, 0).

$$\begin{aligned} y &= a(x-p)^2 + q \\ y &= a(x+2)^2 - 5 \\ 0 &= a(-3+2)^2 - 5 \end{aligned}$$

Step 2
Solve for a.

$$\begin{aligned} 0 &= a(-3+2)^2 - 5 \\ 0 &= a(-1)^2 - 5 \\ 0 &= a - 5 \\ 5 &= a \end{aligned}$$

Therefore, the equation of the parabola is $y = 5(x+2)^2 - 5$.

Step 3
Write the function in the form $y = ax^2 + bx + c$ by expanding.

$$\begin{aligned} y &= 5(x+2)^2 - 5 \\ y &= 5(x^2 + 4x + 4) - 5 \\ y &= 5x^2 + 20x + 20 - 5 \\ y &= 5x^2 + 20x + 15 \end{aligned}$$

Alternate solution:
Step 1
Write the function in the form $y = a(x-s)(x-t)$.

Substitute the values of $s = -3$ and $t = -1$.

$$\begin{aligned} y &= a(x-(-3))(x-(-1)) \\ y &= a(x+3)(x+1) \\ y &= a(x^2 + 4x + 3) \end{aligned}$$

Step 2
Find the value of a by substituting the coordinates of the vertex into $y = a(x^2 + 4x + 3)$.

Substitute $x = -2$ and $y = -5$ into the equation.

$$\begin{aligned} y &= a(x^2 + 4x + 3) \\ -5 &= a((-2)^2 + 4(-2) + 3) \\ -5 &= -a \\ 5 &= a \end{aligned}$$

Step 3
Write the function in the form $y = ax^2 + bx + c$.

Substitute the value for a into $y = a(x^2 + 4x + 3)$, and expand.

$$\begin{aligned} y &= 5(x^2 + 4x + 3) \\ y &= 5x^2 + 20x + 15 \end{aligned}$$

5. Step 1

Write the equation of the function in the form $y = a(x - p) + q$ by completing the square.

$$y = x^2 + 8x + 18$$
$$y = \left(x^2 + 8x + \_\_\_\right) + 18$$
$$y = \left(x^2 + 8x + 16 - 16\right) + 18$$
$$y = \left(x^2 + 8x + 16\right) - 16(1) + 18$$
$$y = (x + 4)^2 + 2$$

Step 2

Identify the characteristics of the function.
The vertex is (–4, 2). The axis of symmetry is $x = -4$. The domain is $x \in \mathbb{R}$. The range is $y \geq 2$.
The parabola opens upward since $a > 0$.
The minimum value of y is 2.

Substitute $x = 0$, and solve for y to find the y-intercept. (You can use the equation in either form.)

$$y = x^2 + 8x + 18$$
$$y = 0^2 + 8(0) + 18$$
$$y = 18$$

The y-intercept is (0, 18).

Substitute $y = 0$, and solve for x to find the x-intercepts. (You can use the equation in either form.)

$$y = (x + 4)^2 + 2$$
$$0 = (x + 4)^2 + 2$$
$$-2 = (x + 4)^2$$
$$\pm\sqrt{-2} = x + 4$$

There are no x-intercepts since it is not possible to take the square root of a negative.

6. Step 1

Find the value of a using the vertex formula.
The x-coordinate of the vertex is determined from $x = \dfrac{-b}{2a}$.

Use the given equation $y = ax^2 - 8x + c$ and the coordinates of the vertex (–4, 2).

$$x = \frac{-b}{2a}$$
$$-4 = \frac{-(-8)}{2a}$$
$$-4 = \frac{8}{2a}$$
$$-8a = 8$$
$$a = -1$$

Step 2

Find the value of c by substituting the value of a and the coordinates of the vertex into the given equation. Substitute $a = -1$, $x = -4$, and $y = 2$ into $y = ax^2 - 8x + c$.

$$y = ax^2 - 8x + c$$
$$y = -x^2 - 8x + c$$
$$2 = -(-4)^2 - 8(-4) + c$$
$$2 = -16 + 32 + c$$
$$2 = 16 + c$$
$$-14 = c$$

Therefore, $a = -1$ and $c = -14$.

7. Step 1

Write the equation in the form $y = ax^2 + bx + c$.

$$\frac{-3x^2 + 5}{2} = x - 1$$
$$2\left(\frac{-3x^2 + 5}{2}\right) = 2(x - 1)$$
$$-3x^2 + 5 = 2(x) - 2(1)$$
$$-3x^2 + 5 = 2x - 2$$
$$0 = 3x^2 + 2x - 7$$

Step 2

Solve the equation using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(3)(-7)}}{2(3)}$$
$$x = \frac{-2 \pm \sqrt{88}}{6}$$
$$x = \frac{-2 \pm 2\sqrt{22}}{6}$$
$$x = \frac{-1 \pm \sqrt{22}}{3}$$

Therefore, $x = \dfrac{-1 + \sqrt{22}}{3}$ or $x = \dfrac{-1 - \sqrt{22}}{3}$.

8. Step 1
Use the given zeros to write the equation of the function in the form $a(x-s)(x-t)=0$, and expand.
$s=-3$ and $t=8$

$$a(x-(-3))(x-8)=y$$
$$a(x+3)(x-8)=y$$
$$a(x^2-8x+3x-24)=y$$
$$a(x^2-5x-24)=y$$

Step 2
Solve for a by substituting the coordinates of the given point for x and y. Substitute $x = 1$ and $y = 1$ into $a(x^2-5x-24)=y$, and solve for a.

$$a(x^2-5x-24)=y$$
$$a((1)^2-5(1)-24)=1$$
$$-28a=1$$
$$a=-\frac{1}{28}$$

Step 3
Write the equation in the form $y = ax^2 + bx + c$. Substitute the value for a into $a(x^2-5x-24)=y$, and expand.

$$a(x^2-5x-24)=y$$
$$-\frac{1}{28}(x^2-5x-24)=y$$
$$-\frac{1}{28}x^2+\frac{5}{28}x+\frac{24}{28}=y$$
$$-\frac{1}{28}x^2+\frac{5}{28}x+\frac{6}{7}=y$$

9. Step 1
Write the equation in the form $ax^2 + bx + c = 0$.

$$2x^2-kx=-6$$
$$2x^2-kx+6=0$$

Step 2
Find the values of k. For the equation to have real and equal roots, the discriminant must equal 0.

$$b^2-4ac=0$$
$$(-k)^2-4(2)(6)=0$$
$$k^2-48=0$$
$$k^2=48$$
$$k=\pm\sqrt{48}$$
$$k=\pm4\sqrt{3}$$

Therefore, $k=4\sqrt{3}$ or $k=-4\sqrt{3}$.

10. Let n represent the number of \$1.00 increases in the price of a ticket, and let R represent the revenue from the ticket sales.

a) Determine the revenue as a function of the number of price increases.

$$R=(\text{number of tickets})(\text{price per ticket})$$
$$R=(6\,000-200n)(20+n)$$
$$R=-200n^2+2\,000n+120\,000$$

b) To find the maximum revenue, R, complete the square.

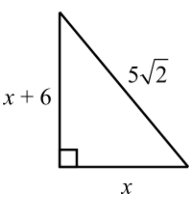
$$R=-200(n^2-10n+\_\_)+120\,000$$
$$R=-200(n^2-10n+25-25)+120\,000$$
$$R=-200(n^2-10n+25)-25(-200)+120\,000$$
$$R=-200(n-5)^2+125\,000$$

The vertex is at (5, 125 000).

Therefore, the ticket price will be $\$(20+n)=\$(20+5)$, which is \$25.
(There should be five increases of \$1.00)

c) The maximum revenue is \$125 000 as given by the y-coordinate of the vertex.

11. Step 1
Draw a diagram.
Let x represent the length of one leg, and let $x + 6$ represent the length of the other leg.



Step 2
Write a quadratic equation in the form $ax^2+bx+c=0$ using the Pythagorean theorem to relate the lengths of all three sides of the right triangle.

$$x^2+(x+6)^2=(5\sqrt{2})^2$$
$$x^2+x^2+12x+36=50$$
$$2x^2+12x-14=0$$

Step 3
Solve the equation by factoring.

$$2x^2+12x-14=0$$
$$2\left(x^2+6x-7\right)=0$$
$$2(x-1)(x+7)=0$$
$$x-1=0 \text{ or } x+7=0$$
$$x=1 \qquad x=-7$$

Since x must be positive, $x = 1$.

Therefore, the lengths of the two legs are 1 cm and 1 + 6 = 7 cm.

12. a) The height of the building will be the height of the rock at $t = 0$.

$$h(0)=-5(0)^2+10(0)+15$$
$$h(0)=15$$

The height of the building is 15 m.

b) The rock is in the air until it hits the ground. Solve the equation for t when $h(t) = 0$.

$$h(t)=-5t^2+10t+15$$
$$0=-5t^2+10t+15$$
$$0=5t^2-10t-15$$
$$0=5\left(t^2-2t-3\right)$$
$$0=5(t+1)(t-3)$$
$$t+1=0 \text{ or } t-3=0$$
$$t=-1 \qquad t=3$$

Since $t \geq 0$, $t = 3$ only.
The height is 0 m at $t = 3$ s.

The rock is in the air for 3 s.

c) Find how long it takes the rock to reach its maximum height by completing the square and identifying the vertex of the graph of the function.

$$h(t)=-5t^2+10t+15$$
$$h(t)=-5\left(t^2-2t\ \_\_\_\right)+15$$
$$h(t)=-5\left(t^2-2t+1\right)-(1)(-5)+15$$
$$h(t)=-5(t-1)^2+20$$

The vertex is at (1, 20). The rock reaches its maximum height 1 s after it is thrown.

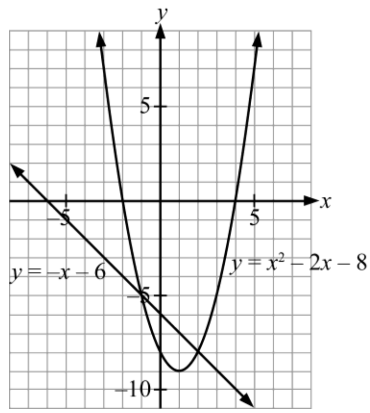
d) Since the y-coordinate of the vertex is 20, the maximum height reached by the rock is 20 m.

LINEAR AND QUADRATIC SYSTEMS

Lesson 1—Solving a System of Equations Graphically

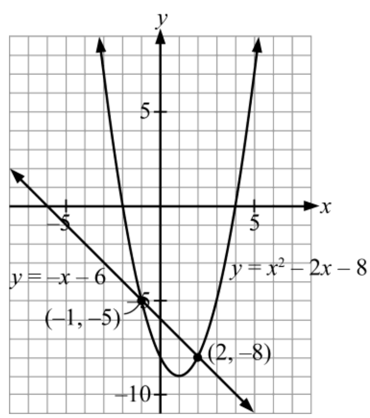
CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Step 1
Graph the functions represented by the given equations.



Step 2
Determine the points of intersection.

From the graph, it can be seen that the points of intersection are at (–1, –5) and (2, –8).



The solution set for the system is {(–1, –5), (2, –8)}.

2. **Step 1**

Write the first equation in the form $y = mx + b$.

$$4x - y = 5$$
$$-y = -4x + 5$$
$$y = 4x - 5$$

Step 2

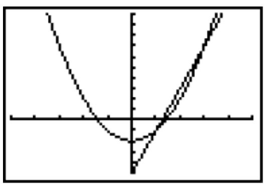
Graph the functions represented by the given equations using a graphing calculator.

Press $\boxed{\text{Y =}}$, and input each function.

$Y_1 = 4x - 5$

$Y_2 = x^2 - 2$

Press $\boxed{\text{GRAPH}}$. The window setting used to display the two graphs is $x:[-5,5,1]$ and $y:[-5,10,1]$.



Step 3

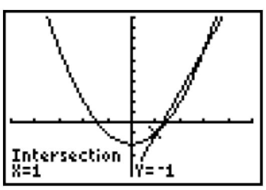
Find the points of intersection.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and choose 5:intersect.

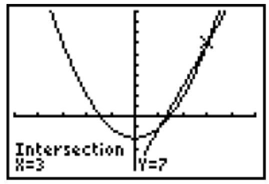
For "First curve?", position the cursor just left or right of the intersection point that is farthest to the left, and press $\boxed{\text{ENTER}}$.

For "Second curve?", position the cursor just left or right of the intersection point that is farthest to the left, and press $\boxed{\text{ENTER}}$.

For "Guess?", press $\boxed{\text{ENTER}}$.



Repeat the process with the intersection point that is farthest to the right.



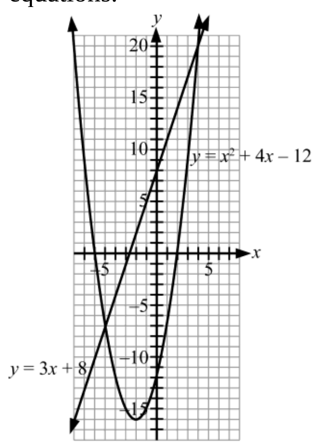
The calculator indicates that the points of intersection of the curve and the line are at (1, –1) and (3, 7).

The solution set for the system of equations is {(1, –1), (3, 7)}.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

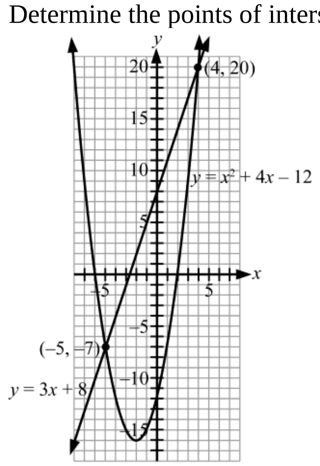
1. **Step 1**

Graph the functions represented by the given equations.



Step 2

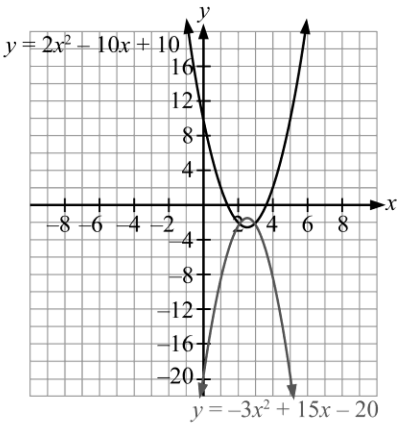
Determine the points of intersection.



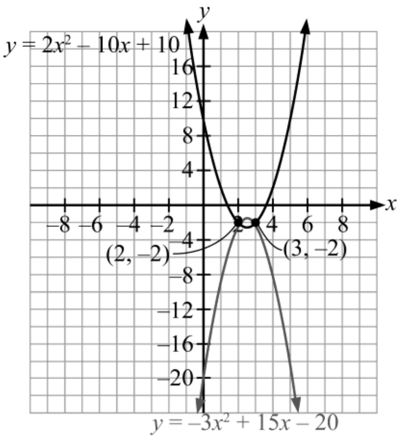
From the graph, it can be seen that the points of intersection are at (–5, –7) and (4, 20).

The solution set for the system is {(–5, –7), (4, 20)}.

2. Step 1
Graph the functions represented by the given equations.



Step 2
Determine the points of intersection.



From the graph, it can be seen that the points of intersection are at (2, –2) and (3, –2).

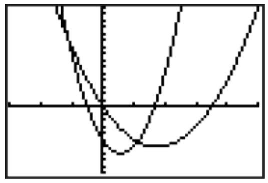
The solution set for the system is {(2, –2), (3, –2)}.

3. Step 1
Graph the functions represented by the given equations using a graphing calculator.

Press $\boxed{\text{Y =}}$, and input each function.

$Y_1 = (3x-5)(2x+1)$
$Y_2 = 2x^2 - 7x$

Press $\boxed{\text{GRAPH}}$. The window setting used to display the two graphs is $x:[-3, 5, 1]$ and $y:[-10, 15, 1]$.



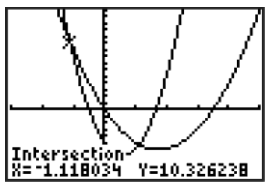
Step 2
Find the points of intersection.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and choose 5:intersect.

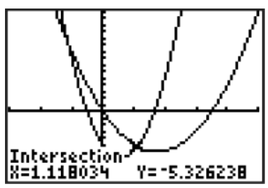
For “First curve?”, position the cursor just left or right of the intersection point that is farthest to the left, and press $\boxed{\text{ENTER}}$.

For “Second curve?”, position the cursor just left or right of the intersection point that is farthest to the left, and press $\boxed{\text{ENTER}}$.

For “Guess?”, press $\boxed{\text{ENTER}}$.



Repeat the process with the intersection point that is farthest to the right.



The calculator indicates that the points of intersection of the two curves are at (–1.12, 10.33) and (1.12, –5.33).

The solution set for the system of equations is {(–1.12, 10.33), (1.12, –5.33)}.

4. Step 1
Write the second equation in the form $y = mx + b$.

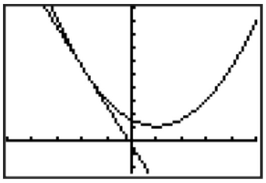
$$0 = 6x + y + 1$$
$$-y = 6x + 1$$
$$y = -6x - 1$$

Step 2
Graph the functions represented by the given equations using a graphing calculator.

Press $\boxed{\text{Y =}}$, and input each function.

$\text{Y}_1 = x^2 - 2x + 3$
$\text{Y}_2 = -6x - 1$

Press $\boxed{\text{GRAPH}}$. The window setting used to display the two graphs is $x:[-5, 5, 1]$ and $y:[-5, 20, 2]$.



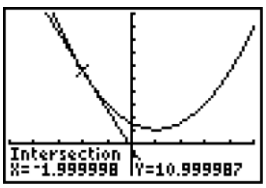
Step 3
Find the point of intersection.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and choose 5:intersect.

For “First curve?”, position the cursor just left or right of the intersection point, and press $\boxed{\text{ENTER}}$.

For “Second curve?”, position the cursor just left or right of the intersection point, and press $\boxed{\text{ENTER}}$.

For “Guess?”, press $\boxed{\text{ENTER}}$.



The calculator indicates that the point of intersection of the curve and the line is (–2.00, 11.00).

The solution set for the system of equations is {(–2.00, 11.00)}.

5. Step 1
Write the first equation in the form $y = ax^2 + bx + c$.

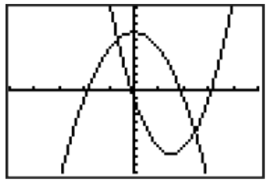
$$y + 9x = 3x^2 - 1$$
$$y = 3x^2 - 9x - 1$$

Step 2
Graph the functions represented by the given equations using a graphing calculator.

Press $\boxed{\text{Y =}}$, and input each function.

$\text{Y}_1 = 3x^2 - 9x - 1$
$\text{Y}_2 = -2x^2 + 7$

Press $\boxed{\text{GRAPH}}$. The window setting used to display the two graphs is $x:[-5, 5, 1]$ and $y:[-10, 10, 1]$.



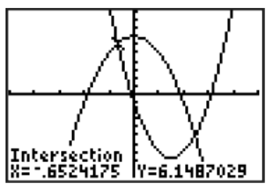
Step 3
Find the points of intersection.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and choose 5:intersect.

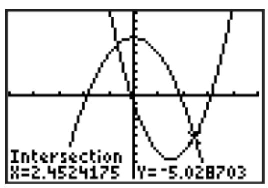
For “First curve?”, position the cursor just left or right of the intersection point that is farthest to the left, and press $\boxed{\text{ENTER}}$.

For “Second curve?”, position the cursor just left or right of the intersection point that is farthest to the left, and press $\boxed{\text{ENTER}}$.

For “Guess?”, press $\boxed{\text{ENTER}}$.



Repeat the process with the intersection point that is farthest to the right.



The calculator indicates that the points of intersection of the two curves are at (–0.65, 6.15) and (2.45, –5.03).

The solution set for the system of equations is {(–0.65, 6.15), (2.45, –5.03)}.

6. Step 1
Write the first equation in the form $y = mx + b$ and the second equation in the form $y = ax^2 + bx + c$.

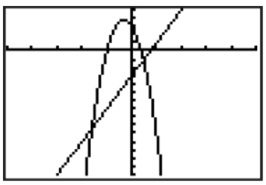
$$y - 4x = -3 \qquad 8x^2 - 3 = -5x - y$$
$$y = 4x - 3 \qquad y = -8x^2 - 5x + 3$$

Step 2
Graph the functions represented by the given equations using a graphing calculator.

Press $\boxed{\text{Y =}}$, and input each function.

$\text{Y}_1 = 4x - 3$
$\text{Y}_2 = -8x^2 - 5x + 3$

Press $\boxed{\text{GRAPH}}$. The window setting used to display the two graphs is $x:[-5, 5, 1]$ and $y:[-15, 5, 1]$.



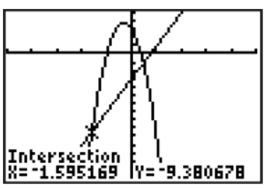
Step 3
Find the points of intersection.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and choose 5:intersect.

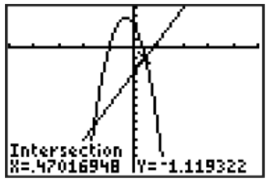
For "First curve?", position the cursor just left or right of the intersection point that is farthest to the left, and press $\boxed{\text{ENTER}}$.

For "Second curve?", position the cursor just left or right of the intersection point that is farthest to the left, and press $\boxed{\text{ENTER}}$.

For "Guess?", press $\boxed{\text{ENTER}}$.



Repeat the process with the intersection point that is farthest to the right.



The calculator indicates that the points of intersection of the curve and the line are at (–1.60, –9.38) and (0.47, –1.12).

The solution set for the system of equations is {(–1.60, –9.38), (0.47, –1.12)}.

Lesson 2—Solving a System of Equations Algebraically

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Step 1
Isolate one variable from equation 1 or 2.

In this case, equation 1 has the *y*-variable isolated.

Step 2
Substitute the expression represented by the isolated variable into equation 2, and simplify.

Substitute x^2 for *y* into equation 2, and bring all terms to one side.

$$x + x^2 = 12$$
$$x^2 + x - 12 = 0$$

Step 3
Solve for *x* by factoring the quadratic equation.

$$x^2 + x - 12 = 0$$
$$x^2 + 4x - 3x - 12 = 0$$
$$x(x+4) - 3(x+4) = 0$$
$$(x+4)(x-3) = 0$$
$$x + 4 = 0 \qquad x - 3 = 0$$
$$x = -4 \qquad x = 3$$

Step 4
Substitute the solved values of *x* into equation 2, and solve for *y*.

$$x + y = 12 \qquad x + y = 12$$
$$-4 + y = 12 \qquad 3 + y = 12$$
$$y = 16 \qquad y = 9$$

Step 5
Write the solutions as ordered pairs in set notation.
{(–4, 16), (3, 9)}

2. **Step 1**
Isolate one variable from equation 1 or 2.

In this case, equation 2 has the *y*-variable isolated.

Step 2
Substitute the expression represented by the isolated variable into equation 1, and simplify.

Substitute $3x^2-17$ for *y* into equation 1, and bring all terms to one side.

$$\begin{aligned}2x^2+3y&=4\\2x^2+3\left(3x^2-17\right)&=4\\2x^2+9x^2-51&=4\\11x^2-55&=0\end{aligned}$$

Step 3
Solve for *x*.

$$\begin{aligned}11x^2-55&=0\\11x^2&=55\\x^2&=5\\x&=\pm\sqrt{5}\end{aligned}$$

Step 4
Substitute the solved values of *x* into equation 2, and solve for *y*.

$$\begin{aligned}y&=3x^2-17\\y&=3\left(\sqrt{5}\right)^2-17\\y&=3(5)-17\\y&=15-17\\y&=-2\end{aligned}\qquad\begin{aligned}y&=3x^2-17\\y&=3\left(-\sqrt{5}\right)^2-17\\y&=3(5)-17\\y&=15-17\\y&=-2\end{aligned}$$

Step 5
Write the solutions as ordered pairs in set notation.

$$\left\{\left(\sqrt{5},-2\right),\ \left(-\sqrt{5},-2\right)\right\}$$

3. **Step 1**
Add equation 1 and equation 2 to eliminate the *y*-variable.

$$\begin{array}{rr}① & x-4y=20\\ ② & x^2+4y=-2\\ \hline & x^2+3x=18\end{array}$$

Step 2
Solve for *x*.

$$\begin{aligned}x^2+3x&=18\\x^2+3x-18&=0\\x^2+6x-3x-18&=0\\x(x+6)-3(x+6)&=0\\(x+6)(x-3)&=0\end{aligned}$$

$$\begin{aligned}x+6&=0\\x&=-6\end{aligned}\qquad\begin{aligned}x-3&=0\\x&=3\end{aligned}$$

Step 3
Substitute the solved values of *x* into equation 1, and solve for *y*.

$$\begin{aligned}3x-4y&=20\\3(-6)-4y&=20\\-18-4y&=20\\-4y&=38\\y&=-\frac{19}{2}\end{aligned}\qquad\begin{aligned}3x-4y&=20\\3(3)-4y&=20\\9-4y&=20\\-4y&=11\\y&=-\frac{11}{4}\end{aligned}$$

Step 4
Write the solutions as ordered pairs in set notation.

$$\left\{\left(-6,-\frac{19}{2}\right),\left(3,-\frac{11}{4}\right)\right\}$$

4. **Step 1**
Multiply equation 1 by 2.

$$①\times 2\quad 2\left(2x^2+y=7\right)\rightarrow 4x^2+2y=14$$

Let $4x^2+2y=14$ represent equation 3.

Step 2
Subtract equation 3 from equation 1 to eliminate the *y*-variable.

$$\begin{array}{rr}② & x+2y=11\\ ③ & 4x^2+2y=14\\ \hline & -4x^2+4x=-3\end{array}$$

Step 3
Solve for *x*.

$$\begin{aligned}-4x^2+4x&=-3\\-4x^2+4x+3&=0\\4x^2-4x-3&=0\\4x^2-6x+2x-3&=0\\2x(2x-3)+(2x-3)&=0\\(2x-3)(2x+1)&=0\end{aligned}$$

$$\begin{aligned}2x-3&=0\\2x&=3\\x&=\frac{3}{2}\end{aligned}\qquad\begin{aligned}2x+1&=0\\2x&=-1\\x&=-\frac{1}{2}\end{aligned}$$

Step 4
Substitute the solved values into equation 2, and solve for y.

$$\begin{aligned} 4x+2y&=11 \\ 4\left(\frac{3}{2}\right)+2y&=11 \\ 6+2y&=11 \\ 2y&=5 \\ y&=\frac{5}{2} \end{aligned} \qquad \begin{aligned} 4x+2y&=11 \\ 4\left(-\frac{1}{2}\right)+2y&=11 \\ -2+2y&=11 \\ 2y&=13 \\ y&=\frac{13}{2} \end{aligned}$$

Step 5
Write the solutions as ordered pairs in set notation.

$$\left\{\left(\frac{3}{2},\frac{5}{2}\right),\left(-\frac{1}{2},\frac{13}{2}\right)\right\}$$

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Isolate one variable from equation 1 or 2.

In this case, both equations already have the y-variable isolated.

Step 2
Substitute the expression represented by the isolated variable from equation 2 into equation 1, and simplify.

Substitute $-4x-25$ for y into equation 1, and bring all terms to one side.

$$\begin{aligned} x^2-35x+5&=y \\ x^2-35x+5&=-4x-25 \\ x^2-31x+30&=0 \end{aligned}$$

Step 3
Solve for x by factoring the quadratic equation.

$$\begin{aligned} x^2-31x+30&=0 \\ x^2-30x-x+30&=0 \\ x(x-30)-1(x-30)&=0 \\ (x-30)(x-1)&=0 \end{aligned}$$

$$\begin{aligned} x-30&=0 \\ x&=30 \end{aligned} \qquad \begin{aligned} x-1&=0 \\ x&=1 \end{aligned}$$

Step 4
Substitute the solved values of x into equation 2, and solve for y.

$$\begin{aligned} y&=-4x-25 \\ y&=-4(30)-25 \\ y&=-120-25 \\ y&=-145 \end{aligned} \qquad \begin{aligned} y&=-4x-25 \\ y&=-4(1)-25 \\ y&=-4-25 \\ y&=-29 \end{aligned}$$

Step 5
Write the solutions as ordered pairs in set notation.
{(30, –145), (1, –29)}

2. **Step 1**
Isolate one variable from equation 1 or 2.

In this case, equation 1 has the y-variable isolated.

Step 2
Substitute the expression represented by the isolated variable into equation 2, and simplify.

Substitute $3-6x$ for y in equation 2, and bring all terms to one side.

$$\begin{aligned} 2x^2+8x&=3-y \\ 2x^2+8x&=3-(3-6x) \\ 2x^2+8x&=3-3+6x \\ 2x^2+8x&=6x \\ 2x^2+2x&=0 \end{aligned}$$

Step 3
Solve for x by factoring the quadratic equation.

$$\begin{aligned} 2x^2+2x&=0 \\ 2x(x+1)&=0 \end{aligned}$$

$$\begin{aligned} 2x&=0 \\ x&=0 \end{aligned} \qquad \begin{aligned} x+1&=0 \\ x&=-1 \end{aligned}$$

Step 4
Substitute the solved values of x into equation 1, and solve for y.

$$\begin{aligned} y&=3-6x \\ y&=3-6(0) \\ y&=3 \end{aligned} \qquad \begin{aligned} y&=3-6x \\ y&=3-6(-1) \\ y&=3+6 \\ y&=9 \end{aligned}$$

Step 5
Write the solutions as ordered pairs in set notation.
{(0, 3), (–1, 9)}

3. **Step 1**
Isolate one variable from equation 1 or 2.

Isolate the y-variable in equation 2.

$$\begin{aligned} -4x^2-x&=y+3 \\ y&=-4x^2-x-3 \end{aligned}$$

Step 2
Substitute the expression represented by the isolated variable from equation 2 into equation 1, and simplify.

Substitute $-4x^2 - x - 3$ for y in equation 1, and bring all terms to one side.

$$\begin{aligned} x &= 5x^2 + y - 32 \\ x &= 5x^2 + \left(-4x^2 - x - 3\right) - 32 \\ x &= 5x^2 - 4x^2 - x - 3 - 32 \\ x &= x^2 - x - 35 \\ 0 &= x^2 - 2x - 35 \end{aligned}$$

Step 3
Solve for x by factoring the quadratic equation.

$$\begin{aligned} 0 &= x^2 - 2x - 35 \\ 0 &= (x-7)(x+5) \end{aligned}$$

$$\begin{array}{ll} x - 7 = 0 & x + 5 = 0 \\ x = 7 & x = -5 \end{array}$$

Step 4
Substitute the solved values of x into equation 1, and solve for y.

$$\begin{array}{ll} x = 5x^2 + y - 32 & x = 5x^2 + y - 32 \\ 7 = 5(7)^2 + y - 32 & -5 = 5(-5)^2 + y - 32 \\ 7 = 245 + y - 32 & -5 = 125 + y - 32 \\ -238 = y - 32 & -130 = y - 32 \\ -206 = y & -98 = y \end{array}$$

Step 5
Write the solutions as ordered pairs in set notation.
{(–5, –98), (7, –206)}

4. **Step 1**
Add equation 1 and equation 2 to eliminate the y-variable.

$$\begin{array}{ll} ① & -x^2 + y = 4x - 4 \\ ② & \underline{-5x - y = -6} \\ & -x^2 - 5x = 4x - 10 \end{array}$$

Step 2
Solve for x.

$$\begin{aligned} -x^2 - 5x &= 4x - 10 \\ 0 &= x^2 + 9x - 10 \\ 0 &= (x-1)(x+10) \end{aligned}$$

$$\begin{array}{ll} x - 1 = 0 & x + 10 = 0 \\ x = 1 & x = -10 \end{array}$$

Step 3
Substitute the solved values of x into equation 2, and solve for y.

$$\begin{array}{ll} -5x - y = -6 & -5x - y = -6 \\ -5(1) - y = -6 & -5(-10) - y = -6 \\ -5 - y = -6 & 50 - y = -6 \\ -y = -1 & -y = -56 \\ y = 1 & y = 56 \end{array}$$

Step 4
Write the solutions as ordered pairs in set notation.
{(1, 1), (–10, 56)}

5. **Step 1**
Multiply equation 2 by 3.

②×3 $3(y + 4x = 3) \rightarrow 3y + 12x = 9$

Let $3y + 12x = 9$ represent equation 3.

Step 2
Subtract equation 3 from equation 1 to eliminate y.

$$\begin{array}{ll} ① & y + 3x = 3x^2 + 15 \\ ③ & \underline{3y + 12x = 9} \\ & -9x = 3x^2 + 6 \end{array}$$

Step 3
Solve for x.

$$\begin{aligned} -9x &= 3x^2 + 6 \\ 0 &= 3x^2 + 9x + 6 \\ 0 &= 3\left(x^2 + 3x + 2\right) \\ 0 &= x^2 + 3x + 2 \\ 0 &= (x+1)(x+2) \end{aligned}$$

$$\begin{array}{ll} x + 1 = 0 & x + 2 = 0 \\ x = -1 & x = -2 \end{array}$$

Step 4
Substitute the solved values of x into equation 2, and solve for y.

$$\begin{array}{ll} y + 4x = 3 & y + 4x = 3 \\ y + 4(-1) = 3 & y + 4(-2) = 3 \\ y - 4 = 3 & y - 8 = 3 \\ y = 7 & y = 11 \end{array}$$

Step 5
Write the solutions as ordered pairs in set notation.
{(–1, 7), (–2, 11)}

6. Step 1
Subtract equation 2 from equation 1 to eliminate the *y*-variable.

$$\text{①} \quad \frac{1}{2}x^2 - 172 = y + 33x$$
$$\text{②} \quad -\frac{3}{2}x^2 + 17x = y - 128$$
$$\overline{2x^2 - 172 - 17x = 33x + 128}$$

Step 2
Solve for *x*.

$$\begin{aligned} 2x^2 - 172 - 17x &= 33x + 128 \\ 2x^2 - 50x - 300 &= 0 \\ 2\left(x^2 - 25x - 150\right) &= 0 \\ 2\left(x - 30\right)\left(x + 5\right) &= 0 \\ \left(x - 30\right)\left(x + 5\right) &= 0 \end{aligned}$$

$$x - 30 = 0 \qquad x + 5 = 0$$
$$x = 30 \qquad x = -5$$

Step 3
Substitute the solved values of *x* into equation 1, and solve for *y*.

$$\begin{aligned} \frac{1}{2}x^2 - 172 &= y + 33x \\ \frac{1}{2}(30)^2 - 172 &= y + 33(30) \\ 450 - 172 &= y + 990 \\ 278 &= y + 990 \\ -712 &= y \end{aligned}$$

$$\begin{aligned} \frac{1}{2}x^2 - 172 &= y + 33x \\ \frac{1}{2}(-5)^2 - 172 &= y + 33(-5) \\ \frac{25}{2} - 172 &= y - 165 \\ -\frac{319}{2} &= y - 165 \\ \frac{11}{2} &= y \end{aligned}$$

Step 4
Write the solutions as ordered pairs in set notation.

$$\left\{(30, -712), \left(-5, \frac{11}{2}\right)\right\}$$

7. Step 1
Set up a system of two equations.

The difference between *y* and 12 times *x* can be represented by $y - 12x = -69$.

The difference between *y* and the square of *x* is represented by $y - x^2 = -33$.

The system of equations is as follows:
① $y - 12x = -69$
② $y - x^2 = -33$

Step 2
Solve the system of equations using substitution.

Isolate the *y*-variable in equation 1.

$$\begin{aligned} y - 12x &= -69 \\ y &= 12x - 69 \end{aligned}$$

Step 3
Substitute the expression represented by the isolated variable from equation 1 into equation 2, and simplify.

Substitute $12x - 69$ for *y* in equation 2, and bring all terms to one side.

$$\begin{aligned} y - x^2 &= -33 \\ 12x - 69 - x^2 &= -33 \\ 0 &= x^2 - 12x + 36 \end{aligned}$$

Step 4
Solve for *x* by factoring the quadratic equation.

$$\begin{aligned} 0 &= x^2 - 12x + 36 \\ 0 &= (x - 6)(x - 6) \\ 0 &= (x - 6)^2 \\ 0 &= x - 6 \\ x &= 6 \end{aligned}$$

Step 5
Substitute the solved value of *x* into equation 1, and solve for *y*.

$$\begin{aligned} y - 12x &= -69 \\ y - 12(6) &= -69 \\ y - 72 &= -69 \\ y &= 3 \end{aligned}$$

The two numbers are 6 and 3.

Lesson 3—Classifying Systems of Equations

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Isolate one variable from equation 1 or 2.

Isolate y in equation 1.
$$y+5x=x^2+6$$
$$y=x^2-5x+6$$

Step 2
Substitute the expression represented by the isolated variable into equation 2, and simplify.

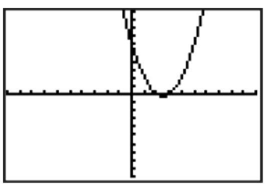
Substitute x^2-5x+6 for y in equation 2.
$$-5x+x^2=-6+y$$
$$-5x+x^2=-6+\left(x^2-5x+6\right)$$
$$-5x+x^2=-6+x^2-5x+6$$
$$-5x+x^2=x^2-5x$$

Step 3
Solve for x.
$$-5x+x^2=x^2-5x$$
$$0=0$$

Since $0=0$ is a true statement for every value of x, there are an infinite number of solutions for all values of x and y in this system of equations.

Step 4
Verify the system of equations has an infinite number of solutions using a graphing calculator.

When the equations are graphed using a graphing calculator, this window is obtained.



The graphs are the same, which is reasonable because the two equations are identical once they are rearranged.

2. **Step 1**
Isolate one variable from equation 1 or 2.

In this case, equation 2 already has the x-variable isolated.

Step 2
Substitute the expression represented by the isolated variable into equation 1, and simplify.

Substitute $8+y$ for x into equation 1, and bring all terms to one side.
$$2x^2=y+1$$
$$2(8+y)^2=y+1$$
$$2\left(64+16y+y^2\right)=y+1$$
$$128+32y+2y^2=y+1$$
$$2y^2+31y+127=0$$

Step 3
Solve for y using the quadratic formula.
$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$y=\frac{-(31)\pm\sqrt{(31)^2-4(2)(127)}}{2(2)}$$
$$y=\frac{-31\pm\sqrt{961-1016}}{4}$$
$$y=\frac{-31\pm\sqrt{-55}}{4}$$

Since you cannot take the square root of a negative number, there is no solution.

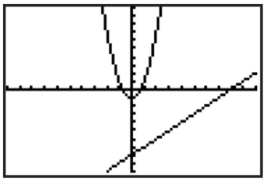
Step 4
Verify the system of equations has no solution using a graphing calculator.

Rewrite equation 1 in the form $y=ax^2+bx+c$ and equation 2 in the form $y=mx+b$.

$2x^2=y+1$	$x=8+y$
$2x^2-1=y$	$x-8=y$

Press $\boxed{\text{Y =}}$, and input each function.
$$Y_1=2x^2-1$$
$$Y_2=x-8$$

Press $\boxed{\text{ZOOM}}$, and select 6:ZStandard to obtain this window.



The graphs do not intersect each other.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Subtract equation 2 from equation 1 to eliminate the *y*-variable.

$$\begin{array}{lr} ① & x-4=x^2-y \\ ② & 5=2x-y \\ \hline & 4x-9=x^2-2x \end{array}$$

Step 2
Solve for *x*.

$$\begin{aligned} 4x-9&=x^2-2x \\ 0&=x^2-6x+9 \\ 0&=(x-3)^2 \\ 0&=x-3 \\ 3&=x \end{aligned}$$

Step 3
Substitute the solved value of *x* into equation 2, and solve for *y*.

$$\begin{aligned} 5&=2x-y \\ 5&=2(3)-y \\ 5&=6-y \\ y&=1 \end{aligned}$$

The system of equations has exactly one solution: {(3, 1)}.

Step 4
Verify the system of equations has one solution using a graphing calculator.

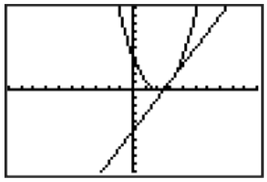
Rewrite equation 1 in the form $y=ax^2+bx+c$ and equation 2 in the form $y=mx+b$.

$$\begin{aligned} 4x-4&=x^2-y \\ y&=x^2-4x+4 \end{aligned} \qquad \begin{aligned} 5&=2x-y \\ y&=2x-5 \end{aligned}$$

Press $\boxed{\text{Y =}}$, and input each function.

$Y_1=x^2-4x+4$
$Y_2=2x-5$

Press $\boxed{\text{ZOOM}}$, and select 6:ZStandard to obtain this window.



The graphs intersect at exactly one point.
Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 5:intersect.
The graphs intersect at (3, 1).

2. **Step 1**
Isolate one variable from equation 1 or 2.

In this case, equation 1 has the *y*-variable isolated.

Step 2
Substitute the expression representing the isolated variable from equation 1 into equation 2, and simplify.

Substitute x^2+3 for *y* into equation 2, and bring all terms to one side.

$$\begin{aligned} x^2&=5-y \\ x^2&=5-(x^2+3) \\ x^2&=5-x^2-3 \\ x^2&=-x^2+2 \\ 2x^2-2&=0 \end{aligned}$$

Step 3
Solve for *x*.

$$\begin{aligned} 2x^2-2&=0 \\ 2x^2&=2 \\ x^2&=1 \\ x&=\pm1 \end{aligned}$$

Step 4
Substitute the solved values of *x* into equation 1, and solve for *y*.

$$\begin{aligned} y&=x^2+3 \\ y&=(1)^2+3 \\ y&=4 \end{aligned} \qquad \begin{aligned} y&=x^2+3 \\ y&=(-1)^2+3 \\ y&=4 \end{aligned}$$

The system of equations has two solutions: {(1, 4), (–1, 4)}.

Step 5
Verify the system of equations has two solutions using a graphing calculator.

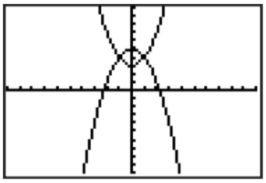
Rewrite equation 2 in the form $y = ax^2 + bx + c$.

$$x^2 = 5 - y$$
$$y = -x^2 + 5$$

Press $\boxed{\text{Y =}}$, and input each function.

$$Y_1 = x^2 + 3$$
$$Y_2 = -x^2 + 5$$

Press $\boxed{\text{ZOOM}}$, and select 6:ZStandard to obtain this window.



The graphs intersect at two points.
Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 5:intersect.
The graphs intersect at (1, 4) and (–1, 4).

3. **Step 1**
Isolate one variable from equation 1 or 2.

In this case, equation 1 already has the *y*-variable isolated.

Step 2
Substitute the expression representing the isolated variable from equation 1 into equation 2, and simplify.

Substitute $(x-3)^2$ for *y* in equation 2, and bring all terms to one side.

$$x^2 = y + 3$$
$$x^2 = (x-3)^2 + 3$$
$$x^2 = x^2 - 6x + 9 + 3$$
$$0 = -6x + 12$$

Step 3
Solve for *x*.

$$0 = -6x + 12$$
$$6x = 12$$
$$x = 2$$

Step 4
Substitute the solved value of *x* into equation 1, and solve for *y*.

$$y = (x-3)^2$$
$$y = (2-3)^2$$
$$y = (-1)^2$$
$$y = 1$$

The system of equations has exactly one solution: {(2, 1)}.

Step 5
Verify the system of equations has one solution using a graphing calculator.

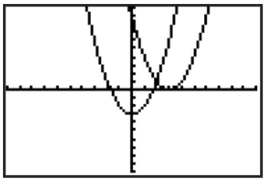
Rewrite equation 2 in the form $y = ax^2 + bx + c$.

$$x^2 = y + 3$$
$$x^2 - 3 = y$$

Press $\boxed{\text{Y =}}$, and input each function.

$$Y_1 = (x-3)^2$$
$$Y_2 = x^2 - 3$$

Press $\boxed{\text{ZOOM}}$, and select 6:ZStandard to obtain this window.



The graphs intersect at exactly one point.
Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 5:intersect.
It can be found that the graphs intersect at (2, 1).

4. **Step 1**
Isolate one variable from equation 1 or 2.

In this case, equation 1 has the *y*-variable isolated.

Step 2
Substitute the expression representing the isolated variable from equation 1 into equation 2, and simplify.

Substitute $7x^2 + 9x + 6$ for *y* in equation 2.

$$3(3x+2) = y - 7x^2$$
$$3(3x+2) = (7x^2 + 9x + 6) - 7x^2$$
$$9x + 6 = 9x + 6$$
$$0 = 0$$

Since 0 = 0 is a true statement for every value of *x*, there are an infinite number of solutions for all values of *x* and *y* in this system of equations.

5. Step 1
Isolate one variable from equation 1 or 2.

In this case, equation 1 already has the *y*-variable isolated.

Step 2
Substitute the expression representing the isolated variable from equation 1 into equation 2, and simplify.

Substitute $-x+4$ for *y* in equation 2, and bring all terms to one side.

$$\begin{aligned} y+1&=-(x-3)^2 \\ (-x+4)+1&=-(x-3)^2 \\ -x+5&=-(x^2-6x+9) \\ -x+5&=-x^2+6x-9 \\ 0&=-x^2+7x-14 \end{aligned}$$

Step 3
Solve for *x* using the quadratic formula.

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$y=\frac{-(7)\pm\sqrt{(7)^2-4(-1)(-14)}}{2(-1)}$$
$$y=\frac{-7\pm\sqrt{49-56}}{-2}$$
$$y=\frac{7\pm\sqrt{-7}}{2}$$

Since you cannot take the square root of a negative number, there is no solution.

Practice Test

ANSWERS AND SOLUTIONS

1. Step 1
Isolate one variable from equation 1 or 2.

In this case, equation 1 has the *y*-variable isolated.

Step 2
Substitute the expression represented by the isolated variable from equation 1 into equation 2, and simplify.

Substitute x^2-9x for *y* in equation 2, and bring all terms to one side.

$$\begin{aligned} y+x&=9 \\ (x^2-9x)+x&=9 \\ x^2-9x+x&=9 \\ x^2-8x&=9 \\ x^2-8x-9&=0 \end{aligned}$$

Step 3
Solve for *x* by factoring the quadratic equation.

$$x^2-8x-9=0$$
$$(x+1)(x-9)=0$$
$$x+1=0 \qquad x-9=0$$
$$x=-1 \qquad x=9$$

Step 4
Substitute the solved values of *x* into equation 2, and solve for *y*.

$$\begin{aligned} y+x&=9 \\ y+(-1)&=9 \\ y-1&=9 \\ y&=10 \end{aligned} \qquad \begin{aligned} y+x&=9 \\ y+9&=9 \\ y&=0 \end{aligned}$$

The solution set is {(–1, 10), (9, 0)}.

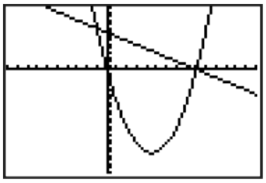
Step 5
Graph the functions represented by the given equations using a graphing calculator.

Rewrite equation 2 in the form $y=mx+b$.

$$\begin{aligned} y+x&=9 \\ y&=-x+9 \end{aligned}$$

Press $\boxed{\text{Y =}}$, and input each function.
$Y_1=x^2-9x$
$Y_2=-x+9$

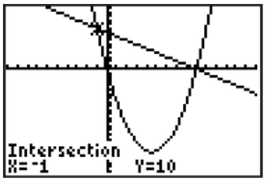
Press $\boxed{\text{GRAPH}}$. The window setting used to display the two graphs is $x:[-10,15,1]$ and $y:[-25,15,1]$.



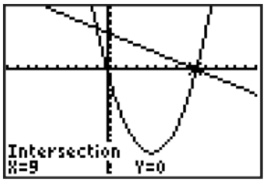
Step 6
Find the points of intersection.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and choose 5:intersect.

Determine the intersection point that is farthest to the left.



Repeat the process to find the intersection point that is farthest to the right.



The calculator verifies that the points of intersection of the curve and the line are at (–1, 10) and (9, 0).

2. Step 1
Subtract equation 2 from equation 1 to eliminate the *y*-variable.

$$\begin{array}{lr} ① & -x^2 = 15x - 160 \\ ② & y - 33x = -2x^2 + 5 \\ \hline & -x^2 + 33x = 15x + 2x^2 - 165 \end{array}$$

Step 2
Solve for *x*.

$$\begin{aligned} -x^2 + 33x &= 15x + 2x^2 - 165 \\ 0 &= 3x^2 - 18x - 165 \\ 0 &= 3\left(x^2 - 6x - 55\right) \\ 0 &= 3\left(x - 11\right)\left(x + 5\right) \\ 0 &= \left(x - 11\right)\left(x + 5\right) \end{aligned}$$

$$\begin{array}{ll} x - 11 = 0 & x + 5 = 0 \\ x = 11 & x = -5 \end{array}$$

Step 3
Substitute the solved values of *x* into equation 1, and solve for *y*.

$$\begin{aligned} y - x^2 &= 15x - 160 \\ y - (11)^2 &= 15(11) - 160 \\ y - 121 &= 165 - 160 \\ y &= 165 - 160 + 121 \\ y &= 126 \end{aligned}$$

$$\begin{aligned} y - x^2 &= 15x - 160 \\ y - (-5)^2 &= 15(-5) - 160 \\ y - 25 &= -75 - 160 \\ y &= -75 - 160 + 25 \\ y &= -210 \end{aligned}$$

Therefore, the solution set is {(11, 126), (–5, –210)}.

Step 4
Graph the functions represented by the given equations using a graphing calculator.

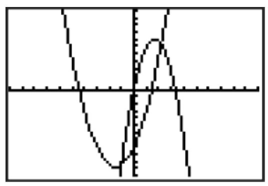
Rewrite equation 1 and equation 2 in the form $y = ax^2 + bx + c$.

$$\begin{array}{ll} y - x^2 = 15x - 160 & y - 33x = -2x^2 + 5 \\ y = x^2 + 15x - 160 & y = -2x^2 + 33x + 5 \end{array}$$

Press $\boxed{\text{Y =}}$, and input each function.

$Y_1 = x^2 + 15x - 160$
$Y_2 = -2x^2 + 33x + 5$

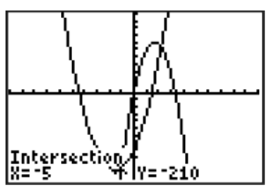
Press $\boxed{\text{GRAPH}}$. The window setting used to display the two graphs is $x:[-50, 50, 5]$ and $y:[-240, 220, 20]$.



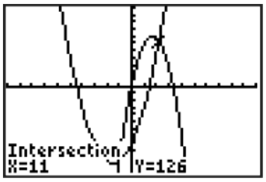
Step 5
Find the points of intersection.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and choose 5:intersect.

Determine the intersection point that is farthest to the left.



Repeat the process to find the intersection point that is farthest to the right.



The calculator verifies that the points of intersection of the two curves are at (11, 126) and (–5, –210).

3. Step 1
Multiply equation 1 by 2.

$$①\times 2 \quad 2\left(y = x^2 - x - 3\right) \rightarrow 2y = 2x^2 - 2x - 6$$

Let $2y = 2x^2 - 2x - 6$ represent equation 3.

Step 2
Subtract equation 3 from equation 2 to eliminate the *y*-variable.

$$\begin{array}{ll} ② & y = 10x + 26 \\ ③ & \underline{2y = 2x^2 - 2x - 6} \\ & 0 = -2x^2 + 12x + 32 \end{array}$$

Step 3
Solve for *x*.

$$0 = -2x^2 + 12x + 32$$
$$0 = -2\left(x^2 - 6x - 16\right)$$
$$0 = -2\left(x - 8\right)\left(x + 2\right)$$
$$0 = \left(x - 8\right)\left(x + 2\right)$$
$$\begin{array}{ll} x - 8 = 0 & x + 2 = 0 \\ x = 8 & x = -2 \end{array}$$

Step 4
Substitute the solved values for *x* into equation 2, and solve for *y*.

$$\begin{array}{ll} 2y = 10x + 26 & 2y = 10x + 26 \\ 2y = 10\left(8\right) + 26 & 2y = 10\left(-2\right) + 26 \\ 2y = 80 + 26 & 2y = -20 + 26 \\ 2y = 106 & 2y = 6 \\ y = 53 & y = 3 \end{array}$$

The solution set is {(–2, 3), (8, 53)}.

Step 5
Graph the functions represented by the given equations using a graphing calculator.

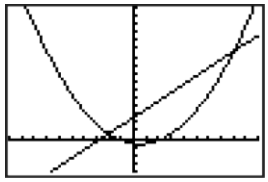
Rewrite equation 2 in the form $y = mx + b$.

$$2y = 10x + 26$$
$$y = 5x + 13$$

Press $\boxed{\text{Y =}}$, and input each function.

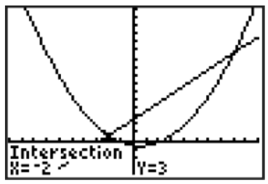
$$Y_1 = x^2 - x - 3$$
$$Y_2 = 5x + 13$$

Press $\boxed{\text{GRAPH}}$. The window setting used to display the two graphs is $x: \left[-10, 10, 1\right]$ and $y: \left[-20, 80, 5\right]$.

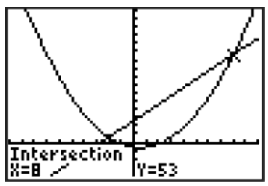


Step 6
Find the points of intersection.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and choose 5:intersect. Determine the intersection point that is farthest to the left.



Repeat the process to find the intersection point that is farthest to the right.



The calculator verifies that the points of intersection of the curve and the line are at (–2, 3) and (8, 53).

4. Step 1
Multiply equation 2 by 2.

$$②\times 2 \quad 2\left(8x - y = -3x^2 - 1\right) \rightarrow 16x - 2y = -6x^2 - 2$$

Let $16x - 2y = -6x^2 - 2$ represent equation 3.

Step 2
Subtract equation 3 from equation 1 to eliminate the y-variable.

$$\begin{array}{rl} ① & 6x^2-2y=-16x-2 \\ ③ & 16x-2y=-6x^2-2 \\ \hline & 6x^2-16x=-16x+6x^2 \end{array}$$

Step 3
Solve for x.

$$\begin{aligned} 6x^2-16x&=-16x+6x^2 \\ 0&=0 \end{aligned}$$

Since $0=0$ is a true statement for every value of x, there are an infinite number of solutions for all values of x and y in this system of equations.

5. Step 1
Isolate one variable from equation 1 or 2. In this case, equation 1 already has the y-variable isolated.

Step 2
Substitute the expression represented by the isolated variable from equation 1 into equation 2, and simplify.

Substitute $25x+47$ for y in equation 2, and bring all terms to one side.

$$\begin{aligned} y+3x^2&=x-1 \\ (25x+47)+3x^2&=x-1 \\ 3x^2+24x+48&=0 \end{aligned}$$

Step 3
Solve for x.

$$\begin{aligned} 3x^2+24x+48&=0 \\ 3(x^2+8x+16)&=0 \\ 3(x+4)(x+4)&=0 \\ 3(x+4)^2&=0 \\ (x+4)^2&=0 \\ x+4&=0 \\ x&=-4 \end{aligned}$$

Step 4
Substitute the solved value of x into equation 1, and solve for y.

$$\begin{aligned} y&=25x+47 \\ y&=25(-4)+47 \\ y&=-100+47 \\ y&=-53 \end{aligned}$$

The system of equations has exactly one solution: {(–4, –53)}.

6. Step 1
Subtract equation 2 from equation 1 to eliminate the y-variable.

$$\begin{array}{rl} ① & -17=3x^2+2x \\ ② & y+2x^2=9 \\ \hline & -17-2x^2=3x^2+2x-9 \end{array}$$

Step 2
Solve for x.

$$\begin{aligned} -17-2x^2&=3x^2+2x-9 \\ 0&=5x^2+2x+8 \end{aligned}$$

Use the quadratic formula to solve the equation $0=5x^2+2x+8$.

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$y=\frac{-(2)\pm\sqrt{(2)^2-4(5)(8)}}{2(5)}$$
$$y=\frac{-2\pm\sqrt{4-160}}{10}$$
$$y=\frac{-2\pm\sqrt{-156}}{10}$$

Since you cannot take the square root of a negative number, there is no solution.

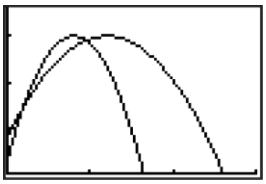
7. The intersection points of the graphs of the functions correspond to the times at which the balls reach the same height.

Graph the equations using a graphing calculator.

Step 1
Press $\boxed{\text{Y =}}$, and input each function.

$Y_1=-7.6(x-1.2)^2+15$
$Y_2=-21.1(x-0.8)^2+15$

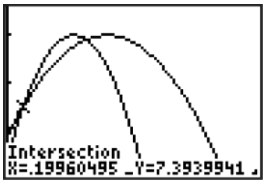
Press $\boxed{\text{GRAPH}}$. The window setting used to display the two graphs is $x:[0,3,1]$ and $y:[0,18,5]$.



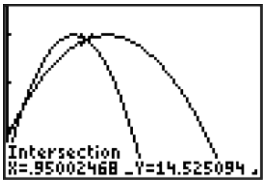
Step 2
Find the points of intersection.

Press [2nd] [TRACE], and choose 5:intersect.

Determine the intersection point that is farthest to the left.



Repeat the process to find the intersection point that is farthest to the right.



The first point of intersection is at approximately (0.20, 7.39), and the second point of intersection is at approximately (0.95, 14.53). The x-coordinate of each intersection point represents the time, t, and the y-coordinate represents the corresponding height, h.

Therefore, the balls reach the same height after 0.20 s and again after 0.95 s.

8. Step 1
Set up a system of equations.

If the car travelled 45 m after 1 s, then substitute 45 for D and 1 for t.

$$D = at^2 + b^2t$$
$$45 = a(1)^2 + b^2(1)$$
$$45 = a + b^2$$

If the car travelled 104 m after 2 s, then substitute 104 for D and 2 for t.

$$D = at^2 + b^2t$$
$$104 = a(2)^2 + b^2(2)$$
$$104 = 4a + 2b^2$$

Therefore, this is the system of equations:

① [redacted] $= a + b^2$

② $104 = 4a + 2b^2$

Step 2
Solve the system of equations by elimination.

Multiply equation 1 by 2.

①×2 $2\left(45 = a + b^2\right) \rightarrow 90 = 2a + 2b^2$

Let $90 = 2a + 2b^2$ represent equation 3.

Step 3
Subtract equation 2 from equation 3 to eliminate $2b^2$.

③ $\quad = 2a + 2b^2$

② $\underline{104 = 4a + 2b^2}$

$-14 = -2a$

Step 4
Solve for a.

$$-14 = -2a$$
$$7 = a$$

Step 5
Substitute the solved value of a into equation 1, and solve for b.

$$45 = a + b^2$$
$$45 = 7 + b^2$$
$$38 = b^2$$
$$b = \pm\sqrt{38}$$

Since b is positive, the value of a is 7 and the value of b is $\sqrt{38}$.

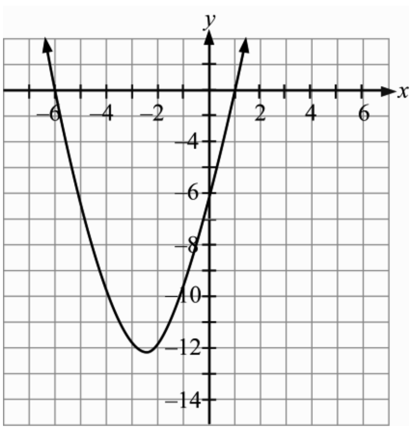
LINEAR AND QUADRATIC INEQUALITIES

Lesson 1—Solving Quadratic Inequalities in One Variable Graphically

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Step 1
Sketch the graph of the corresponding quadratic function.

The quadratic function $f(x)=x^2+5x-6$ is a parabola that opens up.



Step 2
Determine the boundary values.

From the graph of the function, the *x*-intercepts are (–6, 0) and (1, 0).

Therefore, the boundary values are –6 and 1.

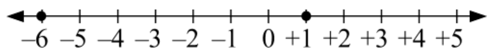
Step 3
Determine the solution to the inequality.

The solutions to the inequality $x^2+5x-6\le 0$ are the *x*-coordinates where $f(x)\le 0$ (below the *x*-axis and equal to 0) for the graph of $f(x)=x^2+5x-6$.

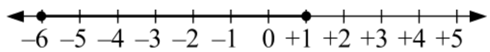
From the graph, the solution is $-6\le x\le 1$.

Step 4
Graph the solution on a number line.

Place the boundary values on the number line using solid dots since these values are included in the inequality.



For the solution $-6\le x\le 1$, shade between the boundary values.



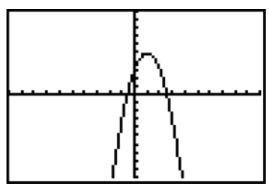
2. Step 1
Write the quadratic inequality in $ax^2+bx+c\ge 0$ form.

Eliminate the denominator by multiplying both sides of the inequality by 3, and rewrite with all the terms on the left side of the inequality.

$$-\frac{2}{3}x^2\ge-\frac{4}{3}x-1$$
$$-2x^2\ge-4x-3$$
$$-2x^2+4x+3\ge 0$$

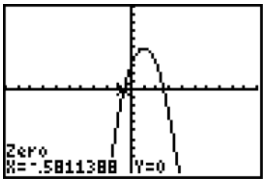
Step 2
Graph the function using a graphing calculator. Press $\boxed{\text{Y=}}$, and enter the function as $Y_1=-2x^2+4x+3$.

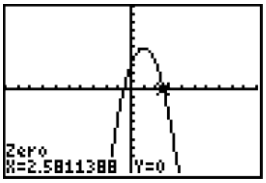
Press $\boxed{\text{ZOOM}}$, and select 6:ZStandard to obtain this screen.



Step 3
Determine the x-intercepts of the graph.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 2:zero. Move the cursor to the left of the first x-intercept, and press $\boxed{\text{ENTER}}$. Next, move the cursor to the right of the same x-intercept, and press $\boxed{\text{ENTER}}$ twice. Repeat to find the other x-intercept.





To the nearest tenth, the x-intercepts are $x \doteq -$[redacted] and $x \doteq 2.6$.

Step 4
Determine the solution to the inequality.

The solution to the inequality $-\frac{2}{3}x^2 \geq -\frac{4}{3}x - 1$ will be the x-coordinates, where $f(x) \geq 0$ (above the x-axis and equal to zero), for the graph of $f(x) = -2x^2 + 4x + 3$.

From the graph, the solution is approximately $0.6 \leq x \leq 2.6$.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

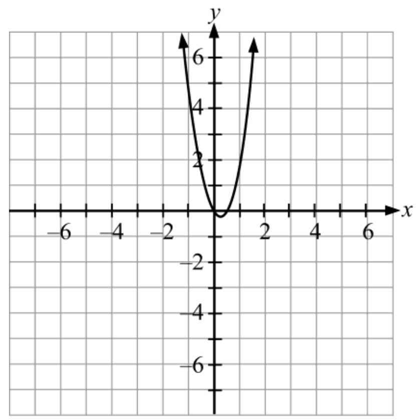
1. Step 1
Write the quadratic inequality in $ax^2 + bx + c < 0$ form.

Rewrite with all the terms on the left side of the inequality.

$$4x^2 < 2x$$
$$4x^2 - 2x < 0$$

Step 2
Sketch the graph of the corresponding quadratic function.

The quadratic function $f(x) = 4x^2 - 2x$ is a parabola that opens up.



Step 3
Determine the boundary values.

From the graph of the function, the x-intercepts are (0, 0) and (0.5, 0).

Therefore, the boundary values are 0 and 0.5.

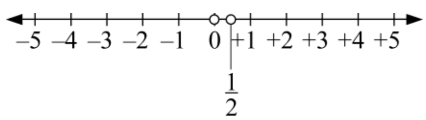
Step 4
Determine the solution to the inequality.

The solutions to the inequality $4x^2 < 2x$ will be the x-coordinates, where $f(x) < 0$ (below the x-axis), for the graph of $f(x) = 4x^2 - 2x$.

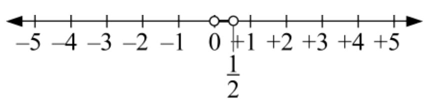
From the graph, the solution is $0 < x < 0.5$.

Step 5
Graph the solution on a number line.

Place the boundary values on the number line using open dots since these values are not included in the inequality.



For the solution $0 < x < 0.5$, shade between the boundary values.



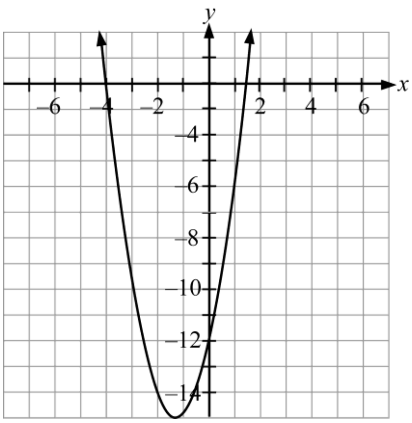
2. Step 1
Write the quadratic inequality in $ax^2+bx+c\geq 0$ form.

Rewrite with all the terms on the left side of the inequality.

$$2x^2+5x\geq 12$$
$$2x^2+5x-12\geq 0$$

Step 2
Sketch the graph of the corresponding quadratic function.

The quadratic function $f(x)=2x^2+5x-12$ is a parabola that opens up.



Step 3
Determine the boundary values.

From the graph of the function, the x-intercepts are (–4, 0) and (1.5, 0).

Therefore, the boundary values are –4 and 1.5.

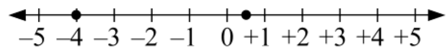
Step 4
Determine the solution set to the inequality.

The solution to the inequality $2x^2+5x\geq 12$ will be the x-coordinates, where $f(x)\geq 0$ (above the x-axis and equal to zero), for the graph of $f(x)=2x^2+5x-12$.

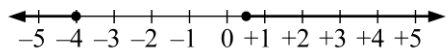
From the graph, the solutions are $x\leq -4$ and $x\geq 1.5$.

Step 5
Graph the solution set on a number line.

Place the boundary values on the number line using solid dots since these values are included in the inequality.



For the solution $x\leq -4$, shade to the left of the boundary value, and for the solution $x\geq 1.5$, shade to the right of the boundary value.



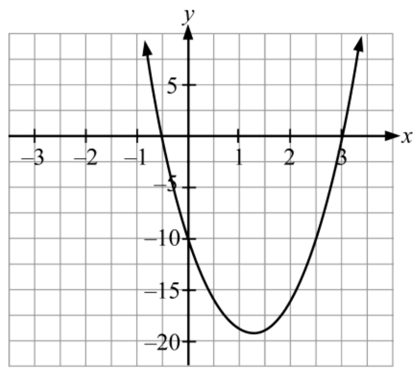
3. Step 1
Write the quadratic inequality in $ax^2+bx+c<0$ form.

Rewrite with all the terms on the left side of the inequality.

$$6x^2<15x+9$$
$$6x^2-15x-9<0$$

Step 2
Sketch the graph of the corresponding quadratic function.

The quadratic function $f(x)=6x^2-15x-9$ is a parabola that opens up.



Step 3
Determine the boundary values.

From the graph of the function, the x-intercepts are $\left(-\frac{1}{2},0\right)$ and (3, 0).

Therefore, the boundary values are $-\frac{1}{2}$ and 3.

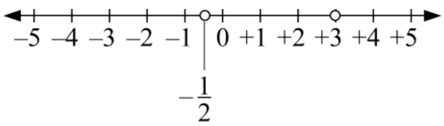
Step 4
Determine the solution set to the inequality.

The solution to the inequality $6x^2 < 15x + 9$ will be the x-coordinates, where $f(x) < 0$ (below the x-axis), for the graph of $f(x) = 6x^2 - 15x - 9$.

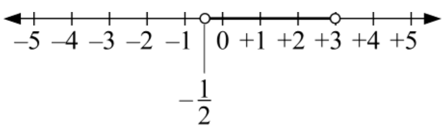
From the graph, the solution is $-\frac{1}{2} < x < 3$.

Step 5
Graph the solution set on a number line.

Place the boundary values on a number line using open dots since these values are not included in the inequality.



For the solution $-\frac{1}{2} < x < 3$, shade between the boundary values.

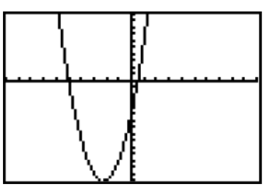


4. Step 1
Graph the function using a graphing calculator.

Press $\boxed{\text{Y=}}$, and enter the function as $Y_1 = 2x^2 + 9x - 5$.

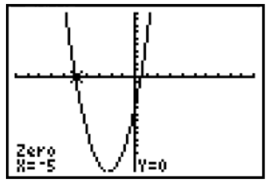
Press $\boxed{\text{WINDOW}}$, and enter a window setting of $x:[-10,10,1]$ and $y:[-15,10,1]$.

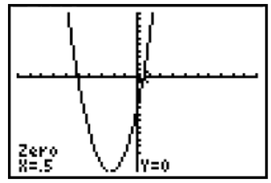
Press $\boxed{\text{GRAPH}}$ to obtain this screen.



Step 2
Determine the x-intercepts of the graph.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 2:zero. Move the cursor to the left of the first x-intercept, and press $\boxed{\text{ENTER}}$. Next, move the cursor to the right of the same x-intercept, and press $\boxed{\text{ENTER}}$ twice. Repeat to find the other x-intercept.





The x-intercepts are $x = –5$ and $x = 0.5$.

Step 3
Determine the solution set to the inequality $2x^2 + 9x - 5 \le 0$.

The solution to the inequality $2x^2 + 9x - 5 \le 0$ will be the x-coordinates, where $f(x) \le 0$ (below the x-axis and zero), for the graph of $f(x) = 2x^2 + 9x - 5$.

From the graph, the solution, to the nearest tenth, is approximately $-5.0 \le x \le 0.5$.

5. Step 1
Write the quadratic inequality in $ax^2 + bx + c > 0$ form.

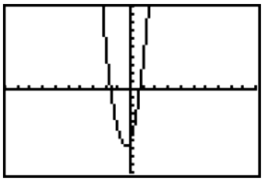
Eliminate the denominator by multiplying both sides of the inequality by 2, and rewrite with all the terms on the left side of the inequality.

$$2x - 3 > -\frac{5}{2}x^2$$
$$4x - 6 > -5x^2$$
$$5x^2 + 4x - 6 > 0$$

Step 2
Graph the function $f(x) = 5x^2 + 4x - 6$ using a graphing calculator.

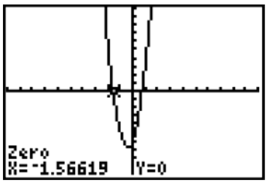
Press $\boxed{\text{Y=}}$, and enter the function as $Y_1 = 5x^2 + 4x - 6$.

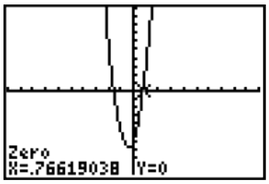
Press $\boxed{\text{ZOOM}}$, and select 6:ZStandard to obtain this screen.



Step 3
Determine the x-intercepts of the graph.

Press $\boxed{\text{2nd}}$ $\boxed{\text{CALC}}$, and select 2:zero. Move the cursor to the left of the first x-intercept, and press $\boxed{\text{ENTER}}$. Next, move the cursor to the right of the same x-intercept, and press $\boxed{\text{ENTER}}$ twice. Repeat to find the other x-intercept.





To the nearest tenth, the x-intercepts are $x \doteq -$[redacted] and $x \doteq 0.8$.

Step 4
Determine the solution set to the inequality.

The solution to the inequality $2x-3>-\frac{5}{2}x^2$ will be the x-coordinates, where $f(x)>0$ (above the x-axis), for the graph of $f(x)=5x^2+4x-6$.

From the graph, the solutions are approximately $x<-1.6$ and $x>0.8$.

6. **Step 1**
Write the quadratic inequality in $ax^2+bx+c\le 0$ form.

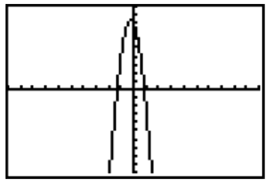
Rewrite with all the terms on the left side of the inequality.

$$-7x^2 \le 3x-8$$
$$-7x^2-3x+8\le 0$$

Step 2
Graph the function using a graphing calculator.

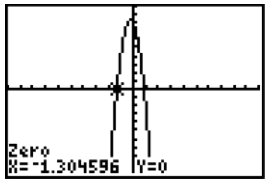
Press $\boxed{\text{Y=}}$, and enter the function as $Y_1=-7x^2-3x+8$.

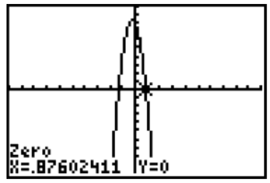
Press $\boxed{\text{ZOOM}}$, and select 6:ZStandard to obtain this window.



Step 3
Determine the x-intercepts of the graph.

Press $\boxed{\text{2nd}}$ $\boxed{\text{CALC}}$, and select 2:zero. Move the cursor to the left of the first x-intercept, and press $\boxed{\text{ENTER}}$. Next, move the cursor to the right of the same x-intercept, and press $\boxed{\text{ENTER}}$ twice. Repeat to find the other x-intercept.





To the nearest tenth, the x-intercepts are $x \doteq -$[redacted] and $x \doteq 0.9$.

Step 4
Determine the solution set to the inequality.

The solution set to the inequality $-7x^2\le 3x-8$ will be the x-coordinates, where $f(x)\le 0$ (below the x-axis and equal to zero), for the graph of $f(x)=-7x^2-3x+8$.

From the graph, the solutions are approximately $x\le -1.3$ and $x\ge 0.9$.

7. **Step 1**
Model the given information with an inequality.

Since the profit must be at least $1 450, an inequality statement that represents the information in the given problem is $-6x^2+240x-350\ge 1450$.

Step 2
Write the quadratic inequality in $ax^2+bx+c>0$ form.

Subtract 1 450 from each side of the inequality $-6x^2+240x-350\ge 1450$.

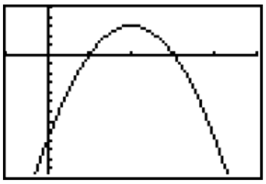
$$-6x^2+240x-350\geq 1450$$
$$-6x^2+240x-350-1450\geq 0$$
$$-6x^2+240x-1800\geq 0$$

Step 3
Graph the function using a graphing calculator.

Press $\boxed{\text{Y=}}$, and enter the function as $Y_1=-6x^2+240x-1800$.

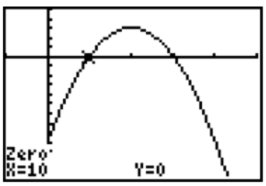
Press $\boxed{\text{WINDOW}}$, and enter a window setting of $x:[-10,50,10]$ and $y:[-2500,1000,200]$.

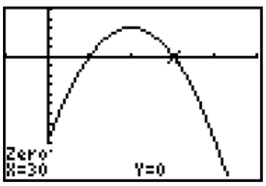
Press $\boxed{\text{GRAPH}}$ to obtain this screen.



Step 4
Determine the *x*-intercepts of the graph.

Press $\boxed{\text{2nd}}$ $\boxed{\text{CALC}}$, and select 2:zero. Move the cursor to the left of the first *x*-intercept, and press $\boxed{\text{ENTER}}$. Next, move the cursor to the right of the same *x*-intercept, and press $\boxed{\text{ENTER}}$ twice. Repeat to find the other *x*-intercept.





The *x*-intercepts of the function are 10 and 30.

Step 5
Determine the solution set to the inequality.

The solution set to the inequality $-6x^2+240x-350\geq 1450$ will be the *x*-coordinates, where $f(x)\geq 0$ (above the *x*-axis and zero), for the graph of $f(x)=-6x^2+240x-1800$.

In order for Naya to make a profit of at least \$1 450, the selling price per necklace must be between \$10 and \$30 inclusive.

Lesson 2—Solving Quadratic Inequalities in One Variable without a Graph

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Factor the quadratic inequality.

$$3x^2-14x\leq 5$$
$$3x^2-14x-5\leq 0$$
$$(3x+1)(x-5)\leq 0$$

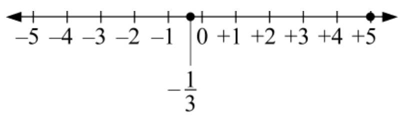
Step 2
Determine the boundary values.

$$3x+1=0 \qquad x-5=0$$
$$3x=-1 \qquad x=5$$
$$x=-\frac{1}{3}$$

The boundary values are $x=-\frac{1}{3}$ and $x=5$.

Step 3
Place the boundary values on a number line, and identify the possible solution regions.

Solid dots are used because $x=-\frac{1}{3}$ and $x = 5$ are included in the inequality.



There are three regions on the number line: $x<-\frac{1}{3}$, $-\frac{1}{3}<x<5$, and $x>5$.

Step 4
Use test points to determine the solution regions on the number line that satisfy the inequality.

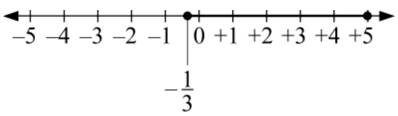
Possible test points for the inequality $(3x+1)(x-5) \le 0$ are −1, 0, and 10.

Region	Test Point	$(3x+1)(x-5)$	Sign
$x < -\frac{1}{3}$	−1	$(3(-1)+1)(-1-5)$ $=(-2)(-6)$ $=12$	+
$-\frac{1}{3} < x < 5$	0	$(3(0)+1)(0-5)$ $=(1)(-5)$ $=-5$	–
$x > 5$	10	$(3(10)+1)(10-5)$ $=(31)(5)$ $=155$	+

Step 5
Identify the solution region.

Since $(3x+1)(x-5) \le 0$ is required, the solution consists of the values in the regions where the product is zero and negative. Thus, the solution is $-\frac{1}{3} \le x \le 5$.

Step 6
Graph the solution on the number line.



PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Factor the quadratic inequality.

$$3x^2 - 2x - 8 \le 0$$
$$(3x+4)(x-2) \le 0$$

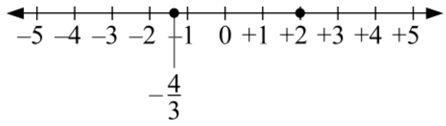
Step 2
Determine the boundary values.

$3x+4=0$ $x-2=0$
$3x=-4$ $x=2$
$x=-\frac{4}{3}$

The boundary values are $x = -\frac{4}{3}$ and $x = 2$.

Step 3
Place the boundary values on a number line, and identify the possible solution regions.

Solid dots are used because $x = -\frac{4}{3}$ and $x = 2$ are included in the inequality.



There are three regions on the number line: $x < -\frac{4}{3}$, $-\frac{4}{3} < x < 2$, and $x > 2$.

Step 4
Use test points to determine the solution regions on the number line that satisfy the inequality.

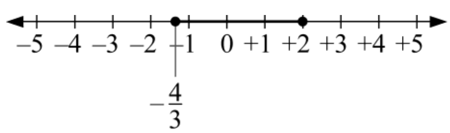
Possible test points for the inequality $(3x+4)(x-2) \le 0$ are −2, 0, and 3.

Region	Test Point	$(3x+4)(x-2)$	Sign
$x < -\frac{4}{3}$	−2	$(3(-2)+4)(-2-2)$ $=(-2)(-4)$ $=8$	+
$-\frac{4}{3} < x < 2$	0	$(3(0)+4)(0-2)$ $=(4)(-2)$ $=-8$	–
$x > 2$	3	$(3(3)+4)(3-2)$ $=(13)(1)$ $=13$	+

Step 5
Identify the solution region.

Since $3x^2 - 2x - 8 \le 0$ is required, the solution consists of the values in the regions where the product is negative or zero. Thus, the solution is $-\frac{4}{3} \le x \le 2$.

Step 6
Graph the solution on the number line.



2. **Step 1**
Factor the quadratic inequality.

$$\begin{aligned} -x+6 &< 2x^2 \\ -2x^2-x+6 &< 0 \\ -\left(2x^2+x-6\right) &< 0 \\ -\left(2x-3\right)\left(x+2\right) &< 0 \\ \left(2x-3\right)\left(x+2\right) &> 0 \end{aligned}$$

Step 2
Determine the boundary values.

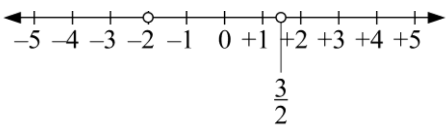
$$\begin{aligned} 2x-3 &= 0 \\ 2x &= 3 \\ x &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned} x+2 &= 0 \\ x &= -2 \end{aligned}$$

The boundary values are $x = –2$ and $x = \frac{3}{2}$.

Step 3
Place the boundary values on a number line, and identify the possible solution regions.

Open dots are used because $x = –2$ and $x = \frac{3}{2}$ are not included in the inequality.



There are three regions on the number line: $x < -2$, $-2 < x < \frac{3}{2}$, and $x > \frac{3}{2}$.

Step 4
Use test points to determine the solution regions on the number line that satisfy the inequality.

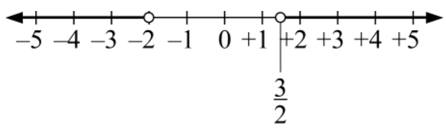
Possible test points for the inequality $\left(2x-3\right)\left(x+2\right) > 0$ are −3, 0, and 2.

Region	**Test Point**	$\mathbf{\left(2x-1\right)\left(x+2\right)}$	**Sign**
$x < -2$	-3	$\left(2(-3)-3\right)\left(-3+2\right)$ $=(-9)(-1)$ $=9$	+
$-2 < x < \frac{3}{2}$	0	$\left(2(0)-3\right)\left(0+2\right)$ $=(-3)(2)$ $=-6$	–
$x > \frac{3}{2}$	2	$\left(2(2)-3\right)\left(2+2\right)$ $=(1)(4)$ $=4$	+

Step 5
Identify the solution region.

Since $\left(2x-3\right)\left(x+2\right) > 0$ is required, the solution consists of the values in the regions where the product is positive. Thus, the solutions are $x < -2$ and $x > \frac{3}{2}$.

Step 6
Graph the solutions on the number line.



3. **Step 1**
Factor the quadratic inequality.

$$\begin{aligned} x^2+x-12 &> 18 \\ x^2+x-30 &> 0 \\ \left(x+6\right)\left(x-5\right) &> 0 \end{aligned}$$

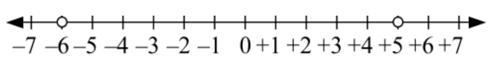
Step 2
Determine the boundary values.

$$\begin{aligned} x+6 &= 0 & \qquad x-5 &= 0 \\ x &= -6 & \qquad x &= 5 \end{aligned}$$

The boundary values are $x = –6$ and $x = 5$.

Step 3
Place the boundary values on a number line, and identify the possible solution regions.

Open dots are used because $x = –6$ and $x = 5$ are not included in the inequality.



There are three regions on the number line: $x < -6$, $-6 < x < 5$, and $x > 5$.

Step 4
Use test points to determine the solution regions on the number line that satisfy the inequality.

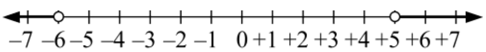
Possible test points for the inequality $(x+6)(x-5) > 0$ are −7, 0, and 6.

Region	Test Point	$(x+6)(x-5)$	Sign
$x < -6$	−7	$(-7+6)(-7-5)$ $=(-1)(-12)$ $=12$	+
$-6 < x < 5$	0	$(0+6)(0-5)$ $=(6)(-5)$ $=-30$	–
$x > 5$	6	$(6+6)(6-5)$ $=(12)(1)$ $=12$	+

Step 5
Identify the solution region.

Since $(x+6)(x-5) > 0$ is required, the solution consists of the values in the regions where the product is positive. Thus, the solutions are $x < -6$ and $x > 5$.

Step 6
Graph the solutions on the number line.



4. Step 1
Factor the quadratic inequality.

$$x^2 \geq -11x - 24$$
$$x^2 + 11x + 24 \geq 0$$
$$(x+3)(x+8) \geq 0$$

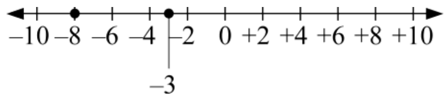
Step 2
Determine the boundary values.

$x+3=0$ $\quad$ $x+8=0$
$x=-3$ $\quad$ $x=-8$

The boundary values are $x = -8$ and $x = -3$.

Step 3
Place the boundary values on a number line, and identify the possible solution regions.

Solid dots are used because $x = -8$ and $x = -3$ are included in the inequality.



There are three regions on the number line: $x < -8$, $-8 < x < -3$, and $x > -3$.

Step 4
Use test points to determine the solution regions on the number line that satisfy the inequality.

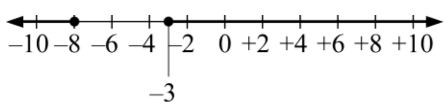
Possible test points for the inequality $(x+3)(x+8) \geq 0$ are −10, –5, and 0.

Region	Test Point	$(x+3)(x+8)$	Sign
$x < -8$	−10	$(-10+3)(-10+8)$ $=(-7)(-2)$ $=14$	+
$-8 < x < -3$	–5	$(-5+3)(-5+8)$ $=(-2)(3)$ $=-6$	–
$x > -3$	0	$(0+3)(0+8)$ $=(3)(8)$ $=24$	+

Step 5
Identify the solution region.

Since $(x+3)(x+8) \geq 0$ is required, the solution consists of the values in the regions where the product is positive or zero. Thus, the solutions are $x \leq -8$ and $x \geq -3$.

Step 6
Graph the solutions on the number line.



5. **Step 1**
Factor the quadratic inequality.

$$-\frac{1}{3}x^2-\frac{2}{3}x>-5$$
$$-\frac{1}{3}x^2-\frac{2}{3}x+5>0$$
$$-x^2-2x+15>0$$
$$-\left(x^2+2x-15\right)>0$$
$$-\left(x-3\right)\left(x+5\right)>0$$
$$\left(x-3\right)\left(x+5\right)<0$$

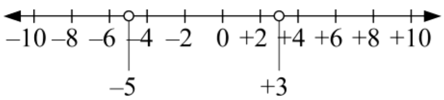
Step 2
Determine the boundary values.

$x-3=0$ $\quad$ $x+5=0$
$x=3$ $\quad$ $x=-5$

The boundary values are x = –5 and x = 3.

Step 3
Place the boundary values on a number line, and identify the possible solution regions.

Open dots are used because x = –5 and x = 3 are not included in the inequality.



There are three regions on the number line: $x<-5$, $-5<x<3$, and $x>3$.

Step 4
Use test points to determine the solution regions on the number line that satisfy the inequality.

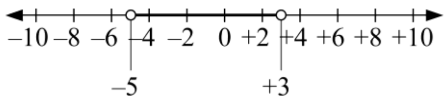
Possible test points for the inequality $\left(x-3\right)\left(x+5\right)<0$ are −10, 0, and 5.

Region	Test Point	$\left(x-3\right)\left(x+5\right)$	Sign
$x<-5$	−10	$\left(-10-3\right)\left(-10+5\right)$ $=\left(-13\right)\left(-5\right)$ $=65$	+
$-5<x<3$	0	$\left(0-3\right)\left(0+5\right)$ $=\left(-3\right)\left(5\right)$ $=-15$	–
$x>3$	5	$\left(5-3\right)\left(5+5\right)$ $=\left(2\right)\left(10\right)$ $=20$	+

Step 5
Identify the solution region.

Since $\left(x-3\right)\left(x+5\right)<0$ is required, the solution consists of the values in the regions where the product is negative. Thus, the solution is $-5<x<3$.

Step 6
Graph the solution on the number line.



6. **Step 1**
Determine the inequality that represents the problem.

Let x and x + 2 represent the integers that differ by 2.

Since the product of the integers is less than 15, the inequality is $x\left(x+2\right)<15$.

Step 2
Factor the quadratic inequality.

$$x\left(x+2\right)<15$$
$$x^2+2x<15$$
$$x^2+2x<15$$
$$x^2+2x-15<0$$
$$(x+5)(x-3)<0$$

Step 3
Determine the boundary values.

$x+5=0$ $\quad$ $x-3=0$
$x=-5$ $\quad$ $x=3$

The boundary values are x = –5 and x = 3.

Step 4
Use test points to determine the solution regions that satisfy the inequality.

There are three regions: $x<-5$, $-5<x<3$, and $x>3$.

Possible test points for the inequality $(x+5)(x-3)<0$ are −10, 0, and 10.

Region	Test Point	$(x+5)(x-3)$	Sign
$x < -5$	-10	$((-10)+5)(-10-3)$ $= (-5)(-13)$ $= 65$	+
$-5 < x < 3$	0	$(0+5)(0-3)$ $= (5)(-3)$ $= -15$	–
$x > 3$	10	$(10+5)(10-3)$ $= (15)(7)$ $= 105$	+

Step 5
Identify the solution region.

Since $(x+5)(x-3) < 0$ is required, the solution consists of the values in the regions where the product is negative. Thus, the solution is $-5 < x < 3$.

Step 6
Determine the possible pairs of integers.

Since the solution for x is any integer between –5 and 3, x can be –4, –3, –2, –1, 0, 1, and 2.

The possible pairs of integers represented by x and $x + 2$ are shown in this chart.

x	x **+ 2**
–4	–2
–3	–1
–2	0
–1	1
0	2
1	3
2	4

Note that the product of each pair is less than 15.

Lesson 3—Linear and Quadratic Inequalities in Two Variables

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Step 1
Isolate the y-variable.

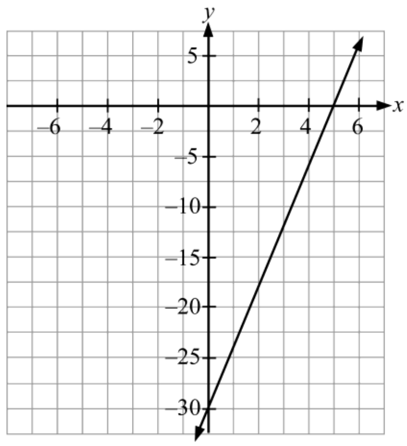
$$\frac{1}{2}y \geq 3x - 15$$
$$y \geq 6x - 30$$

Step 2
Determine whether the boundary line will be dotted or solid.

Since the inequality is ≥, the boundary line is part of the solution set.

The boundary line will be solid.

Step 3
Graph the corresponding linear equation that represents the inequality with the appropriate line.



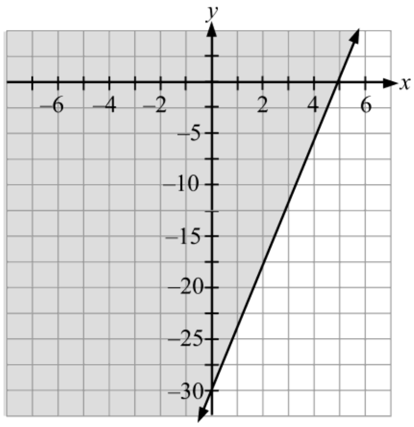
Step 4
Determine which side of the boundary line is to be shaded.

Substitute the test point (0, 0) into the inequality $\frac{1}{2}y \geq 3x - 15$.

$$\frac{1}{2}y \geq 3x - 15$$
$$\frac{1}{2}(0) \geq 3(0) - 15$$
$$0 \geq -15$$

The point (0, 0) satisfies the inequality, so the side that includes (0, 0) should be shaded.

Step 5
Shade the appropriate side of the boundary line.



The solution set to the inequality $\frac{1}{2}y \geq 3x - 15$ is the shaded region of the graph.

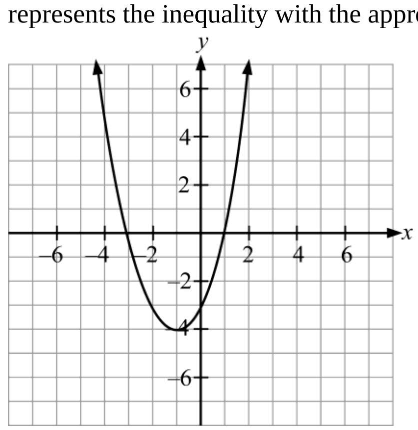
2. **Step 1**
Isolate the *y*-variable.

$$\begin{aligned} -x^2 + 3 &\geq 2x - y \\ y - x^2 + 3 &\geq 2x \\ y + 3 &\geq x^2 + 2x \\ y &\geq x^2 + 2x - 3 \end{aligned}$$

Step 2
Determine whether the boundary line will be dotted or solid.

Since the inequality is ≥, the boundary line is part of the solution set. The boundary line will be solid.

Step 3
Graph the corresponding quadratic equation that represents the inequality with the appropriate line.



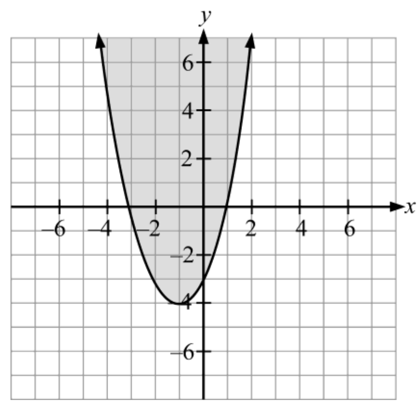
Step 4
Determine which side of the boundary line is to be shaded.

Substitute the test point (0, 0) into the inequality $-x^2 + 3 \geq 2x - y$.

$$\begin{aligned} -x^2 + 3 &\geq 2x - y \\ -(0)^2 + 3 &\geq 2(0) - (0) \\ 0 + 3 &\geq 0 - 0 \\ 3 &\geq 0 \end{aligned}$$

The point (0, 0) satisfies the inequality, so the side that includes (0, 0) should be shaded.

Step 5
Shade the appropriate side of the boundary line.



The solution set to the inequality $-x^2 + 3 \geq 2x - y$ is the shaded region of the graph.

3. **Step 1**
Determine the equation of the boundary line using points on the line.

From the vertex, the values of *p* and *q* are –4 and 5, respectively. The *y*-intercept is at (0, –3), so *x* = 0 and *y* = –3.

Substituting the known values into the equation $y = a(x - p)^2 + q$, find the value of *a*.

$$\begin{aligned} y &= a(x - p)^2 + q \\ -3 &= a(0 - (-4))^2 + 5 \\ -3 &= a(16) + 5 \\ -8 &= 16a \\ -\frac{1}{2} &= a \end{aligned}$$

The equation of the boundary line is
$y=-\frac{1}{2}(x+4)^2+5$.

Step 2
Determine the inequality sign.

Since the boundary line is solid, the inequality sign is ≤ or ≥.

Remove the equal sign in the equation $y=-\frac{1}{2}(x+4)^2+5$. Using a point from the shaded region, such as (–2, 1), evaluate both sides to determine the appropriate inequality sign.

$$y \ \square -\frac{1}{2}(x+4)^2+5$$
$$1 \ \square -\frac{1}{2}(-2+4)^2+5$$
$$1 \ \square -\frac{1}{2}(2)^2+5$$
$$1 \ \square -\frac{4}{2}+5$$
$$1 \ \square -2+5$$
$$1 \ \square \ 3$$

Since 1 is less than 3, the correct sign is ≤.

Therefore, the inequality is $y\le -\frac{1}{2}(x+4)^2+5$.

4. Step1
Isolate the *y*-variable.

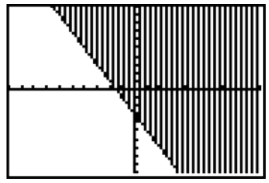
$$-6x-3y<9$$
$$-3y<6x+9$$
$$y>-2x-3$$

Step 2
Press $\boxed{\text{Y=}}$, and enter the equation of the boundary line.
$Y_1=-2x-3$

Step 3
Determine which side of the boundary line is to be shaded.

Move the cursor to the left of $Y_1=-2x-3$, where the dotted line flashes. Since the inequality symbol used is >, press $\boxed{\text{ENTER}}$ twice or until the image  appears.

Step 4
Press $\boxed{\text{ZOOM}}$, and select 6:ZStandard to obtain this screen.



The graph of the solution of $-6x-3y<9$ is displayed.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. Step 1
Isolate the *y*-variable.

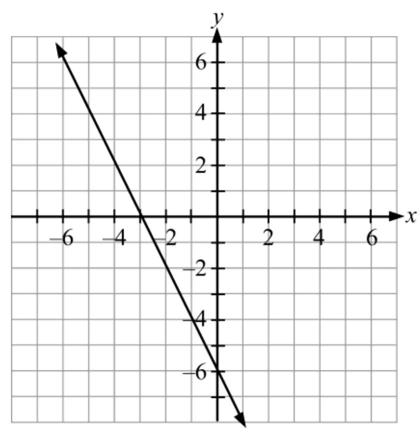
The *y*-variable is already isolated.

Step 2
Determine whether the boundary line will be dotted or solid.

Since the inequality is ≤, the boundary line is part of the solution set.

The boundary line will be solid.

Step 3
Graph the corresponding linear equation that represents the inequality with the appropriate line.



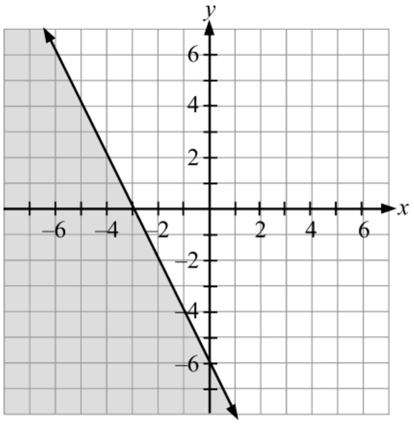
Step 4
Determine which side of the boundary line is to be shaded.

Substitute the test point (0, 0) into the inequality $y \le -2x - 6$.

$$\begin{aligned} y &\le -2x - 6 \\ 0 &\le -2(0) - 6 \\ 0 &\le 0 - 6 \\ 0 &\le -6 \end{aligned}$$

The point (0, 0) does not satisfy the inequality, so the side that includes (0, 0) should not be shaded.

Step 5
Shade the appropriate side of the boundary line.



The solution set to the inequality $y \le -2x - 6$ is the shaded region of the graph.

2. **Step 1**
Isolate the *y*-variable.

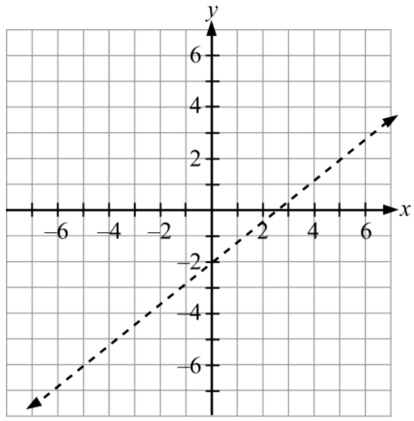
$$\begin{aligned} 3x - 4y - 8 &< 0 \\ -4y &< -3x + 8 \\ y &> \frac{3}{4}x - 2 \end{aligned}$$

Step 2
Determine whether the boundary line will be dotted or solid.

Since the inequality is >, the boundary line is not part of the solution set.

The boundary line will be dotted.

Step 3
Graph the corresponding linear equation that represents the inequality with the appropriate line.



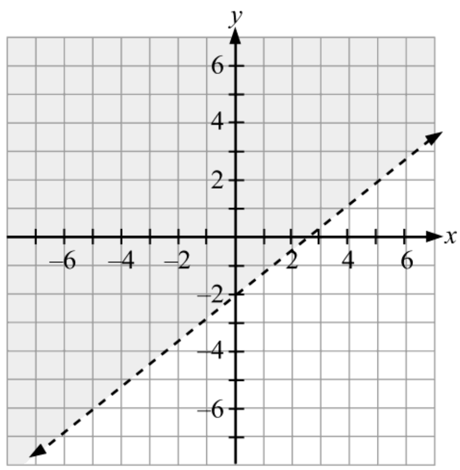
Step 4
Determine which side of the boundary line is to be shaded.

Substitute the test point (0, 0) into the inequality $3x - 4y - 8 < 0$.

$$\begin{aligned} 3x - 4y - 8 &< 0 \\ 3(0) - 4(0) - 8 &< 0 \\ 0 - 0 - 8 &< 0 \\ -8 &< 0 \end{aligned}$$

The point (0, 0) satisfies the inequality, so the side that includes (0, 0) should be shaded.

Step 5
Shade the appropriate side of the boundary line.



The solution set to the inequality $3x - 4y - 8 < 0$ is the shaded region of the graph.

3. Step 1
Isolate the y-variable.

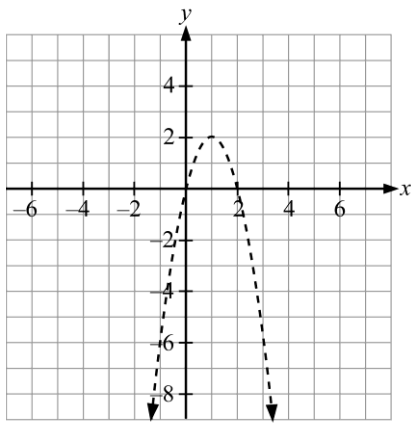
$$-\frac{y}{2} > x^2 - 2x$$
$$-y > 2x^2 - 4x$$
$$y < -2x^2 + 4x$$

Step 2
Determine whether the boundary line will be dotted or solid.

Since the inequality is <, the boundary line is not part of the solution set.

The boundary line will be dotted.

Step 3
Graph the corresponding quadratic equation that represents the inequality with the appropriate line.



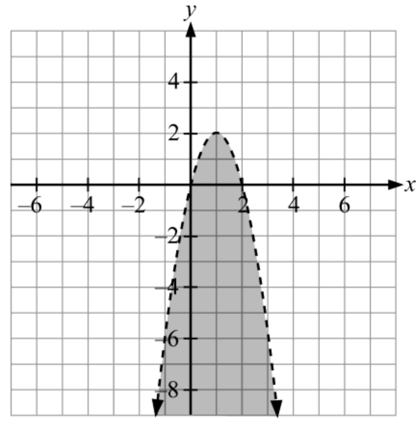
Step 4
Determine which side of the boundary line is to be shaded.

Substitute the test point (1, 1) into the inequality $-\frac{y}{2} > x^2 - 2x$.

$$-\frac{y}{2} > x^2 - 2x$$
$$-\frac{1}{2} > (1)^2 - 2(1)$$
$$-\frac{1}{2} > 1 - 2$$
$$-\frac{1}{2} > -1$$

The point (1, 1) satisfies the inequality, so the side that includes (1, 1) should be shaded.

Step 5
Shade the appropriate side of the boundary line.



The solution set to the inequality $-\frac{y}{2} > x^2 - 2x$ is the shaded region of the graph.

4. Step 1
Isolate the y-variable.

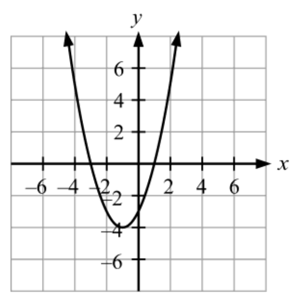
$$-y - 3 \ge -x^2 - 2x$$
$$-y \ge -x^2 - 2x + 3$$
$$y \le x^2 + 2x - 3$$

Step 2
Determine whether the boundary line will be dotted or solid.

Since the inequality is $\le$, the boundary line is part of the solution set.

The boundary line will be solid.

Step 3
Graph the corresponding quadratic equation that represents the inequality with the appropriate line.



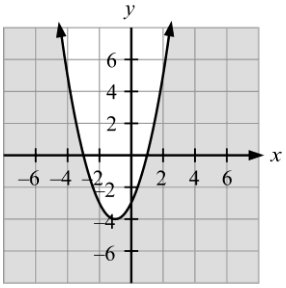
Step 4
Determine which side of the boundary line is to be shaded.

Substitute the test point (0, 0) into the inequality $-y-3\geq -x^2-2x$.

$$
\begin{aligned}
-y-3&\geq -x^2-2x\\
-(0)-3&\geq -(0)^2-2(0)\\
0-3&\geq 0-0\\
-3&\geq 0
\end{aligned}
$$

The point (0, 0) does not satisfy the inequality, so the side that includes (0, 0) should not be shaded.

Step 5
Shade the appropriate side of the boundary line.



The solution set to the inequality $-y-3\geq -x^2-2x$ is the shaded region of the graph.

5. **Step 1**
Determine the equation of the boundary line using points on the line.

The points at (0, 10) and (1, 5) lie on the boundary line.

Use the slope formula to determine the slope, *m*, of the line.

$$
\begin{aligned}
m&=\left(\frac{y_2-y_1}{x_2-x_1}\right)\\
m&=\left(\frac{5-10}{1-0}\right)\\
m&=\frac{-5}{1}\\
m&=-5
\end{aligned}
$$

Since the *y*-intercept from the graph is 10, the equation of the boundary line is $y=-5x+10$.

Step 2
Determine the inequality sign.

Since the boundary line is solid, the inequality sign is ≤ or ≥.

Remove the equal sign in the equation $y=-5x+10$. Using a point from the shaded region, such as (2, 5), evaluate both sides to determine the appropriate inequality sign.

$$
\begin{aligned}
y &\ \square\ -5x+10\\
5 &\ \square\ -5(2)+10\\
5 &\ \square\ -10+10\\
5 &\ \square\ 0
\end{aligned}
$$

Since 5 is greater than 0, the correct sign is ≥.

Therefore, the inequality is $y\geq -5x+10$.

6. **Step 1**
Determine the equation of the boundary line using points on the line.

From the vertex, the values of *p* and *q* are 0 and –2, respectively. A point on the parabola is (2, 2), so *x* = 2 and *y* = 2.

Substituting the known values into the equation $y=a(x-p)+q$, find the value of *a*.

$$
\begin{aligned}
y&=a(x-p)^2+q\\
2&=a(2-0)^2+(-2)\\
2&=a(2)^2-2\\
4&=4a\\
a&=1
\end{aligned}
$$

The equation of the boundary line is $y=x^2-2$.

Step 2
Determine the inequality sign.

Since the boundary line is solid, the inequality sign is ≤ or ≥.

Remove the equal sign in the equation $y=x^2-2$. Using a point from the shaded region, such as (1, –4), evaluate both sides to determine the appropriate inequality sign.

$$
\begin{aligned}
y &\ \square\ x^2-2\\
-4 &\ \square\ (1)^2-2\\
-4 &\ \square\ 1-2\\
-4 &\ \square\ -1
\end{aligned}
$$

Since –4 is less than –1, the correct sign is ≤.

Therefore, the inequality is $y \le x^2 - 2$.

7. **Step1**
Isolate the y-variable.

$$\frac{9}{2}x - 8y \ge 6$$
$$-8y \ge -\frac{9}{2}x + 6$$
$$y \le \frac{9}{16}x - \frac{3}{4}$$

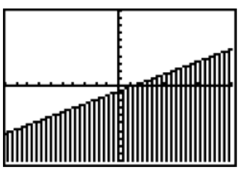
Step 2
Press $\boxed{\text{Y=}}$, and enter the equation of the boundary line.

$$Y_1 = \frac{9}{16}x - \frac{3}{4}$$

Step 3
Determine which side of the boundary line is to be shaded.

Move the cursor to the left of $Y_1 = \frac{9}{16}x - \frac{3}{4}$, where the dotted line flashes. Since the inequality symbol used is ≤, press $\boxed{\text{ENTER}}$ three times or until the image  appears.

Step 4
Press $\boxed{\text{ZOOM}}$, and select 6:ZStandard to obtain this screen.



The graph of the solution of $\frac{9}{2}x - 8 \ge 6$ is displayed.

8. **Step 1**
Isolate the y-variable.

$$\frac{5}{4}x^2 - 1 > -\frac{1}{4}y - \frac{11}{4}x$$
$$5x^2 - 4 > -y - 11x$$
$$y > -5x^2 - 11x + 4$$

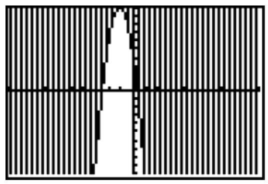
Step 2
Press $\boxed{\text{Y=}}$, and enter the equation of the boundary line.

$$Y_1 = -5x^2 - 11x + 4$$

Step 3
Determine which side of the boundary line is to be shaded.

Move the cursor to the left of $Y_1 = -5x^2 - 11x + 4$, where the dotted line flashes. Since the inequality symbol used is >, press $\boxed{\text{ENTER}}$ twice or until the image  appears.

Step 4
Press $\boxed{\text{ZOOM}}$, and select 6:ZStandard to obtain this screen.



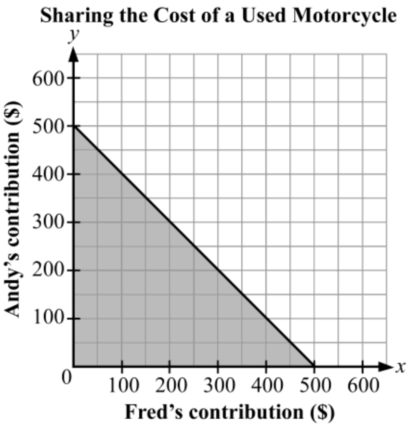
The graph of the solution of $\frac{5}{4}x^2 - 1 > -\frac{1}{4}y - \frac{11}{4}x$ is displayed.

9. If the amount of money that Fred contributes is represented by x and the amount of money Andy contributes is represented by y, then the amount of money they contribute together can be represented by the inequality $x + y \le 500$.

10. The inequality that must be represented is $x + y \le 500$. The solution consists of all ordered pairs on the line $x + y = 500$, as well as ordered pairs that lie in the region bounded by $x + y = 500$, the x-axis, and the y-axis.

Solutions are limited to positive values with two decimal places (dollars and cents).

The ordered pairs represent amounts of money that could be contributed by Fred (x-values) and Andy (y-values).



Practice Test

ANSWERS AND SOLUTIONS

1. The solution to the inequality $f(x) \le 0$ consists of the x-coordinates, where $f(x) \le 0$ (below the x-axis and zero), for the graph of $f(x)$. The graph shows that the solutions are the values of x that are between –2 and 6. Therefore, the solution is $-2 \le x \le 6$.

2. **Step 1**
Factor the quadratic inequality.

$$-x^2 \le 5x - 36$$
$$-x^2 - 5x + 36 \le 0$$
$$-\left(x^2 + 5x - 36\right) \le 0$$
$$-\left(x+9\right)\left(x-4\right) \le 0$$
$$\left(x+9\right)\left(x-4\right) \ge 0$$

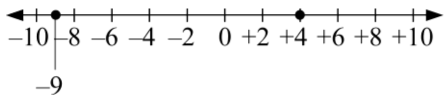
Step 2
Determine the boundary values.

$x + 9 = 0$ $\quad$ $x - 4 = 0$
$x = -9$ $\quad$ $x = 4$

The boundary values are $x = -9$ and $x = 4$.

Step 3
Place the boundary values on a number line, and identify the possible solution regions.

Solid dots are used because $x = -9$ and $x = 4$ are included in the inequality.



There are three regions on the number line: $x < -9$, $-9 < x < 4$, and $x > 4$.

Step 4
Use test points to determine the sign of the expression $\left(x+9\right)\left(x-4\right)$ in each of the three regions.

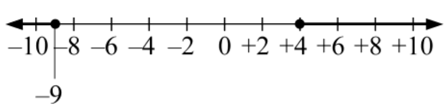
Possible test points are −10, 0, and 10.

Region	Test Point	$\left(x+9\right)\left(x-4\right)$	Sign
$x < -9$	−10	$((-10)+9)(-10-4)$ $=(-1)(-14)$ $=14$	+
$-9 < x < 4$	0	$(0+9)(0-4)$ $=(9)(-4)$ $=-36$	–
$x > 4$	10	$(10+9)(10-4)$ $=(19)(6)$ $=114$	+

Step 5
Identify the solution region.

Since $(x+9)(x-4) \ge 0$ is required, the solution consists of the values in the regions where the product is positive and zero. Thus, the solutions are $x \le -9$ and $x \ge 4$.

Step 6
Graph the solution set on the number line.



3. **Step 1**
Isolate the y-variable.

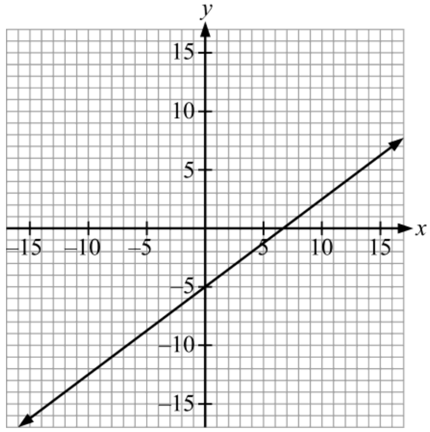
$$\frac{3x-52}{2} \ge 2y - 16$$
$$3x - 52 \ge 4y - 32$$
$$-4y \ge -3x + 52 - 32$$
$$-4y \ge -3x + 20$$
$$y \le \frac{3}{4}x - 5$$

Step 2
Determine whether the boundary line will be dotted or solid.

Since the inequality is ≤, the boundary line is part of the solution set.

The boundary line will be solid.

Step 3
Graph the corresponding linear equation that represents the inequality, with the appropriate line.



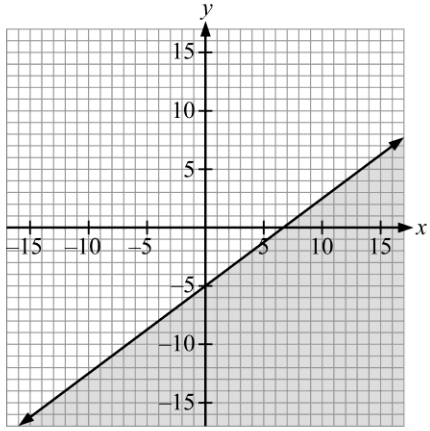
Step 4
Determine which side of the boundary line is to be shaded.

Substitute the test point (0, 0) into the inequality $\frac{3x-52}{2} \geq 2y-16$.

$$\begin{aligned} \frac{3x-52}{2} &\geq 2y-16 \\ \frac{3(0)-52}{2} &\geq 2(0)-16 \\ \frac{0-52}{2} &\geq 0-16 \\ -26 &\geq -16 \end{aligned}$$

The point (0, 0) does not satisfy the inequality, so the side that includes (0, 0) should not be shaded.

Step 5
Shade the appropriate side of the boundary line.



The solution set to the inequality $\frac{3x-52}{2} \geq 2y-16$ is the shaded region of the graph.

4. **Step 1**
Determine the equation of the boundary line using points on the line.

The points at (0, 2) and (1, 0) lie on the boundary line.

Use the slope formula to determine the slope, *m*, of the line.

$$m = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)$$
$$m = \left(\frac{0-2}{1-0}\right)$$
$$m = \left(\frac{-2}{1}\right)$$
$$m = -2$$

Since the *y*-intercept is 2 from the graph, the equation of the boundary line is $y = -2x + 2$.

Step 2
Determine the inequality sign.

Since the boundary line is solid, the inequality sign is ≤ or ≥.

Remove the equal sign in the equation $y = -2x + 2$. Using a point from the shaded region, such as (0, 0), evaluate both sides to determine the appropriate inequality sign.

$$
\begin{aligned}
y &\ \square\ -2x+2\\
0 &\ \square\ -2(0)+2\\
0 &\ \square\ 0+2\\
0 &\ \square\ 2
\end{aligned}
$$

Since 0 is less than 2, the correct inequality sign is ≤.

Therefore, the inequality is $y \le -2x+2$.

5. **Step 1**
Isolate the y-variable.

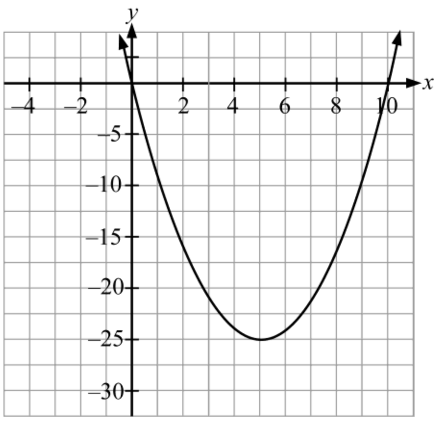
$$
\begin{aligned}
x^2-9x &\le y+x\\
-y &\le -x^2+9x+x\\
-y &\le -x^2+10x\\
y &\ge x^2-10x
\end{aligned}
$$

Step 2
Determine whether the boundary line will be dotted or solid.

Since the inequality is ≥, the boundary line is part of the solution set.

The boundary line will be solid.

Step 3
Graph the corresponding quadratic equation that represents the inequality with the appropriate line.



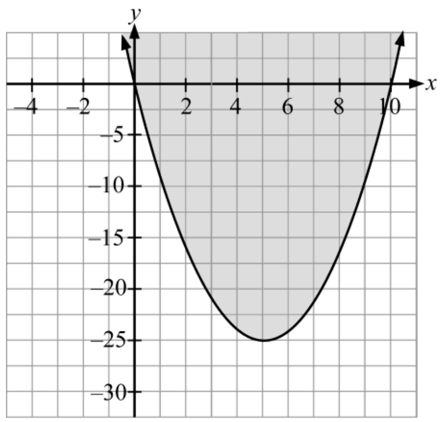
Step 4
Determine which side of the boundary line is to be shaded.

Substitute the test point (1, 1) into the inequality $x^2-9x \le y+x$.

$$
\begin{aligned}
x^2-9x &\le y+x\\
(1)^2-9(1) &\le 1+1\\
-8 &\le 2
\end{aligned}
$$

The point (1, 1) satisfies the inequality, so the side that includes (1, 1) should be shaded.

Step 5
Shade the appropriate side of the boundary line.



The solution set to the inequality $x^2-9x \le y+x$ is the shaded region of the graph.

6. **Step 1**
Model the given information with an inequality.

Since the stone must be less than 39.2 m above the river, an inequality statement that represents the problem is $24.5+19.6t-4.9t^2 < 39.2$.

Step 2
Write the quadratic inequality in $ax^2+bx+c<0$ form.

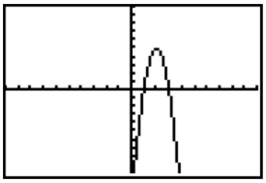
Subtract 39.2 from each side of the inequality $24.5+19.6t-4.9t^2 < 39.2$.

$$
\begin{aligned}
24.5+19.6t-4.9t^2 &< 39.2\\
24.5-39.2+19.6t-4.9t^2 &< 0\\
-14.7+19.6t-4.9t^2 &< 0
\end{aligned}
$$

Step 3
Graph the function using a graphing calculator.

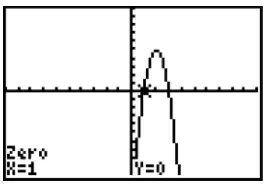
Press $\boxed{\text{Y=}}$, and enter the function as
$Y_1 = -14.7 + 19.6x - 4.9x^2$.

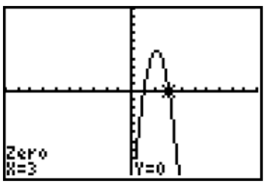
Press $\boxed{\text{ZOOM}}$, and select 6:ZStandard to obtain this screen shot.



Step 4
Determine the *t*-intercepts of the function.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 2:zero. Move the cursor to the left of the first *x*-intercept, and press $\boxed{\text{ENTER}}$. Next, move the cursor to the right of the same *x*-intercept, and press $\boxed{\text{ENTER}}$ twice. Repeat to find the other *x*-intercept.





The *t*-intercepts are $t = 1$ and $t = 3$.

Step 5
Determine the solution to the inequality.

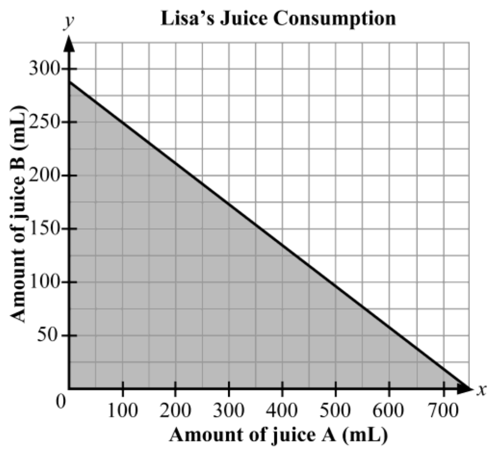
The solution to the inequality
$-14.7 + 19.6t - 4.9t^2 < 0$ consists of the *t*-values, where $h < 0$ (below the *x*-axis), for the graph of $h = -14.7 + 19.6t - 4.9t^2$.

From the graph, it can be seen that the rock is less than 39.2 m above the river when $t < 1$ and $t > 3$. However, since time cannot be negative, the solution is 0 s < t < 1 s and $t > 3$ s.

7. If the amount of juice A, in millilitres, is represented by *x* and the amount of juice B, in millilitres, is represented by *y*, then the amount of juice that Lisa can drink each day can be represented by the inequality $0.15x + 0.40y \le 114$.

8. The inequality that must be represented is $0.15x + 0.40y \le 114$. The solution consists of all ordered pairs on the line $0.15x + 0.40y = 114$, as well as ordered pairs that lie in the region bounded by $0.15x + 0.40y = 114$, the *x*-axis, and the *y*-axis. Solutions are limited to positive values since millilitres (mL) is a positive unit of measure.

The ordered pairs represent the amount of juice A (*x*-coordinates) and juice B(*y*-coordinates) that Lisa can drink.



SEQUENCES AND SERIES

Lesson 1—Arithmetic Sequences

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. The sequence is arithmetic because the constant $\frac{2}{5}$ is added to each term to get the next term.

2. To find the fourth, fifth, and sixth terms, repeatedly add 5.
$\_\_\_, \_\_\_, 8, 13, 18, 23$

To find the first and second terms, work backward by repeatedly subtracting 5.
–2, 3, 8, 13, 18, 23

3. The overtime rate of pay at time and a half is $1.5 \text{ h} \times \$25/\text{h} = \$37.50$.

Working 40 h in week 1, Sonya earns $40 \text{ h} \times \$25/\text{h} = \$1\ 000$.

Working 43 h in week 2, Sonya earns $\$1\ 000 + 3 \text{ h} \times \$37.50/\text{h} = \$1\ 112.50$.

Working 46 h in week 3, Sonya earns $\$1\ 000 + 6 \text{ h} \times \$37.50/\text{h} = \$1\ 225.00$.

The amount that Sonya earns each week forms an arithmetic sequence where d = \$112.50.
\$1 000, \$1 112.50, \$1 225.00

4. The value of t_{13} can be found by adding the common difference six times to the value of t_7, which is the known term closest to t_{13}.

Step 1
Determine the common difference.

$$d = t_2 - t_1$$
$$d = 18 - 24$$
$$d = -6$$

Step 2
Find t_{13}.
–12 + –6 + –6 + –6 + –6 + –6 + –6 = –48

Alternatively, adding six times the common difference to the value of t_7 can generate the 13th term more directly.

$$t_7 + 6d = t_{13}$$
$$-12 + 6(-6) = t_{13}$$
$$-48 = t_{13}$$

5. **Step 1**
Identify the values of a and d for the given sequence.

The value of a (the first term) is $\frac{3}{4}$.

$$d = t_2 - t_1$$
$$d = \frac{15}{4} - \frac{3}{4}$$
$$d = \frac{12}{4}$$
$$d = 3$$

Step 2
Write the general term formula for the given sequence.

Substitute $a = \frac{3}{4}$ and $d = 3$ into the general term formula, and simplify.

$$t_n = a + (n-1)d$$
$$t_n = \frac{3}{4} + (n-1)(3)$$
$$t_n = \frac{3}{4} + 3n - 3$$
$$t_n = 3n - \frac{9}{4}$$

Step 3
Find the required term.

$$t_n = 3n - \frac{9}{4}$$
$$t_{26} = 3(26) - \frac{9}{4}$$
$$t_{26} = \frac{303}{4}$$

6. **Step 1**
Identify the values of a and d for the given sequence.

Between 14 and 110, the first multiple of 3 is 15, so a = 15.

Since the sequence is increasing by multiples of 3, the common difference is 3, so d = 3.

Step 2
Write the general term formula for the given sequence.
$t_n = a + (n-1)d$
$t_n = 15 + (n-1)(3)$
$t_n = 15 + 3n - 3$
$t_n = 3n + 12$

Step 3
Find the number of terms.

The last multiple of 3 between 14 and 110 is 108. Therefore, $t_n = 108$. Substitute $t_n = 108$ into the general term formula, and solve for n.
$t_n = 3n + 12$
$108 = 3n + 12$
$96 = 3n$
$32 = n$

There are 32 multiples of 3 between 14 and 110.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Find the common difference, d.
$d = t_2 - t_1$
$d = -4 - 15$
$d = -19$

Step 2
Add the common difference five times to the last known term value.

Add –19 to –23 to get the next number in the sequence, and then add –19 four more times.
–42, –61, –80, –99, –118

2. **Step 1**
Identify the values of a and d for the given sequence.

The value of a is 3.5, and the value of d is –1.5.

Step 2
Write the general term formula for the given sequence.

Substitute $a = 3.5$ and $d = -1.5$ into the general term formula, and simplify.
$t_n = a + (n-1)d$
$t_n = 3.5 + (n-1)(-1.5)$
$t_n = 3.5 - 1.5n + 1.5$
$t_n = -1.5n + 5$

Step 3
Find the required term.
$t_n = -1.5n + 5$
$t_{30} = -1.5(30) + 5$
$t_{30} = -40$

3. **Step 1**
Find t_1 and t_2 using the given general term formula.

$t_n = 6n - 2$	$t_n = 6n - 2$
$t_1 = 6(1) - 2$	$t_2 = 6(2) - 2$
$t_1 = 4$	$t_2 = 10$

Step 2
Find the common difference, d.
$d = t_2 - t_1$
$d = 10 - 4$
$d = 6$

The first term is 4, and the common difference is 6.

4. **Step 1**
Find the common difference, d.

To insert five terms, use the general term formula to solve for d if $a = 17$ and $t_7 = -19$.
17, __, __, __, __, __, –19
$t_n = a + (n-1)d$
$-19 = 17 + (7-1)d$
$-19 = 17 + 6d$
$-36 = 6d$
$-6 = d$

Step 2
Add the common difference to find the second, third, fourth, fifth, and sixth terms in the sequence.

Add –6 repeatedly.
17, 11, 5, –1, –7, –13, –19

5. **Step 1**
Use the first three terms of the sequence to determine the value of x.

Use the fact that the common difference is constant for all successive terms in an arithmetic sequence to write an equation.

$$\begin{aligned} d &= d \\ t_2 - t_1 &= t_3 - t_2 \\ 7x - (x+6) &= (9x+2) - 7x \\ 7x - x - 6 &= 9x + 2 - 7x \\ 6x - 6 &= 2x + 2 \\ 4x &= 8 \\ x &= 2 \end{aligned}$$

Step 2
Determine the numerical values of the first three terms of the sequence.

Substitute x = 2 into (x + 6), (7x), and (9x + 2).

$x+6$	$7x$	$9x+2$
$=2+6$	$=7(2)$	$=9(2)+2$
$=8$	$=14$	$=20$

The sequence is 8, 14, 20.

6. Substitute d = 0.15, n = 5, and t_5 = 1.8 into the general term formula, and solve for a.

$$\begin{aligned} t_n &= a + (n-1)d \\ 1.8 &= a + (5-1)(0.15) \\ 1.8 &= a + 0.75 - 0.15 \\ 1.2 &= a \end{aligned}$$

Ice-cream cones cost $1.20 in the 1st year.

7. Find t_9 using the general term formula.

$$\begin{aligned} t_n &= a + (n-1)d \\ t_9 &= 1.2 + (8)(0.15) \\ t_9 &= 1.2 + 1.2 \\ t_9 &= 2.4 \end{aligned}$$

In the 9th year, ice-cream cones will cost $2.40.

8. Find n if $t_n = 3.45$ using the general term formula.

$$\begin{aligned} t_n &= a + (n-1)d \\ 3.45 &= 1.2 + (n-1)(0.15) \\ 3.45 &= 1.2 + 0.15n - 0.15 \\ 2.4 &= 0.15n \\ 16 &= n \end{aligned}$$

Ice-cream cones will cost $3.45 in the 16th year.

9. Find the common difference.

$$\begin{aligned} d &= t_2 - t_1 \\ d &= 1 - (-4) \\ d &= 5 \end{aligned}$$

Since the common difference of an arithmetic sequence and the slope of the graph of the related linear function are equal, the slope is 5.

Lesson 2—Arithmetic Series

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Identify the known quantities, and choose a sum formula.

The known quantities are a = 41, n = 12, and d = –2. Therefore, the formula $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$ can be used.

Step 2
Substitute the known quantities into the equation, and solve for S_n.

$$\begin{aligned} S_n &= \frac{n}{2}\left[2a + (n-1)d\right] \\ S_{12} &= \frac{12}{2}\left[2(41) + (12-1)(-2)\right] \\ S_{12} &= 6[60] \\ S_{12} &= 360 \end{aligned}$$

2. **Step 1**
Determine t_5 and t_6.

Find t_6 using S_6 = 108 and S_5 = 75.

$$\begin{aligned} t_6 &= S_6 - S_5 \\ t_6 &= 108 - 75 \\ t_6 &= 33 \end{aligned}$$

Find t_5 using S_5 = 75 and S_4 = 48.

$$\begin{aligned} t_5 &= S_5 - S_4 \\ t_5 &= 75 - 48 \\ t_5 &= 27 \end{aligned}$$

Step 2
Find the common difference, d.

Since $t_5 = 27$ and $t_6 = 33$ in an arithmetic series, then the common difference is $d = 33 - 27$, which is $d = 6$.

Step 3
Use the common difference to find the first four terms in the series by working backward.
$3 + 9 + 15 + 21 + 27 + 33$

3. Find the sum of the earnings over a 12-month period using $S_n = \frac{n}{2}\left[2a+(n-1)d\right]$, where $a = 2\ 200$, $n = 12$, and $d = 50$.

$$\begin{aligned}S_n &= \frac{n}{2}\left[2a+(n-1)d\right]\\ S_{12} &= \frac{12}{2}\left[2(2\ 200)+(12-1)(50)\right]\\ S_{12} &= 6\left[4\ 950\right]\\ S_{12} &= 29\ 700\end{aligned}$$

In one year, the earnings would be \$29 700.

4. Since the sum of the series is known, use the sum formula $S_n = \frac{n}{2}(a+t_n)$ to find the number of terms in the series, where $S_n = -110\ 800$, $t_1 = 43$, and $t_n = -1\ 151$.

$$\begin{aligned}S_n &= \frac{n}{2}(a+t_n)\\ -110\ 800 &= \frac{n}{2}\left(43+(-1\ 151)\right)\\ -110\ 800 &= \frac{n}{2}(-1\ 108)\\ -110\ 800 &= -554n\\ 200 &= n\end{aligned}$$

The series contains 200 terms.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. Find the sum, where $a = 10$, $n = 8$, and $t_8 = -144$.

$$\begin{aligned}S_n &= \frac{n}{2}(a+t_n)\\ S_8 &= \frac{8}{2}\left(10+(-144)\right)\\ S_8 &= 4(-134)\\ S_8 &= -536\end{aligned}$$

2. Find the sum, where $a = \frac{2}{5}$, $n = 11$, and $d = \frac{2}{5}$.

$$\begin{aligned}S_n &= \frac{n}{2}\left[2a+(n-1)d\right]\\ S_{11} &= \frac{11}{2}\left[2\left(\frac{2}{5}\right)+(11-1)\frac{2}{5}\right]\\ S_{11} &= \frac{11}{2}\left(\frac{24}{5}\right)\\ S_{11} &= \frac{132}{5}\end{aligned}$$

3. Find the sum, where $a = 2$, $n = 6$, and $d = 3$.

$$\begin{aligned}S_n &= \frac{n}{2}\left[2a+(n-1)d\right]\\ S_6 &= \frac{6}{2}\left[2(2)+(6-1)3\right]\\ S_6 &= 3(19)\\ S_6 &= 57\end{aligned}$$

Arnold will eat 57 treats in 6 days.

4. If Arnold eats 57 treats, then the remaining number of treats is $71 - 57 = 14$.

Arnold will give 14 treats to his siblings after the 6th day.

5. Find the 9th term of the series, where $S_8 = 14$ and $S_9 = 27$.

$$\begin{aligned}t_n &= S_n - S_{n-1}\\ t_9 &= S_9 - S_8\\ t_9 &= 27-14\\ t_9 &= 13\end{aligned}$$

The 9th term of the series is 13.

6. **Step 1**
Before either sum formula can be used, find the number of terms in the series using the general term formula, where $a = 15$, $d = 3$, and $t_n = 45$.

$$\begin{aligned}t_n &= a+(n-1)d\\ 45 &= 15+(n-1)(3)\\ 45 &= 15+3n-3\\ 33 &= 3n\\ 11 &= n\end{aligned}$$

Step 2

Use the sum formula $S_n = \frac{n}{2}(a + t_n)$ to find the sum of the 11 terms.

$$\begin{aligned} S_n &= \frac{n}{2}(a + t_n) \\ S_{11} &= \frac{11}{2}(15 + 45) \\ S_{11} &= \frac{11}{2}(60) \\ S_{11} &= 330 \end{aligned}$$

7. Since the sum of the series is known, use the sum formula $S_n = \frac{n}{2}[2a + (n-1)d]$ to find the common difference, where $S_{10} = -55$ and $a = t_1 = 8$.

$$\begin{aligned} S_n &= \frac{n}{2}[2a + (n-1)d] \\ -55 &= \frac{10}{2}[2(8) + (10-1)d] \\ -55 &= 5[16 + 9d] \\ -55 &= 80 + 45d \\ -135 &= 45d \\ -3 &= d \end{aligned}$$

Lesson 3—Geometric Sequences

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Find the common ratio.

$$\begin{aligned} r &= \frac{t_2}{t_1} \\ r &= \frac{\sqrt{10}}{\sqrt{5}} \\ r &= \sqrt{2} \end{aligned}$$

Since $r > 1$, the sequence is considered geometric growth.

2. **Step 1**
Find the common ratio.

$$\begin{aligned} r &= \frac{t_2}{t_1} \\ r &= \frac{20}{-5} \\ r &= -4 \end{aligned}$$

Step 2
Find t_7 by repeatedly multiplying the common ratio by the previous term until the 7th term is found.
–5, 20, –80, 320, –1 280, 5 120, –20 480

Therefore, $t_7 = -20\,480$.

3. **Step 1**
Find the common ratio.

The common ratio will be 0.82 since the car is worth 82% of the previous year’s value.

Step 2
Find subsequent car values by repeatedly multiplying the common ratio by the previous car value.

First year	\$25 000
Second year	\$25 000 × 0.82 = \$20 500
Third year	\$20 500 × 0.82 = \$16 810
Fourth year	\$16 810 × 0.82 = \$13 784.20
Fifth year	\$13 784.20 × 0.82 = \$11 303.04
Sixth year	\$11 303.04 × 0.82 = \$9 268.50

By the sixth year, the value of the car drops below \$10 000.

4. **Step 1**
Identify the values of a and r for the given sequence.

The value of a (the first term) is $\frac{3}{16}$.

$$\begin{aligned} r &= \frac{t_2}{t_1} \\ r &= \frac{1}{4} \div \frac{3}{16} \\ r &= \frac{1}{4} \times \frac{16}{3} \\ r &= \frac{4}{3} \end{aligned}$$

Step 2
Write the general term formula for the given sequence.

Substitute $a = \frac{3}{16}$ and $r = \frac{4}{3}$ into the general term formula.

$$\begin{aligned} t_n &= ar^{n-1} \\ t_n &= \frac{3}{16}\left(\frac{4}{3}\right)^{n-1} \end{aligned}$$

5. **Step 1**
Express each term using the general term formula $t_n = ar^{n-1}$.

$$t_4 = ar^{4-1} \quad \text{and} \quad t_6 = ar^{6-1}$$
$$t_4 = ar^3 \qquad t_6 = ar^5$$
$$54 = ar^3 \qquad 486 = ar^5$$

Step 2
Express $486 = ar^5$ and $54 = ar^3$ as two equivalent ratios.

$$\frac{ar^5}{ar^3} = \frac{486}{54}$$

Step 3
Solve for r.

$$\frac{ar^5}{ar^3} = \frac{486}{54}$$
$$r^2 = 9$$
$$r = \pm\sqrt{9}$$
$$r = \pm 3$$

Step 4
Find a.

Substitute r = 3 into the equation $ar^3 = 54$.

$$ar^3 = 54$$
$$a(3)^3 = 54$$
$$27a = 54$$
$$a = 2$$

Substitute r = –3 into the equation $ar^3 = 54$.

$$ar^3 = 54$$
$$a(-3)^3 = 54$$
$$-27a = 54$$
$$a = -2$$

Therefore, a = 2 when r = 3, and a = –2 when r = –3.

Step 5
Write the general term formula for the given sequence.

Substitute the values of a and r into the formula $t_n = ar^{n-1}$ to obtain the two possible general term formulas for the sequence.

The two formulas are $t_n = 2(3)^{n-1}$ and $t_n = (-2)(-3)^{n-1}$.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Find the common ratio.

$$r = \frac{t_2}{t_1}$$
$$r = \frac{3}{\sqrt{3}}$$
$$r = \frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$r = \frac{3\sqrt{3}}{3}$$
$$r = \sqrt{3}$$

If the common ratio is found using t_3 and t_2, rationalizing the denominator is not necessary.

$$r = \frac{t_3}{t_2}$$
$$r = \frac{3\sqrt{3}}{3}$$
$$r = \sqrt{3}$$

Step 2
Repeatedly multiply the common ratio by the previous term to find the next three terms.

$$t_4 = \sqrt{3}\left(3\sqrt{3}\right)$$
$$t_4 = 9$$

$$t_5 = \sqrt{3}(9)$$
$$t_5 = 9\sqrt{3}$$

$$t_6 = \sqrt{3}\left(9\sqrt{3}\right)$$
$$t_6 = 27$$

The geometric sequence is $\sqrt{3}$, 3, $3\sqrt{3}$, 9, $9\sqrt{3}$, 27,….

2. **Step 1**
Find the common ratio.

If David keeps 100% of his current wage and earns a 6%/h increase each year, he will earn 106%/h of what he made the previous year.

Therefore, the common ratio will be 1.06.

Step 2
Find subsequent hourly wages by repeatedly multiplying the common ratio by the hourly wage.
Year 1: wage = $9.85
Year 2: wage = $9.85 × 1.06 = $10.44
Year 3: wage = $10.44 × 1.06 = $11.07

He will earn $11.07/h in year 3.

3. Continue the pattern until the wage exceeds $13.00/h.
Year 4: wage = $11.07 × 1.06 = $11.73
Year 5: wage = $11.73 × 1.06 = $12.44
Year 6: wage = $12.44 × 1.06 = $13.18

In his sixth year, David will earn more than $13.00/h.

4. **Step 1**
Find the common ratio.

If the motorcycle depreciates 16%, it is maintaining 84% of its value. The common ratio is 0.84.

Step 2
Find subsequent motorcycle values by repeatedly multiplying the common ratio by the previous motorcycle value.

Find the year when the value of the motorcycle first drops below $4 000.
Year 1: value = $8 000
Year 2: value = $8 000 × 0.84 = $6 720
Year 3: value = $6 720 × 0.84 = $5 644.80
Year 4: value = $5 644.80 × 0.84 = $4 741.63
Year 5: value = $4 741.63 × 0.84 = $3 982.97

Amorita will own the motorcycle until the fifth year when it becomes worth less than half of its original value and she decides to sell it.

5. Find t_1 by substituting $n = 1$ into the general term formula $t_n = 2(3)^{n-1}$. Repeat for t_2, t_3, and t_4.

$$t_1 = 2(3)^{1-1} = 2$$
$$t_2 = 2(3)^{2-1} = 6$$
$$t_3 = 2(3)^{3-1} = 18$$
$$t_4 = 2(3)^{4-1} = 54$$

The first four terms of the sequence are 2, 6, 18, and 54.

6. **Step 1**
Use the first three terms of the sequence to determine the value of x.

Use the fact that the common ratio is constant between terms to write an equation that can be solved for x.

$$\begin{aligned} r &= r \\ \frac{t_2}{t_1} &= \frac{t_3}{t_2} \\ \frac{x+2}{x} &= \frac{x+5}{x+2} \\ (x+2)(x+2) &= (x)(x+5) \\ x^2+4x+4 &= x^2+5x \\ 4 &= x \end{aligned}$$

Step 2
Find the values of the terms of the sequence.

Substitute $x = 4$ into the first three terms given by x, $x + 2$, and $x + 5$.

$$x = 4 \qquad \begin{aligned} &x+2 \\ &= 4+2 \\ &= 6 \end{aligned} \qquad \begin{aligned} &x+5 \\ &= 4+5 \\ &= 9 \end{aligned}$$

The first three terms of the sequence are 4, 6, and 9.

Step 3
Find the common ratio of the sequence.

$$r = \frac{t_2}{t_1}$$
$$r = \frac{6}{4}$$
$$r = \frac{3}{2}$$

7. Substitute $n = 1$ into the general term to find t_1.

$$t_n = -6\ 161(-0.235)^{n-1}$$
$$t_1 = -6\ 161(-0.235)^{1-1}$$
$$t_1 = -6\ 161$$

8. Step 1
Identify the values of a and r for the given sequence.

The value of a (the first term) is $-\frac{1}{8}$.

$r = \frac{t_2}{t_1}$
$r = \frac{1}{10} \div -\frac{1}{8}$
$r = \frac{1}{10} \times -\frac{8}{1}$
$r = -\frac{4}{5}$

Step 2
Write the general term formula for the given sequence.

Substitute $a = -\frac{1}{8}$ and $r = -\frac{4}{5}$ into the general term formula.
$t_n = ar^{n-1}$
$t_n = -\frac{1}{8}\left(-\frac{4}{5}\right)^{n-1}$

9. Step 1
Express each term using the general term formula $t_n = ar^{n-1}$.

$t_3 = ar^{3-1}$ and $t_6 = ar^{6-1}$
$t_3 = ar^2$ $t_6 = ar^5$
$10\,000 = ar^2$ $5\,120 = ar^5$

Step 2
Express $5\,120 = ar^5$ and $10\,000 = ar^2$ as two equivalent ratios.
$\frac{ar^5}{ar^2} = \frac{5\,120}{10\,000}$

Step 3
Solve for r.
$\frac{ar^5}{ar^2} = \frac{5\,120}{10\,000}$
$r^3 = \frac{64}{125}$
$r = \sqrt[3]{\frac{64}{125}}$
$r = \frac{4}{5}$

Step 4
Write the general term formula for the given sequence.

Find a by substituting $r = \frac{4}{5}$ into the equation $10\ 000 = ar^2$.
$10\ 000 = a\left(\frac{4}{5}\right)^2$
$10\ 000 = \frac{16}{25}a$
$15\ 625 = a$

Substituting the values of a and r into the formula $t_n = ar^{n-1}$, the general term is $t_n = 15\ 625\left(\frac{4}{5}\right)^{n-1}$.

Step 5
Find the required term.

Use the general term for this sequence.
$t_n = 15\ 625\left(\frac{4}{5}\right)^{n-1}$
$t_8 = 15\ 625\left(\frac{4}{5}\right)^{7}$
$t_8 = \frac{16\ 384}{5}$ or $3\ 276.8$

Lesson 4—Geometric Series

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Step 1
Identify the known quantities, and choose a sum formula.

The known quantities are $a = 64$, $t_n = 729$, and $r = \frac{96}{64} = \frac{3}{2}$.

Therefore, the formula $S_n = \frac{rt_n - a}{r-1}$ can be used.

Step 2

Substitute the known quantities into $S_n = \dfrac{rt_n - a}{r-1}$, and solve for S_n.

$$S_n = \frac{rt_n - a}{r-1}$$
$$S_n = \frac{\left(\frac{3}{2}\right)(729) - 64}{\frac{3}{2} - 1}$$
$$S_n = \frac{1\ 029.5}{0.5}$$
$$S_n = 2\ 059$$

The sum of the finite geometric series is 2 059.

2. **Step 1**
Identify the known quantities, and choose a sum formula.

The known quantities are $a = 3$, $S_n = -1\ 023$, and $r = \dfrac{-6}{3} = -2$. Therefore, the formula $S_n = \dfrac{a(r^n - 1)}{r-1}$ should be used to find n.

Step 2
Substitute the known quantities into $S_n = \dfrac{a(r^n - 1)}{r-1}$, and solve for n by guessing and checking.

$$S_n = \frac{a(r^n - 1)}{r-1}$$
$$-1\ 023 = \frac{3\left((-2)^n - 1\right)}{(-2) - 1}$$
$$-1\ 023 = \frac{3\left((-2)^n - 1\right)}{-3}$$
$$-1\ 023 = -\left((-2)^n - 1\right)$$
$$1\ 023 = (-2)^n - 1$$
$$1\ 024 = (-2)^n$$
$$1\ 024 = (-2)^{10}$$
$$1\ 024 = 1\ 024$$

Therefore, the value of n is 10.

3. Given the formula for the sum of the series, the value of t_8 can be found using the formula $t_n = S_n - S_{n-1}$. More specifically, $t_8 = S_8 - S_7$.

Find S_8 and S_7 by substituting $n = 8$ and $n = 7$ into the formula for the sum of the series $S_n = 3n^2 + n - 2$.

$S_8 = 3(8)^2 + 8 - 2$ and $S_7 = 3(7)^2 + 7 - 2$
$S_8 = 198$ $\qquad S_7 = 152$

$$t_8 = S_8 - S_7$$
$$t_8 = 198 - 152$$
$$t_8 = 46$$

Therefore, the 8th term of the series is 46.

4. **Step 1**
Find the first term and the common ratio of the given series.

The value of a (the first term) is 37.5.

$$r = \frac{t_2}{t_1}$$
$$r = \frac{-18.75}{37.5}$$
$$r = -0.5$$

The given series is convergent since $-1 < r < 1$.

Step 2

Apply the formula $S_\infty = \dfrac{a}{1-r}$.

Substitute $a = 37.5$ and $r = -0.5$.

$$S_\infty = \frac{37.5}{1-(-0.5)}$$
$$S_\infty = \frac{37.5}{1.5}$$
$$S_\infty = 25$$

The sum of the geometric series is 25.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. Step 1
Identify the known quantities, and choose a sum formula.

The known quantities are $a = 7$, $t_n = -3\ 584$, and $r = \frac{-14}{7} = -2$. Therefore, the formula $S_n = \frac{rt_n - a}{r - 1}$ should be used to find S_n.

Step 2
Substitute the known quantities into $S_n = \frac{rt_n - a}{r - 1}$, and solve for S_n.

$$S_n = \frac{rt_n - a}{r - 1}$$
$$S_n = \frac{(-2)(-3\ 584) - 7}{-2 - 1}$$
$$S_n = \frac{7\ 161}{-3}$$
$$S_n = -2\ 387$$

The sum of the geometric series is –2 387.

2. Step 1
Use the general term to find a, r, and t_8.

Since the general term $t_n = -4(-3)^{n-1}$ is in the form $t_n = ar^{n-1}$, $a = -4$ and $r = -3$.

Find t_8 by substituting $n = 8$ into the general term.

$$t_n = -4(-3)^{n-1}$$
$$t_8 = -4(-3)^{8-1}$$
$$t_8 = -4(-3)^{7}$$
$$t_8 = 8\ 748$$

Step 2
Substitute the known quantities into $S_n = \frac{rt_n - a}{r - 1}$, and solve for S_n.

$$S_8 = \frac{(-3)(8\ 748) - (-4)}{(-3) - 1}$$
$$S_8 = 6\ 560$$

The sum of the first eight terms is 6 560.

3. Step 1
Identify the known quantities, and choose a sum formula.
The known quantities are $a = -2$, $S_n = -7\ 812$, and $r = 5$. Therefore, the formula $S_n = \frac{a(r^n - 1)}{r - 1}$ should be used to find n.

Step 2
Substitute the known quantities into $S_n = \frac{a(r^n - 1)}{r - 1}$, and solve for n by guessing and checking.

$$-7\ 812 = \frac{-2\left(5^n - 1\right)}{5 - 1}$$
$$-7\ 812 = \frac{-2\left(5^n - 1\right)}{4}$$
$$-7\ 812 = \frac{5^n - 1}{-2}$$
$$15\ 624 = 5^n - 1$$
$$15\ 625 = 5^n$$
$$15\ 625 = 5^6$$
$$15\ 625 = 15\ 625$$

The value of n is 6, so there are six terms in the series.

4. Step 1
Identify the known quantities, and choose a sum formula.

The known quantities are $S_4 = 2\ 862$ and $r = 3.5$. Therefore, the formula $S_n = \frac{a(r^n - 1)}{r - 1}$ should be used to find a.

Step 2
Substitute the known quantities into $S_n = \frac{a(r^n - 1)}{r - 1}$, and solve for a.

$$2\ 862 = \frac{a\left((3.5)^4 - 1\right)}{3.5 - 1}$$
$$2\ 862 = \frac{a(149.0625)}{2.5}$$
$$7\ 155 = a(149.0625)$$
$$48 = a$$

The first term in the series is 48.

5. **Step 1**
Find the value of the common ratio, r.

Since $t_4 = 54$ and $t_6 = 24$, then $ar^3 = 54$ and $ar^5 = 24$. Express $ar^5 = 24$ and $ar^3 = 54$ as two equivalent ratios.

$$\frac{ar^5}{ar^3} = \frac{24}{54}$$
$$r^2 = \frac{4}{9}$$
$$r = \pm\frac{2}{3}$$

Step 2
Find the value of the first term, a.

If $r = +\frac{2}{3}$:

$$ar^3 = 54$$
$$a\left(\frac{2}{3}\right)^3 = 54$$
$$a\left(\frac{8}{27}\right) = 54$$
$$a = 182.25$$

If $r = -\frac{2}{3}$:

$$ar^3 = 54$$
$$a\left(-\frac{2}{3}\right)^3 = 54$$
$$a\left(-\frac{8}{27}\right) = 54$$
$$a = -182.25$$

Therefore, a = 182.25 when $r = \frac{2}{3}$, and

a = –182.25 when $r = -\frac{2}{3}$.

Step 3
Substitute the known quantities into $S_n = \frac{a\left(r^n - 1\right)}{r-1}$, and find S14 for both possible cases.

When a = 182.25 and $r = \frac{2}{3}$:

$$S_{14} = \frac{182.25\left(\left(\frac{2}{3}\right)^{14} - 1\right)}{\frac{2}{3} - 1}$$
$$S_{14} \doteq \frac{-18\ \ 6257}{-0.3333}$$
$$S_{14} \doteq 544.9$$

When a = –182.25 and $r = -\frac{2}{3}$:

$$S_{14} = \frac{-182.25\left(\left(-\frac{2}{3}\right)^{14} - 1\right)}{-\frac{2}{3} - 1}$$
$$S_{14} \doteq \frac{181\ 6257}{-1.6667}$$
$$S_{14} \doteq -109.0$$

Therefore, the sum of the first 14 terms in the series will either be 544.9 or –109.0.

6. **Step 1**
Find the values of t_1 and t_2.

If $S_1 = 18$, $t_1 = 18$.

Using the formula $t_n = S_n - S_{n-1}$, the value of t_2 can be found.

$$t_2 = S_2 - S_1$$
$$t_2 = 6 - 18$$
$$t_2 = -12$$

Step 2
Find the common ratio.

$$r = \frac{t_2}{t_1}$$
$$r = \frac{-12}{18}$$
$$r = \frac{-2}{3}$$

Step 3
Write the general term using the known values of a and r.

The general term is $t_n = 18\left(-\frac{2}{3}\right)^{n-1}$.

7. Step 1
Determine if the infinite series is convergent.

Find the common ratio.

$r = \frac{t_2}{t_1}$

$r = \frac{2\ 500}{3\ 125}$

$r = 0.8$

The series is convergent since $-1 < r < 1$.

Step 2
Find the sum of the series using the formula for the sum of an infinite geometric series $S_\infty = \frac{a}{1-r}$.

Substitute $a = 3\ 125$ and $r = 0.8$.

$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{3\ 125}{1-0.8}$

$S_\infty = \frac{3\ 125}{0.2}$

$S_\infty = 15\ 625$

The sum of the infinite geometric series is 15 625.

8. Step 1
Use the sum formula $S_n = n - 5n^2$ to find S_1, S_2, and S_3.

$S_1 = 1 - 5(1)^2 = -4$

$S_2 = 2 - 5(2)^2 = -18$

$S_3 = 3 - 5(3)^2 = -42$

Step 2
Find the value of t_1, t_2, and t_3 using the formula $t_n = S_n - S_{n-1}$.

$t_1 = S_1 = -4$

$t_2 = S_2 - S_1$

$t_2 = -18 - (-4)$

$t_2 = -14$

$t_3 = S_3 - S_2$

$t_3 = -42 - (-18)$

$t_3 = -24$

The series is –4 –14 –24 …

Step 3
Determine if the series is geometric.

A series is geometric if there is a common ratio.

$\frac{t_2}{t_1} = \frac{-14}{-4} = \frac{7}{2}$

$\frac{t_3}{t_2} = \frac{-24}{-14} = \frac{12}{7}$

Therefore, the series is not geometric because $\frac{t_2}{t_1} \neq \frac{t_3}{t_2}$.

Practice Test

ANSWERS AND SOLUTIONS

1. Since the sequence is decreasing by 2, then 3, and then 4, it is neither geometric nor arithmetic.

2. Since the common ratio is 4, it is a geometric sequence.

3. Since the common difference is –1.5, it is an arithmetic sequence.

4. Since successive terms are written with an additional digit of 1, it is neither geometric nor arithmetic.

5. Since the common ratio is $\sqrt{7}$, it is a geometric sequence.

6. Since the common difference is $\frac{2}{3}$, it is an arithmetic sequence.

7. Step 1
Find the general term, t_n, by substituting the values for a and d into the general term formula.
$a = 3.5$

$d = -1.5 - 3.5$
$d = -5$

$t_n = a + (n-1)d$

$t_n = 3.5 + (n-1)(-5)$

$t_n = -5n + 8.5$

Step 2
Substitute n = 10 into the general term formula to find t_{10}.

$$t_n = -5n + 8.5$$
$$t_{10} = -5(10) + 8.5$$
$$t_{10} = -41.5$$

Step 3
Substitute a = 3.5 and t_{10} = –41.5 into

$S_n = \frac{n}{2}(a + t_n)$ to find S_{10}.

$$S_n = \frac{n}{2}(a + t_n)$$
$$S_{10} = \frac{10}{2}(3.5 + (-41.5))$$
$$S_{10} = 5(-38)$$
$$S_{10} = -190$$

8. Since S_{15} = 48 and S_{14} = 32, t_{15} can be found using the formula $t_n = S_n - S_{(n-1)}$.

$$t_{15} = S_{15} - S_{14}$$
$$t_{15} = 48 - 32$$
$$t_{15} = 16$$

9. If d = 6, n = 10, and t_{10} = 61, use the general term formula to find a.

$$t_n = a + (n-1)d$$
$$61 = a + (10-1)(6)$$
$$61 = a + 54$$
$$7 = a$$

There were 7 members in the first week.

10. If a = 9 and d = 6, use the general term formula to find t_7.

$$t_n = a + (n-1)d$$
$$t_7 = 9 + (7-1)(6)$$
$$t_7 = 45$$

Theresa pulls 45 weeds on the 7th day.

11. If a = 9 and t_7 = 45, use the sum formula

$S_n = \frac{n}{2}(a + t_n)$ to find S_7.

$$S_n = \frac{n}{2}(a + t_n)$$
$$S_7 = \frac{7}{2}(9 + 45)$$
$$S_7 = 189$$

Theresa has pulled 189 weeds in total by the end of the 7th day.

12. **Step 1**
Find d using the general term formula.

Temporarily consider t_4 = 46 and t_{12} = 102 as the first and last terms of a sequence. This means the terms can be rewritten as t_1 = 46 and t_9 = 102 and the general term formula can be used.

$$t_n = a + (n-1)d$$
$$102 = 46 + (9-1)(d)$$
$$102 = 46 + 8d$$
$$56 = 8d$$
$$7 = d$$

Step 2
Find the actual value of a for the sequence where t_4 = 46 and d = 7.

$$t_n = a + (n-1)d$$
$$46 = a + (4-1)(7)$$
$$46 = a + 21$$
$$25 = a$$

Step 3
Solve for n if t_n < 150 using the general term formula.

$$a + (n-1)d = t_n$$
$$25 + (n-1)(7) < 150$$
$$25 + 7n - 7 < 150$$
$$7n < 132$$
$$n < 18.8571...$$

There are 18 terms less than 150 in the sequence.

13. **Step 1**
Find the common ratio, r.

If the car loses 17% of its value each year, it maintains 83%, which means r = 0.83.

Step 2
Find subsequent car values by repeatedly multiplying the common ratio by the previous car value.

Multiply by the common ratio to obtain the value of the car at the end of each year.
Year 1: \$32 000 × 0.83 = \$26 560
Year 2: \$26 560 × 0.83 = \$22 044.80
Year 3: \$22 044.8 × 0.83 = \$18 297.18

Therefore, the car's value at the end of three years is \$18 297.18.

14. Step 1
Identify the values of a and r.

The known values are $a = -3$ and $r = \frac{t_2}{t_1} = \frac{2}{-3}$.

Step 2
Find the general term.

$t_n = a(r)^{n-1}$

$t_n = -3\left(-\frac{2}{3}\right)^{n-1}$

15. Step 1
Identify the values of a and r from the general term.

Since $t_n = -3(-2)^{n-1}$ is in the form $t_n = ar^{n-1}$, $a = -3$ and $r = -2$.

Step 2
Substitute the values for a and r into the sum formula $S_n = \frac{a(r^n - 1)}{r - 1}$ to find S_{10}.

$S_n = \frac{a\left(r^n - 1\right)}{r - 1}$

$S_{10} = \frac{-3\left((-2)^{10} - 1\right)}{-2 - 1}$

$S_{10} = 1\ 023$

16. Step 1
Find a and r using the sums of $S_1 = 36$ and $S_2 = 24$ and the formula $t_n = S_n - S_{(n-1)}$.

$t_1 = S_1 = 36$
$t_2 = S_2 - S_1$
$t_2 = 24 - 36$
$t_2 = -12$

$r = \frac{t_2}{t_1}$

$r = \frac{-12}{36}$

$r = -\frac{1}{3}$

Step 2
Write the general term of the series.

$a = 36$ and $r = -\frac{1}{3}$

$t_n = ar^{n-1}$

$t_n = 36\left(-\frac{1}{3}\right)^{n-1}$

17. Step 1
Find the value of the common ratio, r.
If $t_3 + t_4 = 50$, then $ar^2 + ar^3 = 50$.
If $t_4 + t_5 = 100$, then $ar^3 + ar^4 = 100$.

Express $ar^3 + ar^4 = 100$ and $ar^2 + ar^3 = 50$ as two equivalent ratios.

$\frac{ar^3 + ar^4}{ar^2 + ar^3} = \frac{100}{50}$

$\frac{ar^3(1+r)}{ar^2(1+r)} = 2$

$r = 2$

Step 2
Find the value of the first term, a.

Substitute $r = 2$ and $ar^2 + ar^3 = 50$.

$a(2)^2 + a(2)^3 = 50$
$4a + 8a = 50$
$12a = 50$

$a = \frac{50}{12}$

$a = \frac{25}{6}$

Therefore, $t_1 = \frac{25}{6}$ and $t_2 = \frac{25}{6} \times 2$, which is $t_2 = \frac{25}{3}$.

18. Step 1
Find the total distance that the ball falls.

The series is $12 + 8 + \frac{16}{3} + \ldots$,

where $t_1 = 12$, $t_2 = 8$, and $t_3 = \frac{16}{3}$.

Therefore, $a = 12$ and $r = \frac{2}{3}$.

Since the ball falls 8 times, $n = 8$.

Use the sum formula $S_n = \dfrac{a\left(r^n - 1\right)}{r-1}$ to find S_8.

$$S_n = \frac{a\left(r^n - 1\right)}{r-1}$$
$$S_8 = \frac{12\left(\left(\frac{2}{3}\right)^8 - 1\right)}{\left(\frac{2}{3}\right) - 1}$$
$$S_8 \doteq$$

Step 2
Find the total distance that the ball rises.

If the ball rises to a height of 8 m after the first bounce, then $t_1 = 8$, $t_2 = \frac{16}{3}$, and $t_3 = \frac{32}{9}$.

Therefore, $a = 8$ and $r = \frac{2}{3}$.

Since the ball rises 7 times, $n = 7$.

Use the sum formula $S_n = \dfrac{a\left(r^n - 1\right)}{r-1}$ to find S_7.

$$S_n = \frac{a\left(r^n - 1\right)}{r-1}$$
$$S_7 = \frac{8\left(\left(\frac{2}{3}\right)^7 - 1\right)}{\left(\frac{2}{3}\right) - 1}$$

$S_7 \doteq$ [redacted]

Step 3
Find the total vertical distance travelled by the ball.

total distance = total distance rising + total distance falling
total distance $\doteq$ 22.60 + 34.60 $\doteq$ 57.20

Therefore, the ball has travelled approximately 57.2 m when it hits the ground for the 8th time.

RECIPROCAL FUNCTIONS

Lesson 1—Asymptotes

CLASS EXERCISES
ANSWERS AND SOLUTIONS

1. The horizontal asymptote of the function is at $y = 0$.

Determine the vertical asymptotes.

Set the denominator equal to zero, and solve for x.
$$8x - 4 = 0$$
$$8x = 4$$
$$x = \frac{1}{2}$$
$$x = 0.5$$

The vertical asymptote is at $x = \frac{1}{2}$ or $x = 0.5$.

2. The horizontal asymptote of the function is at $y = 0$.

Determine the vertical asymptotes.

Set the denominator equal to zero, and solve for x.
$$2x^2 + x - 6 = 0$$
$$2x^2 + 4x - 3x - 6 = 0$$
$$2x(x+2) - 3(x+2) = 0$$
$$(2x-3)(x+2) = 0$$
$$2x - 3 = 0 \qquad x + 2 = 0$$
$$2x = 3 \qquad x = -2$$
$$x = \frac{3}{2}$$

The vertical asymptotes are at $x = \frac{3}{2}$ and $x = -2$.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. The horizontal asymptote of the function is at $y = 0$.

Determine the vertical asymptotes.

Set the denominator equal to zero, and solve for *x*.

$$9x + 6 = 0$$
$$9x = -6$$
$$x = \frac{-6}{9}$$
$$x = -\frac{2}{3}$$

The vertical asymptote is at $x = -\frac{2}{3}$.

2. The horizontal asymptote of the function is at $y = 0$.

Determine the vertical asymptotes.

Set the denominator equal to zero, and solve for *x*.

$$10x - 13 = 0$$
$$10x = 13$$
$$x = \frac{13}{10}$$
$$x = 1.3$$

The vertical asymptote is at $x = \frac{13}{10}$ or $x = 1.3$.

3. Rewrite the function $y = -\frac{1}{6x + 54}$ as $y = \frac{1}{-(6x + 54)}$.

The horizontal asymptote of the function is at $y = 0$.

Determine the vertical asymptotes.

Set the denominator equal to zero, and solve for *x*.

$$-(6x + 54) = 0$$
$$6x + 54 = 0$$
$$6x = -54$$
$$x = -9$$

The vertical asymptote is at $x = -9$.

4. The horizontal asymptote of the function is at $y = 0$.

Determine the vertical asymptotes.

Set the denominator equal to zero, and solve for *x*.

$$x^2 + 3x - 4 = 0$$
$$x^2 - x + 4x - 4 = 0$$
$$x(x - 1) + 4(x - 1) = 0$$
$$(x - 1)(x + 4) = 0$$
$$x - 1 = 0 \qquad x + 4 = 0$$
$$x = 1 \qquad x = -4$$

The vertical asymptotes are at $x = 1$ and $x = -4$.

5. The horizontal asymptote of the function is at $y = 0$.

Determine the vertical asymptotes.

Set the denominator equal to zero, and solve for *x*.

$$121x^2 - 144 = 0$$
$$121x^2 = 144$$
$$x^2 = \frac{144}{121}$$
$$x = \pm\sqrt{\frac{144}{121}}$$
$$x = \pm\frac{12}{11}$$

The vertical asymptotes are at $x = \frac{12}{11}$ and $x = -\frac{12}{11}$.

6. The horizontal asymptote of the function is at $y = 0$.

Determine the vertical asymptotes.

Set the denominator equal to zero, and solve for *x*.

$$10x^2 - 14x + 4 = 0$$
$$2(5x^2 - 7x + 2) = 0$$
$$5x^2 - 7x + 2 = 0,$$
$$5x^2 - 5x - 2x + 2 = 0$$
$$5x(x - 1) - 2(x - 1) = 0$$
$$(x - 1)(5x - 2) = 0$$

$$x-1=0 \qquad 5x-2=0$$
$$x=1 \qquad 5x=2$$
$$x=\frac{2}{5}$$
$$x=0.4$$

The vertical asymptotes are at $x=1$ and $x=\frac{2}{5}$ or $x=0.4$.

Lesson 2—The Behaviour of the Graph of a Reciprocal Function

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Invariant points will occur when the y-coordinate of any ordered pair on the graph of $f(x)=x^2-16$ is equal to 1 or –1.

If $y=1$:

$$1=x^2-16$$
$$17=x^2$$
$$\pm\sqrt{17}=x$$

If $y=-1$:

$$-1=x^2-16$$
$$15=x^2$$
$$\pm\sqrt{15}=x$$

Therefore, $\left(-\sqrt{17},1\right)$, $\left(\sqrt{17},1\right)$, $\left(-\sqrt{15},-1\right)$, and $\left(\sqrt{15},-1\right)$ are invariant points.

2. **Step 1**
Determine the vertical asymptotes.

Set the denominator equal to zero, and solve for x.

$$x^2-x-6=0$$
$$x^2-3x+2x-6=0$$
$$x(x-3)+2(x-3)=0$$
$$(x-3)(x+2)=0$$
$$x-3=0 \qquad x+2=0$$
$$x=3 \qquad x=-2$$

The vertical asymptotes are at $x = –2$ and $x = 3$.

Step 2
Determine the behaviour of the graph on the left of $x = –2$.

Calculate values of y as x approaches –2 from the left.

If $x = –2.01$:

$$y=\frac{1}{x^2-x-6}$$
$$y=\frac{1}{(-2.01)^2-\left(-2.01\right)-6}$$
$$y=\frac{1}{(-2.01)^2+2.01-6}$$
$$y=\frac{1}{0\ 0501}$$
$$y\doteq 19.960\ 079\ 84$$

If $x = –2.001$:

$$y=\frac{1}{x^2-x-6}$$
$$y=\frac{1}{(-2.001)^2-\left(-2.001\right)-6}$$
$$y=\frac{1}{(-2.001)^2+2.001-6}$$
$$y=\frac{1}{0\ 005\ 001}$$
$$y\doteq 199.960\ 008$$

The y-values get larger and larger toward positive infinity as x approaches –2 from the left.

Step 3
Determine the behaviour of the graph on the right of $x = –2$.

Calculate values of y as x approaches –2 from the right.
If $x = –1.99$:

$$y=\frac{1}{x^2-x-6}$$
$$y=\frac{1}{(-1.99)^2-\left(-1.99\right)-6}$$
$$y=\frac{1}{(-1.99)^2+1.99-6}$$
$$y=\frac{1}{-0\ 0499}$$
$$y\doteq -20.040\ 080\ 16$$

If $x = –1.999$:

$$y=\frac{1}{x^2-x-6}$$
$$y=\frac{1}{(-1.999)^2-\left(-1.999\right)-6}$$
$$y=\frac{1}{(-1.999)^2+1.999-6}$$
$$y=\frac{1}{-0\ 004\ 999}$$
$$y\doteq -200.040\ 008$$

The y-values get smaller and smaller toward negative infinity as x approaches –2 from the right.

Step 4
Determine the behaviour of the graph on the left of $x = 3$.

Calculate values of y as x approaches 3 from the left.

If $x = 2.99$:

$$y = \frac{1}{x^2 - x - 6}$$
$$y = \frac{1}{(2.99)^2 - (2.99) - 6}$$
$$y = \frac{1}{-0\ 0499}$$
$$y \doteq -20.040\ 080\ 16$$

If $x = 2.999$:

$$y = \frac{1}{x^2 - x - 6}$$
$$y = \frac{1}{(2.999)^2 - (2.999) - 6}$$
$$y = \frac{1}{-0\ 004\ 999}$$
$$y \doteq -200.040\ 008$$

The y-values get smaller and smaller toward negative infinity as x approaches 3 from the left.

Step 5
Determine the behaviour of the graph on the right of $x = 3$.

Calculate values of y as x approaches 3 from the right.
If $x = 3.01$:

$$y = \frac{1}{x^2 - x - 6}$$
$$y = \frac{1}{(3.01)^2 - (3.01) - 6}$$
$$y = \frac{1}{0\ 0501}$$
$$y \doteq 19.960\ 079\ 84$$

If $x = 3.001$:

$$y = \frac{1}{x^2 - x - 6}$$
$$y = \frac{1}{(3.001)^2 - (3.001) - 6}$$
$$y = \frac{1}{0\ 005\ 001}$$
$$y \doteq 199.960\ 008$$

The y-values get larger and larger toward positive infinity as x approaches 3 from the right.

Step 6
Determine the behaviour of the graph as $x \to -\infty$.
If $x = -10$:

$$y = \frac{1}{x^2 - x - 6}$$
$$y = \frac{1}{(-10)^2 - (-10) - 6}$$
$$y = \frac{1}{104}$$

If $x = -200$:

$$y = \frac{1}{x^2 - x - 6}$$
$$y = \frac{1}{(-200)^2 - (-200) - 6}$$
$$y = \frac{1}{40\ 194}$$

The y-values approach zero, but remain positive. The x-axis becomes a horizontal asymptote as $x \to -\infty$.

Step 7
Determine the behaviour of the graph as $x \to \infty$.
If $x = 10$:

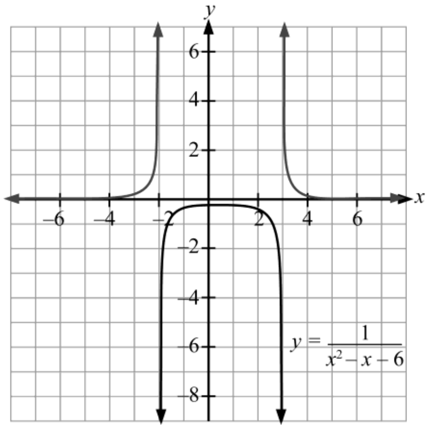
$$y = \frac{1}{x^2 - x - 6}$$
$$y = \frac{1}{(10)^2 - (10) - 6}$$
$$y = \frac{1}{84}$$

If $x = 200$:

$$y = \frac{1}{x^2 - x - 6}$$
$$y = \frac{1}{(200)^2 - (200) - 6}$$
$$y = \frac{1}{39\ 794}$$

The y-values approach zero, but remain positive. The x-axis becomes a horizontal asymptote as $x \to \infty$.

A possible sketch of $y = \dfrac{1}{x^2 - x - 6}$ is shown.



PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. Invariant points will occur when the y-coordinate of any ordered pair on the graph of $f(x) = x + 5$ is equal to 1 or –1.

If $y = 1$:
$$1 = x + 5$$
$$-4 = x$$

If $y = -1$:
$$-1 = x + 5$$
$$-6 = x$$

Therefore, (–4, 1) and (–6, –1) are invariant points.

Step 1
Determine the vertical asymptote. Set the denominator equal to zero, and solve for x.
$$x + 5 = 0$$
$$x = -5$$

There is a vertical asymptote at $x = -5$.

Step 2
Determine the behaviour of the graph on the left of $x = -5$. Calculate values of y as x approaches –5 from the left.

If $x = -5.01$:
$$y = \frac{1}{x+5}$$
$$y = \frac{1}{-5.01+5}$$
$$y = \frac{1}{-0.01}$$
$$y = -100$$

If $x = -5.0001$:
$$y = \frac{1}{x+5}$$
$$y = \frac{1}{-5.0001+5}$$
$$y = \frac{1}{-0.0001}$$
$$y = -10\ 000$$

The y-values get smaller and smaller toward negative infinity.

Step 3
Determine the behaviour of the graph on the right of $x = -5$.

Calculate values of y as x approaches –5 from the right.

If $x = -4.99$:
$$y = \frac{1}{x+5}$$
$$y = \frac{1}{(-4.99)+5}$$
$$y = \frac{1}{0.01}$$
$$y = 100$$

If $x = -4.9999$:
$$y = \frac{1}{x+5}$$
$$y = \frac{1}{(-4.9999)+5}$$
$$y = \frac{1}{0.0001}$$
$$y = 10\ 000$$

The y-values get larger and larger toward positive infinity.

Step 4
Determine the behaviour of the graph as $x \to -\infty$.

If $x = -10$:
$$y = \frac{1}{x+5}$$
$$y = \frac{1}{-10+5}$$
$$y = \frac{1}{-5}$$

If $x = -200$:
$$y = \frac{1}{x+5}$$
$$y = \frac{1}{-200+5}$$
$$y = \frac{1}{-195}$$

The y-values approach zero, but remain negative. The x-axis becomes a horizontal asymptote as $x \to -\infty$.

Step 5
Determine the behaviour of the graph as $x \to \infty$.

If $x = 10$:
$$y = \frac{1}{x+5}$$
$$y = \frac{1}{10+5}$$
$$y = \frac{1}{15}$$

If $x = 200$:
$$y = \frac{1}{x+5}$$
$$y = \frac{1}{200+5}$$
$$y = \frac{1}{205}$$

The y-values approach zero, but remain positive. The x-axis becomes the horizontal asymptote as $x \to \infty$.

2. Invariant points will occur when the y-coordinate of any ordered pair on the graph of $f(x) = 4x - 28$ is equal to 1 or –1.

If $y = 1$:
$$\begin{aligned} 1 &= 4x - 28 \\ 29 &= 4x \\ \frac{29}{4} &= x \\ 7.25 &= x \end{aligned}$$

If $y = -1$:
$$\begin{aligned} -1 &= 4x - 28 \\ 27 &= 4x \\ \frac{27}{4} &= x \\ 6.75 &= x \end{aligned}$$

Therefore, (7.25, 1) and (6.75, –1) are invariant points.

Step 1
Determine the vertical asymptote.

Set the denominator equal to zero, and solve for x.
$$\begin{aligned} 4x - 28 &= 0 \\ 4x &= 28 \\ x &= 7 \end{aligned}$$

There is a vertical asymptote at $x = 7$.

Step 2
Determine the behaviour of the graph on the left of $x = 7$.

Calculate values of y as x approaches 7 from the left.

If $x = 6.99$:
$$\begin{aligned} y &= \frac{1}{4x - 28} \\ y &= \frac{1}{4(6.99) - 28} \\ y &= \frac{1}{-0.04} \\ y &= -25 \end{aligned}$$

If $x = 6.9999$:
$$\begin{aligned} y &= \frac{1}{4x - 28} \\ y &= \frac{1}{4(6.9999) - 28} \\ y &= \frac{1}{-0.0004} \\ y &= -2\ 500 \end{aligned}$$

The y-values get smaller and smaller toward negative infinity.

Step 3
Determine the behaviour of the graph on the right of $x = 7$.

Calculate values of y as x approaches 7 from the right.

If $x = 7.01$:
$$\begin{aligned} y &= \frac{1}{4x - 28} \\ y &= \frac{1}{4(7.01) - 28} \\ y &= \frac{1}{0.04} \\ y &= 25 \end{aligned}$$

If $x = 7.0001$:
$$\begin{aligned} y &= \frac{1}{4x - 28} \\ y &= \frac{1}{4(7.0001) - 28} \\ y &= \frac{1}{0.0004} \\ y &= 2\ 500 \end{aligned}$$

The y-values get larger and larger toward positive infinity.

Step 4
Determine the behaviour of the graph as $x \to -\infty$.

If $x = -10$:
$$\begin{aligned} y &= \frac{1}{4x - 28} \\ y &= \frac{1}{4(-10) - 28} \\ y &= \frac{1}{-68} \end{aligned}$$

If $x = -200$:
$$\begin{aligned} y &= \frac{1}{4x - 28} \\ y &= \frac{1}{4(-200) - 28} \\ y &= \frac{1}{-828} \end{aligned}$$

The y-values approach zero, but remain negative. The x-axis becomes a horizontal asymptote as $x \to -\infty$.

Step 5
Determine the behaviour of the graph as $x \to \infty$.

If $x = 10$:
$$\begin{aligned} y &= \frac{1}{4x - 28} \\ y &= \frac{1}{4(10) - 28} \\ y &= \frac{1}{12} \end{aligned}$$

If $x = 200$:
$$\begin{aligned} y &= \frac{1}{4x - 28} \\ y &= \frac{1}{4(200) - 28} \\ y &= \frac{1}{772} \end{aligned}$$

The y-values approach zero, but remain positive. The x-axis becomes the horizontal asymptote as $x \to \infty$.

3. Invariant points will occur when the y-coordinate of any ordered pair on the graph of $f(x) = 2x^2 - 8$ is equal to 1 or –1.

If $y = 1$:
$$\begin{aligned} 1 &= 2x^2 - 8 \\ 9 &= 2x^2 \\ 4.5 &= x^2 \\ \pm\sqrt{4.5} &= x \end{aligned}$$

If $y = -1$:
$$\begin{aligned} -1 &= 2x^2 - 8 \\ 7 &= 2x^2 \\ 3.5 &= x^2 \\ \pm\sqrt{3.5} &= x \end{aligned}$$

Therefore, $\left(-\sqrt{4.5}, 1\right)$, $\left(\sqrt{4.5}, 1\right)$, $\left(-\sqrt{3.5}, -1\right)$, and $\left(\sqrt{3.5}, -1\right)$ are invariant points.

Step 1
Determine the vertical asymptotes.

Set the denominator equal to zero, and solve for x.

$$\begin{aligned} 0 &= 2x^2 - 8 \\ 8 &= 2x^2 \\ 4 &= x^2 \\ \pm\sqrt{4} &= x \\ \pm 2 &= x \end{aligned}$$

The vertical asymptotes are at $x = -2$ and $x = 2$.

Step 2
Determine the behaviour of the graph on the left of $x = -2$.

Calculate values of y as x approaches -2 from the left.

If $x = -2.01$:

$$y = \frac{1}{2x^2 - 8}$$
$$y = \frac{1}{2(-2.01)^2 - 8}$$
$$y = \frac{1}{0\ 0802}$$
$$y \doteq 12.468\ 827\ 93$$

If $x = -2.001$:

$$y = \frac{1}{2x^2 - 8}$$
$$y = \frac{1}{2(-2.001)^2 - 8}$$
$$y = \frac{1}{0\ 008\ 002}$$
$$y \doteq 124.968\ 757\ 8$$

The y-values get larger and larger toward positive infinity.

Step 3
Determine the behaviour of the graph on the right of $x = -2$.

Calculate values of y as x approaches -2 from the right.

If $x = -1.99$:

$$y = \frac{1}{2x^2 - 8}$$
$$y = \frac{1}{2(-1.99)^2 - 8}$$
$$y = \frac{1}{-0\ 0798}$$
$$y \doteq -12.531\ 328\ 32$$

If $x = -1.9999$:

$$y = \frac{1}{2x^2 - 8}$$
$$y = \frac{1}{2(-1.999)^2 - 8}$$
$$y = \frac{1}{-0\ 007\ 998}$$
$$y \doteq -125.031\ 257\ 8$$

The y-values get smaller and smaller toward negative infinity.

Step 4
Determine the behaviour of the graph on the left of $x = 2$.

Calculate values of y as x approaches 2 from the left.

If $x = 1.99$:

$$y = \frac{1}{2x^2 - 8}$$
$$y = \frac{1}{2(1.99)^2 - 8}$$
$$y = \frac{1}{-0\ 0798}$$
$$y \doteq -12.531\ 328\ 32$$

If $x = 1.999$:

$$y = \frac{1}{2x^2 - 8}$$
$$y = \frac{1}{2(1.999)^2 - 8}$$
$$y = \frac{1}{-0\ 007\ 998}$$
$$y \doteq -125.031\ 257\ 8$$

The y-values get smaller and smaller toward negative infinity.

Step 5
Determine the behaviour of the graph on the right of $x = 2$.

Calculate values of y as x approaches 2 from the right.

If $x = 2.01$:

$$y = \frac{1}{2x^2 - 8}$$
$$y = \frac{1}{2(2.01)^2 - 8}$$
$$y = \frac{1}{0\ 0802}$$
$$y \doteq 12.468\ 827\ 93$$

If $x = 2.001$:

$$y = \frac{1}{2x^2 - 8}$$
$$y = \frac{1}{2(2.001)^2 - 8}$$
$$y = \frac{1}{0\ 008\ 002}$$
$$y \doteq 124.968\ 757\ 8$$

The y-values get larger and larger toward positive infinity.

Step 6
Determine the behaviour of the graph as $x \to -\infty$.

If $x = -10$:

$$y = \frac{1}{2x^2 - 8}$$
$$y = \frac{1}{2(-10)^2 - 8}$$
$$y = \frac{1}{192}$$

If $x = -200$:

$$y = \frac{1}{2x^2 - 8}$$
$$y = \frac{1}{2(-200)^2 - 8}$$
$$y = \frac{1}{79\ 992}$$

The y-values approach zero, but remain positive. The x-axis becomes a horizontal asymptote as $x \to -\infty$.

Step 7
Determine the behaviour of the graph as $x \to \infty$.

If $x = 10$:

$$y = \frac{1}{2x^2 - 8}$$
$$y = \frac{1}{2(10)^2 - 8}$$
$$y = \frac{1}{192}$$

If $x = 200$:

$$y = \frac{1}{2x^2 - 8}$$
$$y = \frac{1}{2(200)^2 - 8}$$
$$y = \frac{1}{79\ 992}$$

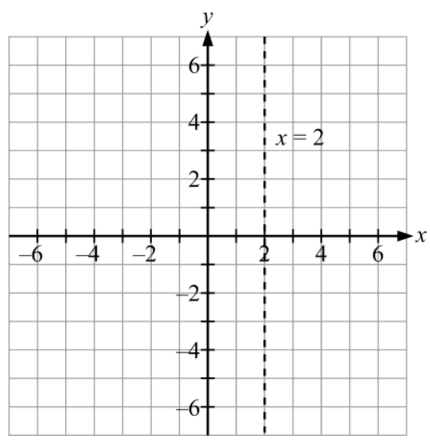
The y-values approach zero, but remain positive. The x-axis becomes a horizontal asymptote as $x \to \infty$.

Lesson 3—Graphing Reciprocal Functions

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Determine the vertical asymptotes of the function $y = \dfrac{1}{-2x+4}$.

Because the graph of $f(x) = -2x+4$ has an x-intercept at (2, 0), the graph of $y = \dfrac{1}{-2x+4}$ will have a vertical asymptote at $x = 2$. Sketch the asymptote as a dotted line.



Step 2
Find the invariant points.

The invariant points will occur when the y-coordinate of any ordered pair on the graph of $f(x) = -2x+4$ is equal to 1 or –1.

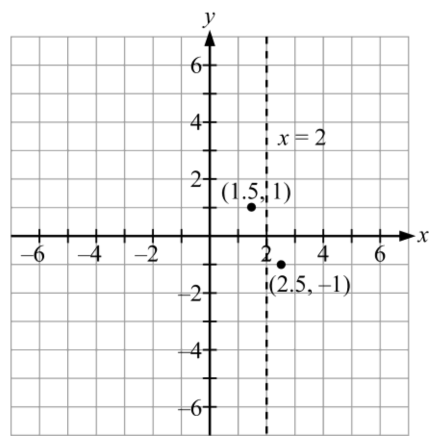
If $y = 1$:

$$1 = -2x+4$$
$$-3 = -2x$$
$$\frac{3}{2} = x$$
$$1.5 = x$$

If $y = -1$:

$$-1 = -2x+4$$
$$-5 = -2x$$
$$\frac{5}{2} = x$$
$$2.5 = x$$

The invariant points are (1.5, 1) and (2.5, –1).



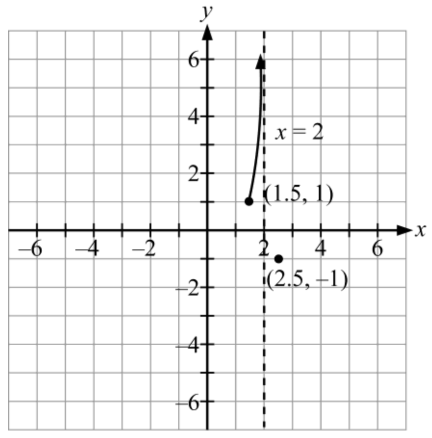
Step 3
Determine the y-intercept of the graph of $y = \dfrac{1}{-2x+4}$.

For $y = f(x) = -2x+4$, solve for y when $x = 0$.

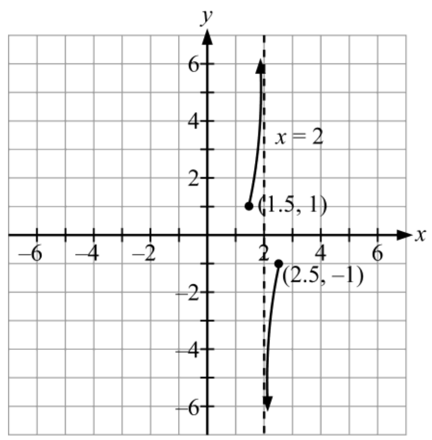
$$y = -2x+4$$
$$y = -2(0)+4$$
$$y = 4$$

The y-intercept of the graph of $f(x) = -2x+4$ is (0, 4). Therefore, the y-intercept of the graph of $y = \dfrac{1}{-2x+4}$ is $\left(0, \dfrac{1}{4}\right)$ or (0, 0.25).

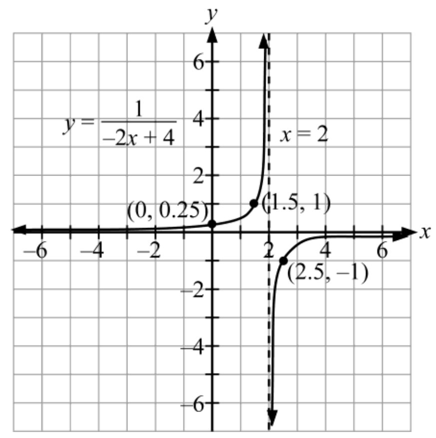
Step 4
From the invariant point (1.5, 1), show the graph of $y = \dfrac{1}{-2x+4}$ increasing as x approaches the vertical asymptote $x = 2$ from the left.



Step 5
From the invariant point (2.5, –1), show the graph of $y = \dfrac{1}{-2x+4}$ decreasing as x approaches the vertical asymptote $x = 2$ from the right.



Step 6
From the invariant point (1.5, 1) on the left, show the graph of $y = \dfrac{1}{-2x+4}$ approaching the horizontal asymptote $y = 0$ from above the x-axis and passing through the y-intercept (0, 0.25).
From the invariant point (2.5, –1) on the right, show the graph of $y = \dfrac{1}{-2x+4}$ approaching the horizontal asymptote $y = 0$ from below the x-axis.



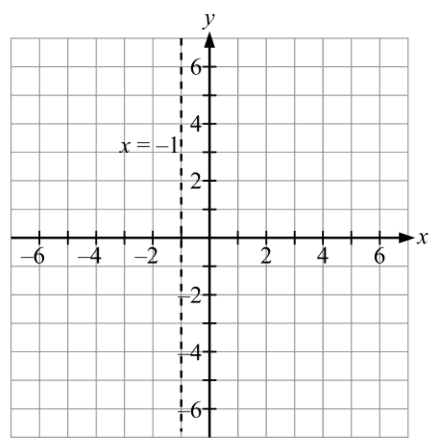
PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Determine the vertical asymptote of the function $y = \dfrac{1}{2x+2}$.

Solve for x when $2x+2=0$.

$$2x+2=0$$
$$2x=-2$$
$$x=-1$$

Therefore, the vertical asymptote occurs at $x=-1$.
Sketch the vertical asymptote as a dotted line.



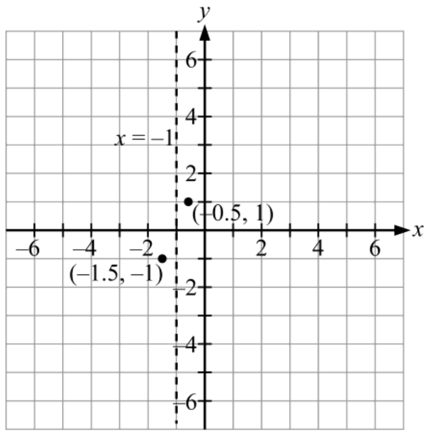
Step 2
Find the invariant points.

Invariant points will occur when the y-coordinate of any ordered pair on the graph of $f(x) = 2x + 2$ is equal to 1 or –1.

If $y = 1$:
$$1 = 2x + 2$$
$$-1 = 2x$$
$$-\frac{1}{2} = x$$
$$-0.5 = x$$

If $y = -1$:
$$-1 = 2x + 2$$
$$-3 = 2x$$
$$-\frac{3}{2} = x$$
$$-1.5 = x$$

Therefore, (–0.5, 1) and (–1.5, –1) are invariant points. Plot these points as shown.



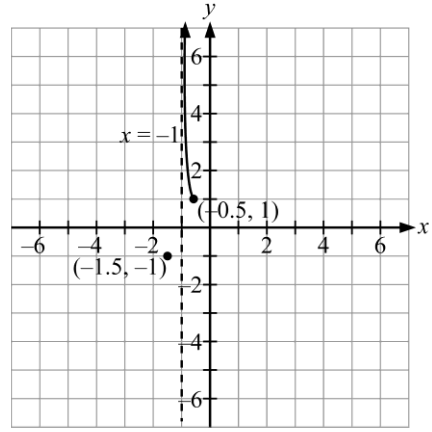
Step 3
Determine the y-intercept of the graph of $y = \frac{1}{2x + 2}$.

For $y = f(x) = 2x + 2$, solve for y when $x = 0$.
$$y = 2x + 2$$
$$y = 2(0) + 2$$
$$y = 2$$

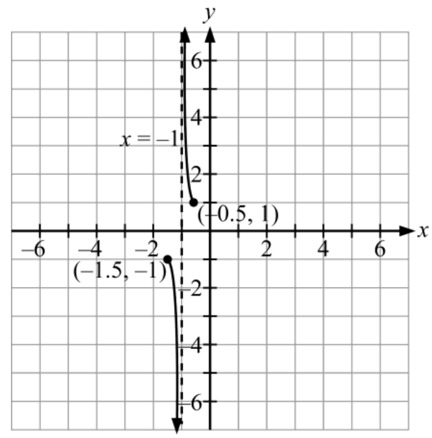
The y-intercept of the graph of $f(x) = 2x + 2$ is (0, 2).

Therefore, the y-intercept of the graph of $y = \frac{1}{2x + 2}$ is $\left(0, \frac{1}{2}\right)$ or (0, 0.5).

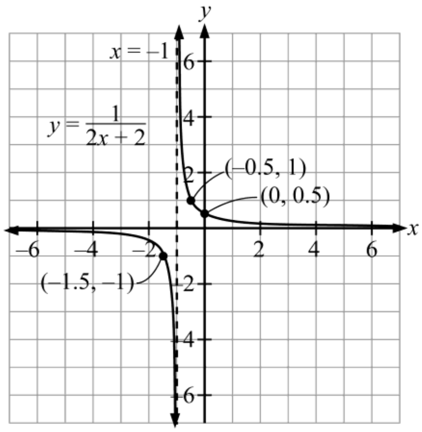
Step 4
From the invariant point (–0.5, 1), show the graph of $y = \frac{1}{2x + 2}$ increasing as x approaches the vertical asymptote $x = -1$ from the right.



Step 5
From the invariant point (–1.5, –1), show the graph of $y = \frac{1}{2x + 2}$ decreasing as x approaches the vertical asymptote $x = -1$ from the left.

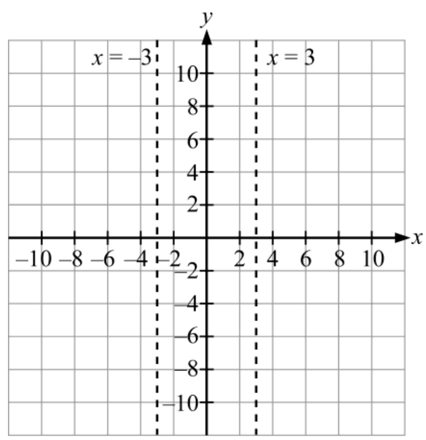


Step 6
From the invariant point (–1.5, –1) on the left, show the graph of $y=\frac{1}{2x+2}$ approaching the horizontal asymptote $y=0$ from below the x-axis. From the invariant point (0.5, 1) on the right, show the graph of $y=\frac{1}{2x+2}$ approaching the horizontal asymptote $y=0$ from above the x-axis and passing through the y-intercept (0, 0.5).



2. **Step 1**
Determine the vertical asymptotes of the function $y=\frac{1}{x^2-9}$.

The x-intercepts of the graph of $f(x)=x^2-9$ are (–3, 0) and (3, 0). Therefore, the graph of $y=\frac{1}{x^2-9}$ will have vertical asymptotes at $x=-3$ and $x=3$. Sketch the asymptotes as dotted lines.

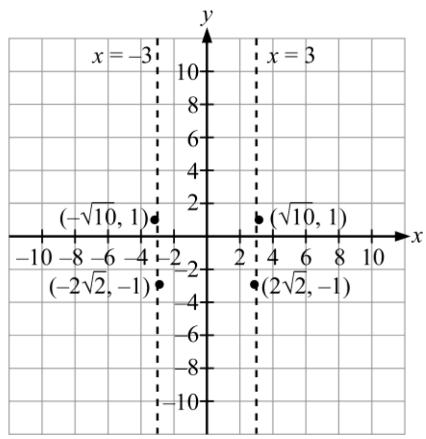


Step 2
Find the invariant points.

The invariant points will occur when the y-coordinate of any ordered pair on the graph of $f(x)=x^2-9$ is equal to 1 or –1.

If y = 1:	If y = –1:
$1=x^2-9$	$-1=x^2-9$
$10=x^2$	$8=x^2$
$\pm\sqrt{10}=x$	$\pm\sqrt{8}=x$
	$\pm2\sqrt{2}=x$

The invariant points are $\left(-\sqrt{10},1\right)$, $\left(\sqrt{10},1\right)$, $\left(-2\sqrt{2},-1\right)$, and $\left(2\sqrt{2},-1\right)$. Plot the points as shown.



Step 3
Determine the y-intercept of $y=\frac{1}{x^2-9}$.

For $y=f(x)=x^2-9$, solve for y when $x=0$.

$$y=x^2-9$$
$$y=(0)^2-9$$
$$y=-9$$

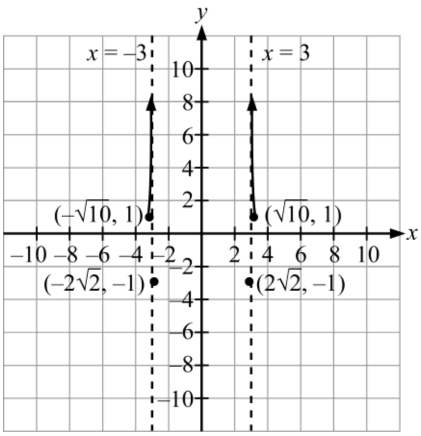
The y-intercept of the graph of $f(x)=x^2-9$ is (0, –9).

Therefore, the y-intercept of the graph of $y=\frac{1}{x^2-9}$ is $\left(0,-\frac{1}{9}\right)$.

Step 4

From the invariant point $\left(-\sqrt{10}, 1\right)$, show the graph of $y=\dfrac{1}{x^2-9}$ increasing as x approaches the vertical asymptote $x=-3$ from the left.

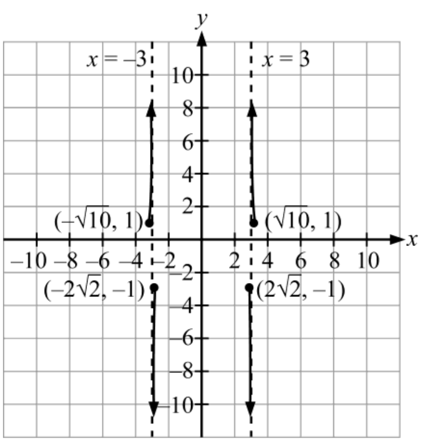
From the invariant point $\left(\sqrt{10}, 1\right)$, show the graph of $y=\dfrac{1}{x^2-9}$ increasing as x approaches the vertical asymptote $x = 3$ from the right.



Step 5

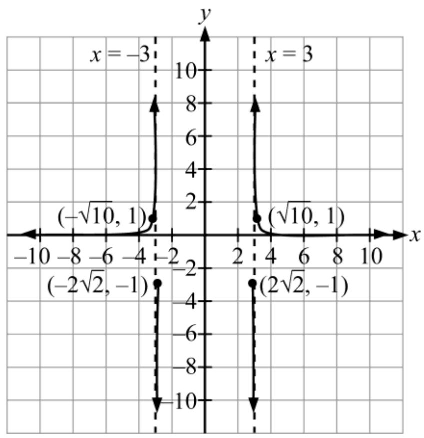
From the invariant point $\left(-2\sqrt{2}, -1\right)$, show the graph of $y=\dfrac{1}{x^2-9}$ decreasing as x approaches the vertical asymptote $x=-3$ from the right.

From the invariant point $\left(2\sqrt{2}, -1\right)$, show the graph of $y=\dfrac{1}{x^2-9}$ decreasing as x approaches the vertical asymptote $x=3$ from the left.



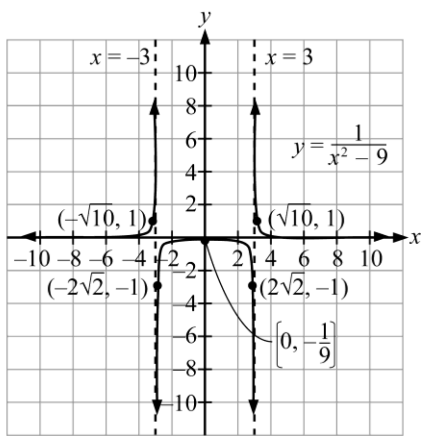
Step 6

From the invariant point $\left(-\sqrt{10},1\right)$ on the left and the invariant point $\left(\sqrt{10},1\right)$ on the right, show the graph of $y=\dfrac{1}{x^2-9}$ approaching the horizontal asymptote $y=0$ from above the x-axis.



Step 7

From the invariant points $\left(-2\sqrt{2},-1\right)$ and $\left(2\sqrt{2},-1\right)$, show the graph of $y=\dfrac{1}{x^2-9}$ increasing to the y-intercept $\left(0,-\dfrac{1}{9}\right)$.



Practice Test

ANSWERS AND SOLUTIONS

1. The horizontal asymptote of the function is at $y = 0$.

Determine the vertical asymptotes.

Set the denominator equal to zero, and solve for x.

$$\begin{aligned}0 &= 4x^2 - 4x - 48\\0 &= 4(x^2 - x - 12)\\0 &= x^2 - x - 12\\0 &= x^2 - 4x + 3x - 12\\0 &= x(x-4) + 3(x-4)\\0 &= (x-4)(x+3)\end{aligned}$$

$$x - 4 = 0 \qquad x + 3 = 0$$
$$x = 4 \qquad x = -3$$

The vertical asymptotes are at $x = 4$ and $x = -3$.

2. A function in the form $y = \dfrac{1}{f(x)}$ has a vertical asymptote when $f(x) = 0$.

Since the vertical asymptote is at $x = 6$, then $f(6) = 0$.

Substitute 6 for x into the equation $ax - 5 = 0$, and solve for a.

$$\begin{aligned}ax + b &= 0\\a(6) - 5 &= 0\\6a &= 5\\a &= \frac{5}{6}\\a &\doteq 0.833\ 333\end{aligned}$$

To the nearest hundredth, the value of a is 0.83.

3. The points where the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ intersect are called invariant points, and they occur when the y-coordinate of any ordered pair on the graph of $y = f(x)$ is either 1 or –1. The corresponding x-coordinate can be determined by substituting 1 or –1 for y in the equation $y = f(x)$ and then solving for x.

Step 1
Substitute 1 for y in the equation $y = f(x) = x + 3$, and solve for x.

$$\begin{aligned}1 &= x + 3\\-2 &= x\end{aligned}$$

The graphs intersect at (–2, 1).

Step 2
Substitute –1 for y in the equation $y = f(x) = x + 3$, and solve for x.

$$\begin{aligned}-1 &= x + 3\\-4 &= x\end{aligned}$$

The graphs also intersect at (–4, –1).

Thus, the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ intersect at (–2, 1) and (–4, –1).

4. **B**
An invariant point for the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ will occur when the y-coordinate of any ordered pair on the graph of $f(x) = x^2 - 3x - 17$ is equal to 1, since $y > 0$.

Determine the value of the x-coordinate that will give a y-coordinate of 1.

Let the equation of the function equal 1, and solve for x.

$$\begin{aligned}x^2 - 3x - 17 &= 1\\x^2 - 3x - 18 &= 0\\(x-6)(x+3) &= 0\\x &= 6, -3\end{aligned}$$

The points are (6, 1) and (–3, 1).

5. **Step 1**
Determine the vertical asymptote.

Set the denominator equal to zero, and solve for x.

$$\begin{aligned}3x - 48 &= 0\\3x &= 48\\x &= 16\end{aligned}$$

There is a vertical asymptote at $x = 16$.

Step 2
Determine the behaviour of the graph on the left of $x = 16$.

Calculate values of y as x approaches 16 from the left.

If $x = 15.99$:

$$y = \frac{1}{3x - 48}$$
$$y = \frac{1}{3(15.99) - 48}$$
$$y = \frac{1}{-0.0\overline{3}}$$
$$y = -33.\overline{33}$$

If $x = 15.9999$:

$$y = \frac{1}{3x - 48}$$
$$y = \frac{1}{3(15.9999) - 48}$$
$$y = \frac{1}{-0.000\overline{3}}$$
$$y = -3\ 333.\overline{33}$$

The y-values get smaller and smaller toward negative infinity.

Step 3
Determine the behaviour of the graph on the right of $x = 16$.

Calculate values of y as x approaches 16 from the right.

If $x = 16.01$:

$$y = \frac{1}{3x - 48}$$
$$y = \frac{1}{3(16.01) - 48}$$
$$y = \frac{1}{0.0\overline{3}}$$
$$y = 33.\overline{33}$$

If $x = 16.0001$:

$$y = \frac{1}{3x - 48}$$
$$y = \frac{1}{3(16.0001) - 48}$$
$$y = \frac{1}{0.000\overline{3}}$$
$$y = 3\ 333.\overline{33}$$

The y-values get larger and larger toward positive infinity.

Step 4
Determine the behaviour of the graph as $x \to -\infty$.

If $x = -20$:

$$y = \frac{1}{3x - 48}$$
$$y = \frac{1}{3(-20) - 48}$$
$$y = \frac{1}{-108}$$

If $x = -200$:

$$y = \frac{1}{3x - 48}$$
$$y = \frac{1}{3(-200) - 48}$$
$$y = \frac{1}{-648}$$

The y-values approach zero, but remain negative. The x-axis becomes a horizontal asymptote as $x \to -\infty$.

Step 5
Determine the behaviour of the graph as $x \to \infty$.

If $x = 20$:

$$y = \frac{1}{3x - 48}$$
$$y = \frac{1}{3(20) - 48}$$
$$y = \frac{1}{12}$$

If $x = 200$:

$$y = \frac{1}{3x - 48}$$
$$y = \frac{1}{3(200) - 48}$$
$$y = \frac{1}{552}$$

The y-values approach zero, but remain positive. The x-axis becomes the horizontal asymptote as $x \to \infty$.

6. Asymptotes on the graph of $y = \frac{1}{f(x)}$ occur when $f(x) = 0$; these are the x-intercepts of the graph of $y = f(x)$.

The x-intercepts of the graph of $y = f(x)$ are (–3, 0) and (1, 0). Therefore, the graph of $y = \frac{1}{f(x)}$ will have vertical asymptotes at $x = -3$ and $x = 1$.

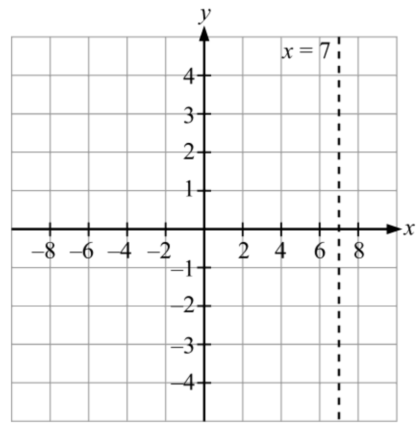
7. **Step 1**
Determine the vertical asymptotes of the function $y = \frac{1}{x - 7}$.

Solve for x when $x - 7 = 0$.

$$x - 7 = 0$$
$$x = 7$$

Therefore, the vertical asymptote is at $x = 7$.

Sketch the asymptote as a dotted line.



Step 2
Find the invariant points.

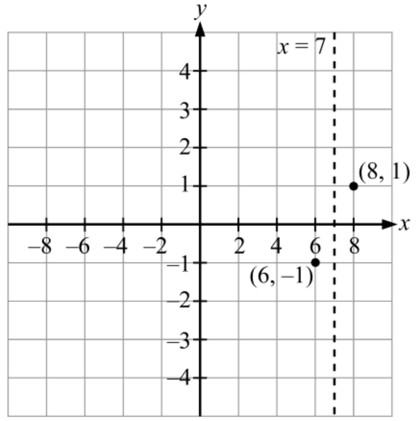
The invariant points will occur when the y-coordinate of any ordered pair on the graph of $f(x) = x - 7$ is equal to 1 or –1.

If $y = 1$:
$1 = x - 7$
$8 = x$

If $y = -1$:
$-1 = x - 7$
$6 = x$

The invariant points are (8, 1) and (6, –1).

Plot the points as shown.

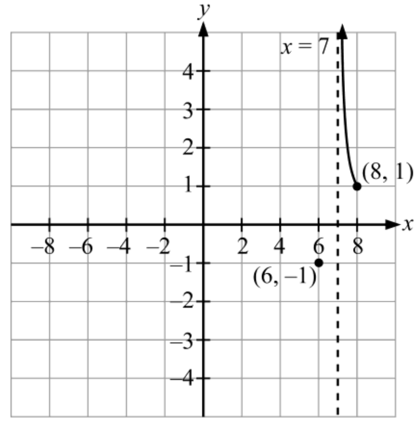


Step 3
Determine the y-intercept of the graph of $y = \frac{1}{x-7}$.

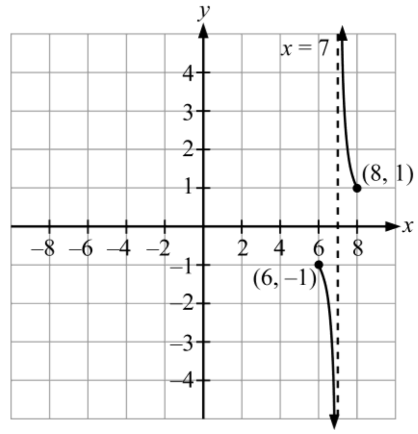
For $y = f(x) = x - 7$, solve for y when $x = 0$.
$y = x - 7$
$y = (0) - 7$
$y = -7$

The y-intercept of the graph of $f(x) = x - 7$ is (0, –7). Therefore, the y-intercept of the graph of $y = \frac{1}{x-7}$ is $\left(0, -\frac{1}{7}\right)$.

Step 4
From the invariant point (8, 1), show the graph of $y = \frac{1}{x-7}$ increasing as x approaches the vertical asymptote $x = 7$ from the right.

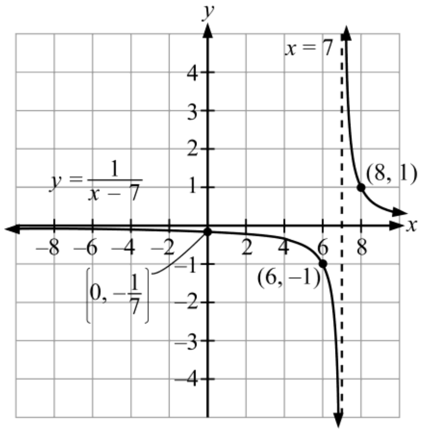


Step 5
From the invariant point (6, –1), show the graph of $y = \frac{1}{x-7}$ decreasing as x approaches the vertical asymptote $x = 7$ from the left.



Step 6
From the invariant point (6, –1) on the left, show the graph of $y = \dfrac{1}{x-7}$ approaching the horizontal asymptote $y = 0$ from below the x-axis and passing through the y-intercept $\left(0, -\dfrac{1}{7}\right)$.

From the invariant point (8, 1) on the right, show the graph of $y = \dfrac{1}{x-7}$ approaching the horizontal asymptote $y = 0$ from above the x-axis.



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NOTES