S N A P

Student Notes and Problems

ANSWERS AND SOLUTIONS

CASTLE ROCK
RESEARCH CORP

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CASTLE ROCK
RESEARCH CORP

Dedicated to the memory of Dr. V. S. Rao

TRANSFORMATIONS OF FUNCTIONS

Lesson 1—Horizontal and Vertical Translations

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Since the graph of $y=|x|$ is translated 8 units right, replace x with $x-8$.

$y=|x|$
$y=|x-8|$

Therefore, the equation of the transformed graph is $y=|x-8|$.

2. Since the graph of $y=2x^3+5$ is translated 4 units down, replace y with $y-(-4)$.

$$\begin{aligned} y &= 2x^3+5 \\ y-(-4) &= 2x^3+5 \\ y+4 &= 2x^3+5 \\ y &= 2x^3+1 \end{aligned}$$

Therefore, the equation of the transformed graph is $y=2x^3+1$.

3. Step 1
Determine how to obtain the graph of $y-4=f(x-3)$ from the graph of $y=f(x)$.

The equation $y-4=f(x-3)$ is of the form $y-k=f(x-h)$, where $h=3$ and $k=4$. Therefore, the graph of $y-4=f(x-3)$ or $y=f(x-3)+4$ is formed from $y=f(x)$ by a horizontal translation of 3 units right and a vertical translation 4 units up.

Step 2
Identify points on the graph of $y=f(x)$.

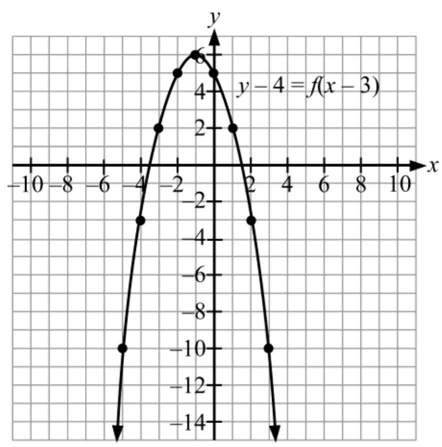
Points on the graph of $y=f(x)$ are (–8, –14), (–7, –7), (–6, –2), (–5, 1), (–4, 2), (–3, 1), (–2, –2), (–1, –7), and (0, –14).

Step 3
Transform points on the graph of $y=f(x)$ to obtain points on the graph of $y-4=f(x-3)$.

Increase the *x*- and *y*-coordinates by 3 and 4 units, respectively.

$$\begin{aligned} (-8+3, -14+4) &\to (-5, -10) \\ (-7+3, -7+4) &\to (-4, -3) \\ (-6+3, -2+4) &\to (-3, 2) \\ (-5+3, 1+4) &\to (-2, 5) \\ (-4+3, 2+4) &\to (-1, 6) \\ (-3+3, 1+4) &\to (0, 5) \\ (-2+3, -2+4) &\to (1, 2) \\ (-1+3, -7+4) &\to (2, -3) \\ (0+3, -14+4) &\to (3, -10) \end{aligned}$$

Step 4
Plot and join the transformed points.



PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. Step 1
Add 15 to both sides of the equation.

$$\begin{aligned} y+1 &= f(x)-15 \\ y+16 &= f(x) \end{aligned}$$

Step 2
Identify the transformations on the graph of $y=f(x)$ to obtain the graph of $y+16=f(x)$.

The equation $y+16=f(x)$ is in the form $y-k=f(x)$, where $k=-16$. Therefore, the graph of $y+1=f(x)-15$, or $y+16=f(x)$, is formed from $y=f(x)$ by a vertical translation of 16 units down.

2. **Step 1**
Determine the *y*-intercept on the graph of $y = -18x - 7$.

The *y*-intercept occurs when $x = 0$.

$$y = -18x - 7$$
$$y = -18(0) - 7$$
$$y = -7$$

The *y*-intercept is (0, –7).

Step 2
Transform the *y*-intercept.

Since the graph of $y = -18x - 7$ is translated 7 units left and 3 units down, every point (*x*, *y*) on the graph of $y = -18x - 7$ will be transformed to the point (*x* – 7, *y* – 3).

The *y*-intercept on the graph of $y = -18x - 7$ will transform to (0 – 7, –7 – 3) = (–7, –10).

3. **Step 1**
Apply the horizontal translation.

Since the graph of $y = \sqrt{(x-8)} - 12$ is translated 9 units left, replace *x* with $x - (-9)$.

$$y = \sqrt{(x-8)} - 12$$
$$y = \sqrt{((x-(-9))-8)} - 12$$
$$y = \sqrt{((x+9)-8)} - 12$$
$$y = \sqrt{(x+1)} - 12$$

Step 2
Apply the vertical translation.
Since the graph of $y = \sqrt{(x-8)} - 12$ is translated 10 units down, replace *y* in $y = \sqrt{(x+1)} - 12$ with $y - (-10)$.

$$y = \sqrt{(x+1)} - 12$$
$$y - (-10) = \sqrt{(x+1)} - 12$$

Step 3
Isolate *y*.

$$y - (-10) = \sqrt{(x+1)} - 12$$
$$y + 10 = \sqrt{(x+1)} - 12$$
$$y = \sqrt{(x+1)} - 22$$

Therefore, the equation of the transformed graph is $y = \sqrt{(x+1)} - 22$.

4. **Step 1**
Determine how to obtain the graph of $y + 3 = f(x-2)$ from the graph of $y = f(x)$.

The equation $y + 3 = f(x-2)$ is of the form $y - k = f(x-h)$, where $h = 2$ and $k = -3$.
Therefore, the graph of $y + 3 = f(x-2)$, or $y = f(x-2) - 3$, is formed from $y = f(x)$ by a horizontal translation 2 units right and a vertical translation 3 units down.

Step 2
Identify points on the graph of $y = f(x)$.

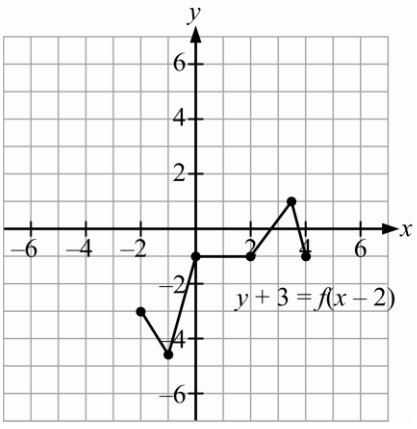
Points on the graph of $y = f(x)$ are (–4, 0), (–3, –1.5), (–2, 2), (0, 2), (1.5, 4), and (2, 2).

Step 3
Transform points on the graph of $y = f(x)$ to obtain points on the graph of $y + 3 = f(x-2)$.

Increase the *x*-coordinates by 2 units and decrease the *y*-coordinates by 3 units.

$$(-4+2,\ 0-3) \to (-2,\ -3)$$
$$(-3+2,\ -1.5-3) \to (-1,\ -4.5)$$
$$(-2+2,\ 2-3) \to (0,\ -1)$$
$$(0+2,\ 2-3) \to (2,\ -1)$$
$$(1.5+2,\ 4-3) \to (3.5,\ 1)$$
$$(2+2,\ 2-3) \to (4,\ -1)$$

Step 4
Plot and join the transformed points.



Lesson 2—Horizontal and Vertical Stretches

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. When the graph of $y = f(x)$ is transformed to the graph of $y = 5f(x)$, the graph of $y = f(x)$ will be stretched vertically by a factor of 5. Thus, every point (x, y) on the graph of $y = f(x)$ will be transformed to the point $(x, 5y)$.

The corresponding point on the graph of $y = 5f(x)$ is $(-4, 10\times 5) = (-4, 50)$.

2. Since the graph of $y = 11x^2 + 1$ is stretched horizontally by a factor of 6, the equation of the transformed graph can be obtained by replacing x with $\frac{1}{6}x$ (the reciprocal of 6 is $\frac{1}{6}$) in the equation $y = 11x^2 + 1$.

$$y = 11x^2 + 1$$
$$y = 11\left(\frac{1}{6}x\right)^2 + 1$$
$$y = 11\left(\frac{1}{36}x^2\right) + 1$$
$$y = \frac{11}{36}x^2 + 1$$

Therefore, the equation of the transformed graph is $y = \frac{11}{36}x^2 + 1$.

3. **Step 1**
Determine how to obtain the graph of $y = 2f\left(\frac{1}{2}x\right)$ from the graph of $y = f(x)$.

The equation $y = 2f\left(\frac{1}{2}x\right)$ is of the form $y = af(bx)$, where $a = 2$ and $b = \frac{1}{2}$.

Therefore, the graph of $y = 2f\left(\frac{1}{2}x\right)$ is formed from $y = f(x)$ by a horizontal stretch factor of 2 and a vertical stretch factor of 2.

Step 2
Identify points on the graph of $y = f(x)$.

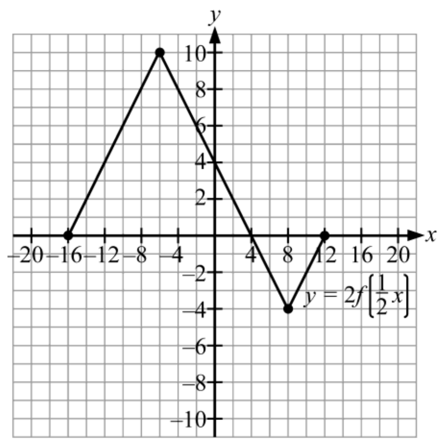
Points on the graph of $y = f(x)$ are (–8, 0), (–3, 5), (4, –2), and (6, 0).

Step 3
Transform points on the graph of $y = f(x)$ to obtain points on the graph of $y = 2f\left(\frac{1}{2}x\right)$.

Multiply the *x*- and *y*-coordinates of $y = f(x)$ by 2.

$$(-8\times 2,\ 0\times 2) \to (-16,\ 0)$$
$$(-3\times 2,\ 5\times 2) \to (-6,\ 10)$$
$$(4\times 2,\ -2\times 2) \to (8,\ -4)$$
$$(6\times 2,\ 0\times 2) \to (12,\ 0)$$

Step 4
Plot and join the transformed points.



PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Divide both sides of the equation by 4.

$$4y = f(3x)$$
$$y = \frac{f(3x)}{4}$$
$$y = \frac{1}{4}f(3x)$$

Step 2
Identify the transformations on the graph of $y = f(x)$ to obtain the graph of $y = \frac{1}{4}f(3x)$.

The equation of $y = \frac{1}{4}f(3x)$ is in the form $y = af(bx)$, where $a = \frac{1}{4}$ and $b = 3$. Therefore, the graph of $y = \frac{1}{4}f(3x)$ is formed from $y = f(x)$ by a vertical stretch factor of $\frac{1}{4}$ and a horizontal stretch factor of $\frac{1}{3}$.

2. **Step 1**
Apply the horizontal stretch.

Since the graph of $y = 3x^2 + 2x$ is horizontally stretched by a factor of $\frac{7}{2}$, replace *x* with $\frac{2}{7}x$ and simplify.

$$y = 3x^2 + 2x$$
$$y = 3\left(\frac{2}{7}x\right)^2 + 2\left(\frac{2}{7}x\right)$$
$$y = 3\left(\frac{4}{49}x^2\right) + \frac{4}{7}x$$
$$y = \frac{12}{49}x^2 + \frac{4}{7}x$$

Step 2
Apply the vertical translation.

Since the graph of $y = 3x^2 + 2x$ is vertically stretched by a factor of 9, replace *y* in $y = \frac{12}{49}x^2 + \frac{4}{7}x$ with $\frac{1}{9}y$.

$$y = \frac{12}{49}x^2 + \frac{4}{7}x$$
$$\frac{1}{9}y = \frac{12}{49}x^2 + \frac{4}{7}x$$

Step 3
Isolate *y*, and simplify.

$$\frac{1}{9}y = \frac{12}{49}x^2 + \frac{4}{7}x$$
$$9\left(\frac{1}{9}y\right) = 9\left(\frac{12}{49}x^2 + \frac{4}{7}x\right)$$
$$y = \frac{108}{49}x^2 + \frac{36}{7}x$$

Therefore, the equation of the transformed graph is $y = \frac{108}{49}x^2 + \frac{36}{7}x$.

3. **Step 1**
Determine how to obtain the graph of $y = \frac{3}{2}f\left(\frac{1}{2}x\right)$ from the graph of $y = f(x)$.

The equation $y = \frac{3}{2}f\left(\frac{1}{2}x\right)$ is of the form $y = af(bx)$, where $a = \frac{3}{2}$ and $b = \frac{1}{2}$. Therefore, the graph of $y = \frac{3}{2}f\left(\frac{1}{2}x\right)$ is formed from $y = f(x)$ by a horizontal stretch factor of 2 and a vertical stretch factor of $\frac{3}{2}$.

Step 2
Identify points on the graph of $y = f(x)$.

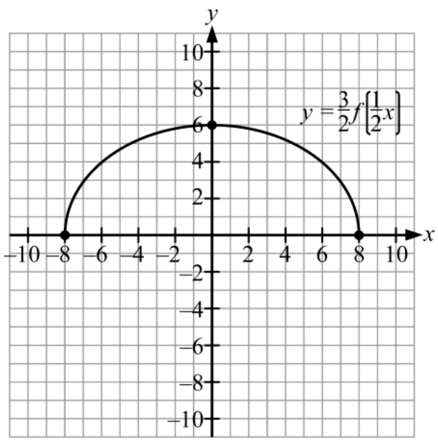
Points on the graph of $y = f(x)$ are (–4, 0), (0, 4), and (4, 0).

Step 3
Transform points on the graph of $y = f(x)$ to obtain points on the graph of $y = \frac{3}{2}f\left(\frac{1}{2}x\right)$.

Multiply the *x*-coordinates by 2 and the *y*-coordinates by $\frac{3}{2}$ or 1.5.

$$(-4\times 2,\ 0\times 1.5) \to (-8,\ 0)$$
$$(0\times 2,\ 4\times 1.5) \to (0,\ 6)$$
$$(4\times 2,\ 0\times 1.5) \to (8,\ 0)$$

Step 4
Plot and join the transformed points.



4. **Step 1**
Determine the *x*-intercepts on the graph of $y = x^2 + 7x - 44$.

The *x*-intercepts occur when $y = 0$.

$$y = x^2 + 7x - 44$$
$$0 = x^2 + 7x - 44$$

Factor and solve for *x*.

$$0 = x^2 + 7x - 44$$
$$0 = (x-4)(x+11)$$
$$x - 4 = 0 \qquad x + 11 = 0$$
$$x = 4 \qquad x = -11$$

The *x*-intercepts of $y = x^2 + 7x - 44$ are (4, 0) and (–11, 0).

Step 2
Transform the *x*-intercepts.

Since the graph of $y = x^2 + 7x - 44$ is stretched horizontally by a factor of $\frac{7}{5}$ and stretched vertically by a factor of 5, every point (*x*, *y*) on the graph of $y = x^2 + 7x - 44$ will be transformed to the point $\left(\frac{7}{5}x, 5y\right)$.

The *x*-intercepts on the graph of $y = x^2 + 7x - 44$ will transform to $\left(\frac{7}{5}\times 4, 5\times 0\right) = \left(\frac{28}{5}, 0\right)$ and $\left(\frac{7}{5}\times(-11), 5\times 0\right) = \left(-\frac{77}{5}, 0\right)$.

Lesson 3—Translations and Stretches

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Rewrite the equation $y = 2f(3x+12)$ as $y = 2f(3(x+4))$. The equation $y = 2f(3(x+4))$ is of the form $y - k = af(b(x-h))$, where *a* = 2, *b* = 3, and *h* = –4. Therefore, the graph of $y = 2f(3x+12)$ is formed from $y = f(x)$ by a vertical stretch factor of 2, a horizontal stretch factor of $\frac{1}{3}$, and a horizontal translation 4 units left.

2. **a)** **Step 1**
Determine how the graph of $y + 2 = 4f(x)$ is obtained from the graph of $y = f(x)$.

The equation $y + 2 = 4f(x)$ is of the form $y - k = af(x)$, where $a = 4$ and $k = -2$. Therefore, the graph of $y + 2 = 4f(x)$ is formed from $y = f(x)$ by a vertical stretch factor of 4 and a vertical translation 2 units down.

Step 2
Identify points on the graph of $y = f(x)$.

Points on the graph of $y = f(x)$ are (–7, –4), (–2, 1), (0, 6), and (3, 7).

Step 3
Apply the vertical stretch.

Multiply the *y*-coordinates of $y = f(x)$ by 4.

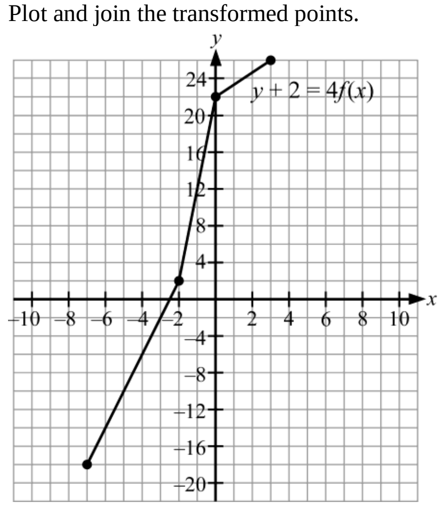
$$(-7, -4\times 4) \rightarrow (-7, -16)$$
$$(-2, 1\times 4) \rightarrow (-2, 4)$$
$$(0, 6\times 4) \rightarrow (0, 24)$$
$$(3, 7\times 4) \rightarrow (3, 28)$$

Step 4
Apply the vertical translation.

Decrease the *y*-coordinates obtained in step 3 by 2.

$$(-7, -16-2) \rightarrow (-7, -18)$$
$$(-2, 4-2) \rightarrow (-2, 2)$$
$$(0, 24-2) \rightarrow (0, 22)$$
$$(3, 28-2) \rightarrow (3, 26)$$

Step 5
Plot and join the transformed points.



b) **Step 1**
Determine how to obtain the graph of $y = f\left(\frac{1}{3}x\right) + 3$, or $y - 3 = f\left(\frac{1}{3}x\right)$, from the graph of $y = f(x)$.

The equation $y - 3 = f\left(\frac{1}{3}x\right)$ is of the form $y - k = f(bx)$, where $b = \frac{1}{3}$ and $k = 3$.

Therefore, the graph of $y = f\left(\frac{1}{3}x\right) + 3$ is formed from $y = f(x)$ by a horizontal stretch factor of 3 and a vertical translation 3 units up.

Step 2
Identify points on the graph of $y = f(x)$.

Points on the graph of $y = f(x)$ are (–7, –4), (–2, 1), (0, 6), and (3, 7).

Step 3
Apply the horizontal stretch.

Multiply the *x*-coordinates of $y = f(x)$ by 3.

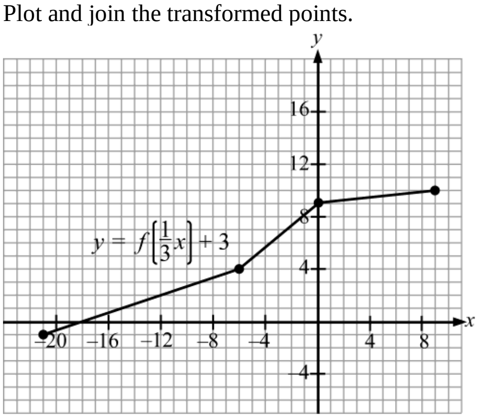
$$(-7\times 3, -4) \rightarrow (-21, -4)$$
$$(-2\times 3, 1) \rightarrow (-6, 1)$$
$$(0\times 3, 6) \rightarrow (0, 6)$$
$$(3\times 3, 7) \rightarrow (9, 7)$$

Step 4
Apply the vertical translation.

Increase the *y*-coordinates by 3.

$$(-21, -4+3) \rightarrow (-21, -1)$$
$$(-6, 1+3) \rightarrow (-6, 4)$$
$$(0, 6+3) \rightarrow (0, 9)$$
$$(9, 7+3) \rightarrow (9, 10)$$

Step 5
Plot and join the transformed points.



3. **Step 1**
Apply the vertical stretch.

Since the graph of $y = x^3 + 4$ is vertically stretched by a factor of $\frac{1}{2}$, replace *y* with $2y$.

$$y = x^3 + 4$$
$$2y = x^3 + 4$$

Step 2
Apply the vertical translation.

Since the graph of $y = x^3 + 4$ is translated 12 units up, replace *y* in $2y = x^3 + 4$ with $y - 12$.

$$2y = x^3 + 4$$
$$2(y - 12) = x^3 + 4$$

Step 3
Isolate *y*.

$$2(y - 12) = x^3 + 4$$
$$y - 12 = \frac{1}{2}x^3 + 2$$
$$y = \frac{1}{2}x^3 + 14$$

Therefore, the equation of the transformed graph is $y = \frac{1}{2}x^3 + 14$.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. The equation $y + 8 = 4f\left(\frac{1}{7}(x - 2)\right)$ is of the form $y - k = af\left(b(x - h)\right)$, where $a = 4$, $b = \frac{1}{7}$, $h = 2$, and $k = -8$.

Therefore, the graph $y + 8 = 4f\left(\frac{1}{7}(x - 2)\right)$ is formed from $y = f(x)$ by a vertical stretch factor of 4, a horizontal stretch factor of 7, a horizontal translation 2 units right, and a vertical translation 8 units down.

2. **Step 1**
Determine how to obtain the graph of
$y-3=\frac{1}{4}f\left(x+5\right)$ from the graph of $y=f\left(x\right)$.

The equation $y-3=\frac{1}{4}f\left(x+5\right)$ is of the form $y-k=af\left(x-h\right)$, where $a=\frac{1}{4}$, $h=-5$, and $k=3$. Therefore, the graph $y-3=\frac{1}{4}f\left(x+5\right)$ is formed from $y=f\left(x\right)$ by a vertical stretch factor of $\frac{1}{4}$, a horizontal translation 5 units left, and a vertical translation 3 units up.

Step 2
Apply the vertical stretch.

The vertical stretch by a factor of $\frac{1}{4}$ will transform the point (12, –1) to $\left(12, -1\times\frac{1}{4}\right)=\left(12, -\frac{1}{4}\right)$.

Step 3
Apply the horizontal and vertical translations.

The horizontal translation 5 units left and vertical translation 3 units up will transform the point $\left(12, -\frac{1}{4}\right)$ to $\left(12-5, -\frac{1}{4}+3\right)=\left(7, \frac{11}{4}\right)$.

Therefore, the corresponding transformed point on the graph of $y-3=\frac{1}{4}f\left(x+5\right)$ is $\left(7, \frac{11}{4}\right)$.

3. **Step 1**
Apply the horizontal stretch.

Replace x in the equation $y=-3x+7$ with $\frac{3}{2}x$.

$$y=-3\left(\frac{3}{2}x\right)+7$$

Step 2
Apply the horizontal translation.

Replace x in $y=-3\left(\frac{3}{2}x\right)+7$ with $x-\left(-1\right)$.

$$y=-3\left(\frac{3}{2}x\right)+7$$
$$y=-3\left(\frac{3}{2}\left(x-\left(-1\right)\right)\right)+7$$

Step 3
Apply the vertical translation.

Replace y in $y=-3\left(\frac{3}{2}\left(x-\left(-1\right)\right)\right)+7$ with $y-\left(-11\right)$.

$$y-\left(-11\right)=-3\left(\frac{3}{2}\left(x-\left(-1\right)\right)\right)+7$$

Step 4
Isolate y, and simplify.

$$\begin{aligned}y-\left(-11\right)&=-3\left(\frac{3}{2}\left(x-\left(-1\right)\right)\right)+7\\ y+11&=-3\left(\frac{3}{2}\left(x+1\right)\right)+7\\ y&=-\frac{9}{2}\left(x+1\right)-4\\ y&=-\frac{9}{2}x-\frac{9}{2}-4\\ y&=-\frac{9}{2}x-\frac{17}{2}\end{aligned}$$

Therefore, the equation of the transformed graph is $y=-\frac{9}{2}x-\frac{17}{2}$.

4. **Step 1**
Determine how to obtain the graph of
$y-5=\frac{1}{2}f\left(x-3\right)$ from the graph of $y=f\left(x\right)$.

The equation $y-5=\frac{1}{2}f\left(x-3\right)$ is of the form $y-k=af\left(b\left(x-h\right)\right)$, where $a=\frac{1}{2}$, $h=3$, and $k=5$. Therefore, the graph of $y-5=\frac{1}{2}f\left(x-3\right)$ is formed from $y=f\left(x\right)$ by a vertical stretch factor of $\frac{1}{2}$, a horizontal translation 3 units right, and a vertical translation 5 units up.

Step 2
Identify points on the graph of $y = f(x)$.

Points on the graph of $y = f(x)$ are (–4, 0), (0, 2), (0, –2), (5, 3), and (–5, –3).

Step 3
Apply the vertical stretch.

Multiply the *y*-coordinates of $y = f(x)$ by $\frac{1}{2}$.

$$\left(-4,\ 0\times\frac{1}{2}\right)\to(-4,\ 0)$$
$$\left(0,\ 2\times\frac{1}{2}\right)\to(0,\ 1)$$
$$\left(0,\ -2\times\frac{1}{2}\right)\to(0,\ -1)$$
$$\left(5,\ 3\times\frac{1}{2}\right)\to(5,\ 1.5)$$
$$\left(5,\ -3\times\frac{1}{2}\right)\to(5,\ -1.5)$$

Step 4
Apply the vertical translation.

Increase the *y*-coordinates from step 3 by 5.

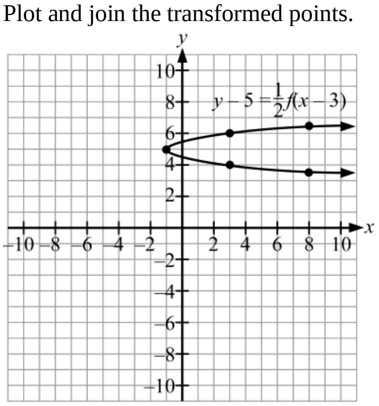
$$(-4,\ 0+5)\to(-4,\ 5)$$
$$(0,\ 1+5)\to(0,\ 6)$$
$$(0,\ -1+5)\to(0,\ 4)$$
$$(5,\ 1.5+5)\to(5,\ 6.5)$$
$$(5,\ -1.5+5)\to(5,\ 3.5)$$

Step 5
Apply the horizontal translation.

Increase the *x*-coordinates from step 4 by 3.

$$(-4+3,\ 5)\to(-1,\ 5)$$
$$(0+3,\ 6)\to(3,\ 6)$$
$$(0+3,\ 4)\to(3,\ 4)$$
$$(5+3,\ 6.5)\to(8,\ 6.5)$$
$$(5+3,\ 3.5)\to(8,\ 3.5)$$

Step 6
Plot and join the transformed points.



Lesson 4—Reflections

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Determine how to obtain the graph of $y = -f(x)$ from the graph of $y = f(x)$.

The equation $y = -f(x)$ is obtained by substituting –*y* for *y* in the equation $y = f(x)$. Therefore, the graph of $y = f(x)$ will be reflected in the *x*-axis to obtain the graph of $y = -f(x)$.

Step 2
Identify points on the graph of $y = f(x)$.

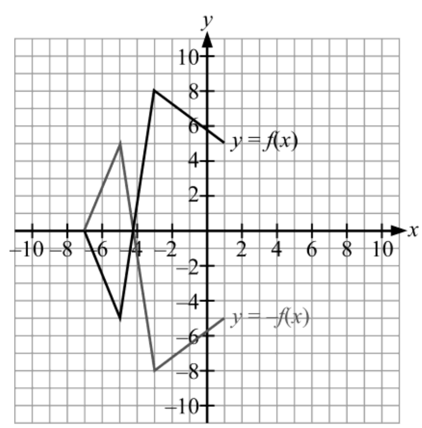
Points on the graph of $y = f(x)$ are (–7, 0), (–5, –5), (–3, 8), and (1, 5).

Step 3
Transform points on the graph of $y = f(x)$ to obtain points on the graph of $y = -f(x)$.

Multiply the *y*-coordinates of $y = f(x)$ by –1.

$$(-7,\ 0\times(-1))\to(-7,\ 0)$$
$$(-5,\ -5\times(-1))\to(-5,\ 5)$$
$$(-3,\ 8\times(-1))\to(-3,\ -8)$$
$$(1,\ 5\times(-1))\to(1,\ -5)$$

Step 4
Plot and join the transformed points.



2. **Step 1**
Determine how to obtain the equation of the transformed graph from the equation $y=\left|\frac{1}{4}x\right|+12x.$

Since the graph is reflected in the *x*-axis, the equation of the transformed graph is obtained by replacing *y* with –*y* in the equation $y=\left|\frac{1}{4}x\right|+12x.$

$$y=\left|\frac{1}{4}x\right|+12x$$
$$-y=\left|\frac{1}{4}x\right|+12x$$

Step 2
Isolate *y*.

$$-y=\left|\frac{1}{4}x\right|+12x$$
$$y=-\left(\left|\frac{1}{4}x\right|+12x\right)$$
$$y=-\left|\frac{1}{4}x\right|-12x$$

Therefore, the equation of the transformed graph is $y=-\left|\frac{1}{4}x\right|-12x.$

3. **Step 1**
Determine how to obtain the graph of $y=f(-x)$ from the graph of $y=f(x)$.

The equation $y=f(-x)$ is obtained by substituting –*x* for *x* in the equation $y=f(x)$. Therefore, the graph of $y=f(x)$ will be reflected in the *y*-axis to obtain the graph of $y=f(-x)$.

Step 2
Identify points on the graph of $y=f(x)$.

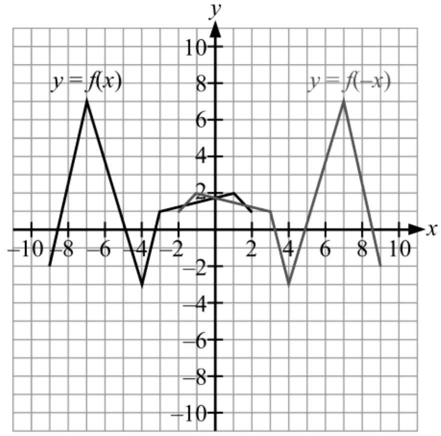
Points on the graph of $y=f(x)$ are (–9, –2), (–7, 7), (–4, –3), (–3, 1), (1, 2), and (2, 1).

Step 3
Transform points on the graph of $y=f(x)$ to obtain points on the graph of $y=f(-x)$.

Multiply the *x*-coordinates of $y=f(x)$ by –1.

$$(-9\times(-1),-2)\rightarrow(9,-2)$$
$$(-7\times(-1),7)\rightarrow(7,7)$$
$$(-4\times(-1),-3)\rightarrow(4,-3)$$
$$(-3\times(-1),1)\rightarrow(3,1)$$
$$(1\times(-1),2)\rightarrow(-1,2)$$
$$(2\times(-1),1)\rightarrow(-2,1)$$

Step 4
Plot and join the transformed points.



4. Since the transformed graph is a reflection in the *y*-axis of the graph $y=|x+6|+3$, the equation of the transformed graph is obtained by replacing *x* with –*x* in the equation $y=|x+6|+3$.

$$y=|x+6|+3$$
$$y=|(-x)+6|+3$$
$$y=|-x+6|+3$$

Therefore, the equation of the transformed graph is $y=|-x+6|+3$.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. The graph of $y=-f(x)$ is a reflection in the *x*-axis of the graph of $y=f(x)$. Therefore, a reflection in the *x*-axis will transform the point (5, –3) to $(5,-3\times(-1))=(5,3)$.

2. **Step 1**
Determine the equation of the transformed graph from the equation $y = x^2 + 2x - 9$.

Since the graph is reflected in the y-axis, the equation of the transformed graph is obtained by replacing x with $-x$ in the equation $y = x^2 + 2x - 9$.

$$y = x^2 + 2x - 9$$
$$y = (-x)^2 + 2(-x) - 9$$

Step 2
Simplify $y = (-x)^2 + 2(-x) - 9$.

$$y = (-x)^2 + 2(-x) - 9$$
$$y = x^2 - 2x - 9$$

Therefore, the equation of the transformed graph is $y = x^2 - 2x - 9$.

3. **Step 1**
Determine how to obtain the graph of $y = f(-x)$ from the graph of $y = f(x)$.

The equation $y = f(-x)$ is obtained by substituting $-x$ for x in the equation $y = f(x)$. Therefore, the graph of $y = f(x)$ will be reflected in the y-axis to obtain the graph of $y = f(-x)$.

Step 2
Identify points on the graph of $y = f(x)$.

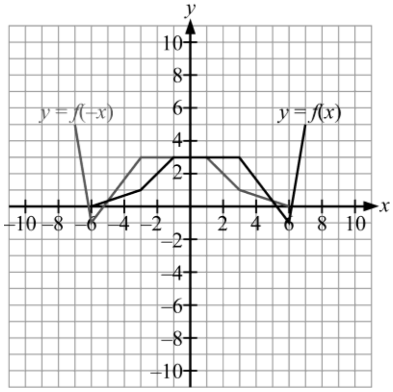
Points on the graph of $y = f(x)$ are (–6, 0), (–3, 1), (–1, 3), (3, 3), (6, –1), and (7, 5).

Step 3
Transform points on the graph of $y = f(x)$ to obtain points on the graph of $y = f(-x)$.

Multiply the x-coordinates of $y = f(x)$ by –1.

$$(-6\times(-1),\ 0) \rightarrow (6,\ 0)$$
$$(-3\times(-1),\ 1) \rightarrow (3,\ 1)$$
$$(-1\times(-1),\ 3) \rightarrow (1,\ 3)$$
$$(3\times(-1),\ 3) \rightarrow (-3,\ 3)$$
$$(6\times(-1),\ -1) \rightarrow (-6,\ -1)$$
$$(7\times(-1),\ 5) \rightarrow (-7,\ 5)$$

Step 4
Plot and join the transformed points.



4. **Step 1**
Determine how to obtain the graph of $y = -f(x)$ from the graph of $y = f(x)$.

The equation $y = -f(x)$ is obtained by substituting $-y$ for y in the equation $y = f(x)$. Therefore, the graph of $y = f(x)$ will be reflected in the x-axis to obtain the graph of $y = -f(x)$.

Step 2
Identify points on the graph of $y = f(x)$.

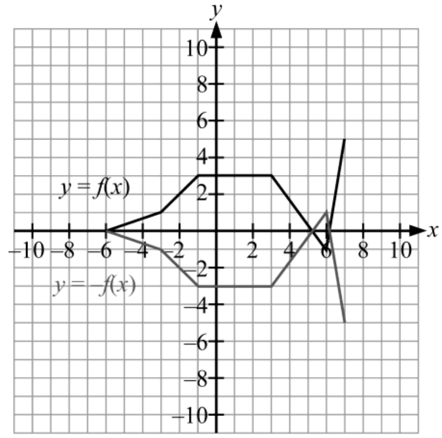
Points on the graph of $y = f(x)$ are (–6, 0), (–3, 1), (–1, 3), (3, 3), (6, –1), and (7, 5).

Step 3
Transform points on the graph of $y = f(x)$ to obtain points on the graph of $y = -f(x)$.

Multiply the y-coordinates of $y = f(x)$ by –1.

$$(-6,\ 0\times(-1)) \rightarrow (-6,\ 0)$$
$$(-3,\ 1\times(-1)) \rightarrow (-3,\ -1)$$
$$(-1,\ 3\times(-1)) \rightarrow (-1,\ -3)$$
$$(3,\ 3\times(-1)) \rightarrow (3,\ -3)$$
$$(6,\ -1\times(-1)) \rightarrow (6,\ 1)$$
$$(7,\ 5\times(-1)) \rightarrow (7,\ -5)$$

Step 4
Plot and join the transformed points.



5. Since the transformed graph is a reflection in the x-axis of the graph $y=\sqrt{x+5}$, the equation of the transformed graph is obtained by replacing y with $-y$ in the equation $y=\sqrt{x+5}$.

$$-y=\sqrt{x+5}$$

Isolate y.

$$\begin{aligned}-y&=\sqrt{x+5}\\ y&=-\sqrt{x+5}\end{aligned}$$

Therefore, the equation of the transformed graph is $y=-\sqrt{x+5}$.

Practice Test

ANSWERS AND SOLUTIONS

1. The equation $2y+4=f(x-1)$ can be written as $y+2=\frac{1}{2}f(x-1)$. The equation $y+2=\frac{1}{2}f(x-1)$ is of the form $y-k=af(x-h)$, where $a=\frac{1}{2}$, $h=1$, and $k=-2$. Therefore, the graph $2y+4=f(x-1)$ is formed from $y=f(x)$ by a vertical stretch factor of $\frac{1}{2}$, a horizontal translation 1 unit right, and a vertical translation 2 units down.

2. **Step 1**
Apply the vertical stretch.

Replace y in the equation $y=\frac{1}{x+5}$ with $\frac{1}{3}y$.

$$\begin{aligned}y&=\frac{1}{x+5}\\ \frac{1}{3}y&=\frac{1}{x+5}\end{aligned}$$

Step 2
Apply the horizontal translation.

Replace x in $\frac{1}{3}y=\frac{1}{x+5}$ with $x-(-7)$.

$$\begin{aligned}\frac{1}{3}y&=\frac{1}{x+5}\\ \frac{1}{3}y&=\frac{1}{(x-(-7))+5}\end{aligned}$$

Step 3
Isolate y, and simplify.

$$\begin{aligned}\frac{1}{3}y&=\frac{1}{(x-(-7))+5}\\ \frac{1}{3}y&=\frac{1}{x+7+5}\\ \frac{1}{3}y&=\frac{1}{x+12}\\ y&=\frac{3}{x+12}\end{aligned}$$

Therefore, the equation of the transformed graph is $y=\frac{3}{x+12}$.

3. The graph of $y=\left|\frac{3}{2}x\right|$ is obtained from the graph of $y=|x|$ by a horizontal stretch factor of $\frac{2}{3}$. Under a horizontal stretch, the y-intercept on the graph of $y=|x|$ is an invariant point.

Therefore, the invariant point is (0, 0).

4. **Step 1**
Determine how to obtain the graph of $y = f(3x+9)+4$ from the graph of $y = f(x)$.

The equation $y = f(3x+9)+4$ is equivalent to $y-4 = f(3(x+3))$. The equation $y-4 = f(3(x+3))$ is of the form $y-k = f(b(x-h))$, where $b = 3$, $h = -3$, and $k = 4$. Therefore, the graph $y = f(3x+9)+4$ is formed from $y = f(x)$ by a horizontal stretch factor of $\frac{1}{3}$, a horizontal translation 3 units left, and a vertical translation 4 units up.

Step 2
Apply the horizontal stretch.

The horizontal stretch by a factor of $\frac{1}{3}$ will transform the point (–18, 16) to $\left(-18\times\frac{1}{3}, 16\right) = (-6, 16)$.

Step 3
Apply the horizontal and vertical translations.

The horizontal translation 3 units left and vertical translation 4 units up will transform the point (–6, 16) to (–6 – 3, 16 + 4) = (–9, 20).

Therefore, the corresponding transformed point on the graph of $y = f(3x+9)+4$ is (–9, 20).

5. **Step 1**
Determine how to obtain the graph of $y = f\left(\frac{1}{3}x-\frac{1}{2}\right)$ from the graph of $y = f(x)$.

The equation $y = f\left(\frac{1}{3}x-\frac{1}{2}\right)$ is equivalent to $y = f\left(\frac{1}{3}\left(x-\frac{3}{2}\right)\right)$. The equation $y = f\left(\frac{1}{3}\left(x-\frac{3}{2}\right)\right)$ is of the form $y = f(b(x-h))$, where $b = \frac{1}{3}$ and $h = \frac{3}{2}$.

Therefore, the graph of $y = f\left(\frac{1}{3}\left(x-\frac{3}{2}\right)\right)$ is formed from $y = f(x)$ by a horizontal stretch factor of 3 and a horizontal translation $\frac{3}{2}$ units right.

Step 2
Identify points on the graph of $y = f(x)$.

Points on the graph of $y = f(x)$ are (–7, –3), (–4, –4), (–2, 0), (0, 0), (2, 4), and (5, 3).

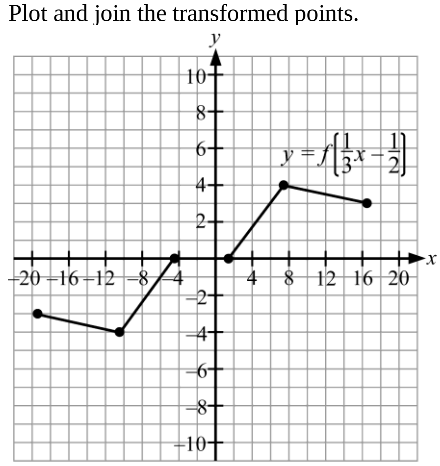
Step 3
Apply the horizontal stretch.
Multiply the *x*-coordinates of $y = f(x)$ by 3.

$$(-7\times3, -3)\to(-21, -3)$$
$$(-4\times3, -4)\to(-12, -4)$$
$$(-2\times3, 0)\to(-6, 0)$$
$$(0\times3, 0)\to(0, 0)$$
$$(2\times3, 4)\to(6, 4)$$
$$(5\times3, 3)\to(15, 3)$$

Step 4
Apply the horizontal translation.
Increase the *x*-coordinates from step 3 by $\frac{3}{2}$ or 1.5.

$$(-21+1.5, -3)\to(-19.5, -3)$$
$$(-12+1.5, -4)\to(-10.5, -4)$$
$$(-6+1.5, 0)\to(-4.5, 0)$$
$$(0+1.5, 0)\to(1.5, 0)$$
$$(6+1.5, 4)\to(7.5, 4)$$
$$(15+1.5, 3)\to(16.5, 3)$$

Step 5
Plot and join the transformed points.



6. Step 1
Determine how to obtain the graph of $y = -f(x)$ from the graph of $y = f(x)$.

The equation $y = -f(x)$ is obtained by substituting $-y$ for y in the equation $y = f(x)$. Therefore, the graph of $y = f(x)$ will be reflected in the x-axis to obtain the graph of $y = -f(x)$.

Step 2
Identify points on the graph of $y = f(x)$.

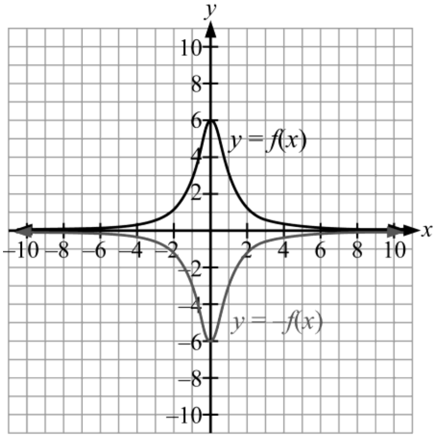
Points on the graph of $y = f(x)$ are (–1, 3), (0, 6), and (1, 3).

Step 3
Transform points on the graph of $y = f(x)$ to obtain points on the graph of $y = -f(x)$.

Multiply the y-coordinates of $y = f(x)$ by –1.

$$(-1,\ 3\times(-1)) \to (-1, -3)$$
$$(0,\ 6\times(-1)) \to (0, -6)$$
$$(1,\ 3\times(-1)) \to (1, -3)$$

Step 4
Plot and join the transformed points.



7. Step 1
Determine the x-intercepts on the graph of $x^2 + y^2 = 16$.

The x-intercepts occur when $y = 0$.

$$\begin{aligned} x^2 + y^2 &= 16 \\ x^2 + (0)^2 &= 16 \\ x^2 &= 16 \\ x &= \pm 4 \end{aligned}$$

Therefore, the x-intercepts on the graph of $x^2 + y^2 = 16$ are (–4, 0) and (4, 0).

Step 2
Transform the x-intercepts.

Since the graph of $x^2 + y^2 = 16$ is horizontally stretched by a factor of $\frac{1}{2}$, every point (x, y) on the graph of $x^2 + y^2 = 16$ will be transformed to the point $\left(\frac{1}{2}x, y\right)$. Therefore, the x-intercepts on the graph of $x^2 + y^2 = 16$ will transform to $\left(-4\times\frac{1}{2},\ 0\right) = (-2,\ 0)$ and $\left(4\times\frac{1}{2},\ 0\right) = (2,\ 0)$.

Alternative Method

Step 1
Determine the equation of the transformed graph.

Replace x in the equation $x^2 + y^2 = 16$ with $2x$.

$$\begin{aligned} x^2 + y^2 &= 16 \\ (2x)^2 + y^2 &= 16 \end{aligned}$$

Step 2
Determine the x-intercepts on the graph of $(2x)^2 + y^2 = 16$.

The x-intercepts occur when $y = 0$.

$$\begin{aligned} (2x)^2 + y^2 &= 16 \\ (2x)^2 + (0)^2 &= 16 \\ (2x)^2 &= 16 \\ 4x^2 &= 16 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

Therefore, the x-intercepts on the graph of $x^2 + y^2 = 16$ will transform to (–2, 0) and (2, 0).

OPERATIONS ON FUNCTIONS

Lesson 1—Combining Functions

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **a)** **Step 1**

Substitute x^2-2x+7 for $f(x)$ and $3x^2+2x-1$ for $g(x)$ into the expression $f(x)+g(x)$.

$f(x)+g(x)=(x^2-2x+7)+(3x^2+2x-1)$

Step 2

Combine like terms, and simplify.

$f(x)+g(x)$
$=(x^2-2x+7)+(3x^2+2x-1)$
$=x^2-2x+7+3x^2+2x-1$
$=4x^2+6$

b) **Step 1**

Substitute x^2-2x+7 for $f(x)$ and $3x^2+2x-1$ for $g(x)$ into the expression $f(x)-g(x)$.

$f(x)-g(x)=(x^2-2x+7)-(3x^2+2x-1)$

Step 2

Combine like terms, and simplify.

$f(x)-g(x)$
$=(x^2-2x+7)-(3x^2+2x-1)$
$=x^2-2x+7-3x^2-2x+1$
$=-2x^2-4x+8$

2. **a)** **Step 1**

Substitute x^2-49 for $f(x)$ and $x+7$ for $g(x)$ into the expression $f(x)g(x)$.

$f(x)g(x)=(x^2-49)(x+7)$

Step 2

Expand and simplify.

$f(x)g(x)$
$=(x^2-49)(x+7)$
$=x^3+7x^2-49x-343$

b) **Step 1**

Substitute x^2-49 for $f(x)$ and $x+7$ for $g(x)$ into the expression $\frac{g(x)}{f(x)}$.

$$\frac{g(x)}{f(x)}=\frac{x+7}{x^2-49}$$

Step 2

Factor the denominator of the rational expression.

$\frac{g(x)}{f(x)}$
$=\frac{x+7}{x^2-49}$
$=\frac{x+7}{(x+7)(x-7)}$

Step 3

State the non-permissible values.

$(x+7)(x-7)\neq 0$

$x+7\neq 0$ $\quad$ $x-7\neq 0$
$x\neq -7$ $\quad$ $x\neq 7$

The non-permissible values of x are 7 and –7.

Step 4

Reduce the rational expression.

$\frac{x+7}{(x+7)(x-7)}$
$=\frac{1}{x-7}$

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**

Substitute $4x^2+11x-3$ for $f(x)$ and $x+3$ for $g(x)$ into the expression $f(x)+g(x)$.

$f(x)+g(x)=(4x^2+11x-3)+(x+3)$

Step 2

Combine like terms, and simplify.

$f(x)+g(x)$
$=(4x^2+11x-3)+(x+3)$
$=4x^2+11x-3+x+3$
$=4x^2+12x$

2. **Step 1**

Substitute $4x^2+11x-3$ for $f(x)$ and $x+3$ for $g(x)$ into the expression $f(x)-g(x)$.

$f(x)-g(x)=(4x^2+11x-3)-(x+3)$

Step 2
Combine like terms, and simplify.

$$f(x)-g(x)$$
$$=(4x^2+11x-3)-(x+3)$$
$$=4x^2+11x-3-x-3$$
$$=4x^2+10x-6$$

3. **Step 1**
Substitute $4x^2+11x-3$ for $f(x)$ and $x+3$ for $g(x)$ into the expression $f(x)g(x)$.

$$f(x)g(x)=(4x^2+11x-3)(x+3)$$

Step 2
Expand and simplify.

$$f(x)g(x)$$
$$=(4x^2+11x-3)(x+3)$$
$$=4x^3+11x^2-3x+12x^2+33x-9$$
$$=4x^3+23x^2+30x-9$$

4. **Step 1**
Substitute $4x^2+11x-3$ for $f(x)$ and $x+3$ for $g(x)$ into the expression $\frac{g(x)}{f(x)}$.

$$\frac{g(x)}{f(x)}=\frac{x+3}{4x^2+11x-3}$$

Step 2
Factor the denominator of the rational expression.

$$\frac{g(x)}{f(x)}$$
$$=\frac{x+3}{4x^2+11x-3}$$
$$=\frac{x+3}{(x+3)(4x-1)}$$

Step 3
State the non-permissible values.

$$(x+3)(4x-1)\neq 0$$
$$x+3\neq 0 \qquad 4x-1\neq 0$$
$$x\neq -3 \qquad 4x\neq 1$$
$$x\neq \frac{1}{4}$$

The non-permissible values of x are –3 and $\frac{1}{4}$.

Step 4
Reduce the rational expression.

$$\frac{g(x)}{f(x)}$$
$$=\frac{x+3}{4x^2+11x-3}$$
$$=\frac{x+3}{(x+3)(4x-1)}$$
$$=\frac{1}{4x-1}$$

Lesson 2—Compositions of Functions

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **a)** **Step 1**
Replace variable x in $g(x)$ with the function $f(x)$.

$$g(x)=-x+4$$
$$g(f(x))=-(3x^2+5)+4$$

Step 2
Simplify the expression.

$$g(f(x))=-(3x^2+5)+4$$
$$g(f(x))=-3x^2-5+4$$
$$g(f(x))=-3x^2-1$$

b) **Step 1**
Replace x in $f(x)$ with the function $g(x)$.

$$f(x)=3x^2+5$$
$$f\circ g=3(-x+4)^2+5$$

Step 2
Simplify the expression.

$$f\circ g=3(-x+4)^2+5$$
$$f\circ g=3(-x+4)(-x+4)+5$$
$$f\circ g=3(x^2-8x+16)+5$$
$$f\circ g=3x^2-24x+48+5$$
$$f\circ g=3x^2-24x+53$$

c) **Step 1**
Replace x in $g(x)$ with $-x+4$.

$$g(x)=-x+4$$
$$g\circ g=-(-x+4)+4$$

Step 2

Simplify the expression.

$$g \circ g = -(-x+4)+4$$
$$g \circ g = x-4+4$$
$$g \circ g = x$$

2. **Step 1**

Replace variable x in $g(x)$ with the function $f(x)$.

$$g(x) = \frac{1}{x^2 - x}$$
$$g \circ f = \frac{1}{(x+3)^2 - (x+3)}$$

Step 2

State the non-permissible values.

$$(x+3)^2 - (x+3) \neq 0$$
$$x^2 + 3x + 3x + 9 - x - 3 \neq 0$$
$$x^2 + 5x + 6 \neq 0$$
$$(x+2)(x+3) \neq 0$$
$$x+2 \neq 0 \qquad x+3 \neq 0$$
$$x \neq -2 \qquad x \neq -3$$

The non-permissible values of x are –2 and –3.

Step 3

Simplify the expression.

$$g \circ f = \frac{1}{(x+3)^2 - (x+3)}$$
$$g \circ f = \frac{1}{x^2 + 3x + 3x + 9 - x - 3}$$
$$g \circ f = \frac{1}{x^2 + 5x + 6}$$

Therefore, $g \circ f = \frac{1}{x^2 + 5x + 6}$, where $x \neq -2$ and $x \neq -3$.

3. **a)** **Step 1**

Calculate the value of $g(2)$.

$$g(x) = x+3$$
$$g(2) = 2+3$$
$$g(2) = 5$$

Step 2

Calculate the value of $f(g(2))$.

Since $g(2) = 5$, find the value of $f(5)$.

$$f(x) = \frac{2}{x+1}$$
$$f(5) = \frac{2}{5+1}$$
$$f(5) = \frac{2}{6}$$
$$f(5) = \frac{1}{3}$$

Therefore, $f(g(2)) = \frac{1}{3}$.

b) **Step 1**

Calculate the value of $f(–5)$.

$$f(x) = \frac{2}{x+1}$$
$$f(-5) = \frac{2}{-5+1}$$
$$f(-5) = -\frac{2}{4}$$
$$f(-5) = -\frac{1}{2}$$

Step 2

Calculate the value of $g(f(-5))$.

Since $f(-5) = -\frac{1}{2}$, find the value of $g\left(-\frac{1}{2}\right)$.

$$g(x) = x+3$$
$$g\left(-\frac{1}{2}\right) = -\frac{1}{2} + 3$$
$$g\left(-\frac{1}{2}\right) = -\frac{1}{2} + \frac{6}{2}$$
$$g\left(-\frac{1}{2}\right) = \frac{5}{2}$$

Therefore, $g(f(-5)) = \frac{5}{2}$, or 2.5.

c) **Step 1**

Calculate the value of $f(7)$.

$$f(x) = \frac{2}{x+1}$$
$$f(7) = \frac{2}{7+1}$$
$$f(7) = \frac{2}{8}$$
$$f(7) = \frac{1}{4}$$

Step 2

Calculate the value of $f(f(7))$.

Since $f(7) = \frac{1}{4}$, find the value of $f\left(\frac{1}{4}\right)$.

$$f(x) = \frac{2}{x+1}$$
$$f\left(\frac{1}{4}\right) = \frac{2}{\frac{1}{4}+1}$$
$$f\left(\frac{1}{4}\right) = \frac{2}{\frac{1}{4}+\frac{4}{4}}$$
$$f\left(\frac{1}{4}\right) = \frac{2}{\frac{5}{4}}$$
$$f\left(\frac{1}{4}\right) = \frac{8}{5}$$

Therefore, $f(f(7)) = \frac{8}{5}$, or 1.6.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Replace variable x in $f(x)$ with the function $g(x)$.

$$f(x) = -4x + 11$$
$$f \circ g = -4\left(\frac{x-1}{5}\right) + 11$$

Step 2
Simplify the expression.

$$f \circ g = -4\left(\frac{x-1}{5}\right) + 11$$
$$f \circ g = \frac{-4x+4}{5} + 11$$
$$f \circ g = \frac{-4x+4}{5} + \frac{55}{5}$$
$$f \circ g = \frac{-4x+4+55}{5}$$
$$f \circ g = \frac{-4x+59}{5}$$

2. **Step 1**
Replace variable x in $g(x)$ with the function $f(x)$.

$$g(x) = \frac{x-1}{5}$$
$$g \circ f = \frac{(-4x+11)-1}{5}$$

Step 2
Simplify the expression.

$$g \circ f = \frac{(-4x+11)-1}{5}$$
$$g \circ f = \frac{-4x+11-1}{5}$$
$$g \circ f = \frac{-4x+10}{5}$$

3. **Step 1**
Replace variable x in $f(x)$ with $-4x+11$.

$$f(x) = -4x + 11$$
$$f \circ f = -4(-4x+11) + 11$$

Step 2
Simplify the expression.

$$f \circ f = -4(-4x+11) + 11$$
$$f \circ f = 16x - 44 + 11$$
$$f \circ f = 16x - 33$$

4. **Step 1**
Replace variable x in $g(x)$ with $\frac{x-1}{5}$.

$$g(x) = \frac{x-1}{5}$$
$$g \circ g = \frac{\left(\frac{x-1}{5}\right) - 1}{5}$$

Step 2
Simplify the expression.

$$g \circ g = \frac{\left(\frac{x-1}{5}\right) - 1}{5}$$
$$g \circ g = \frac{\frac{x-1}{5} - \frac{5}{5}}{5}$$
$$g \circ g = \frac{\frac{x-1-5}{5}}{5}$$
$$g \circ g = \frac{\frac{x-6}{5}}{5}$$
$$g \circ g = \frac{x-6}{25}$$

5. **Step 1**
Calculate the value of $g(5)$.

$$g(x) = \frac{2}{3}x + 11$$
$$g(5) = \frac{2}{3}(5) + 11$$
$$g(5) = \frac{10}{3} + 11$$
$$g(5) = \frac{10}{3} + \frac{33}{3}$$
$$g(5) = \frac{43}{3}$$

Step 2

Calculate the value of $f\left(g(5)\right)$.

Since $g(5)=\frac{43}{3}$, find the value of $f\left(\frac{43}{3}\right)$.

$$f(x)=-17-9x$$
$$f\left(\frac{43}{3}\right)=-17-9\left(\frac{43}{3}\right)$$
$$f\left(\frac{43}{3}\right)=-17-3(43)$$
$$f\left(\frac{43}{3}\right)=-17-129$$
$$f\left(\frac{43}{3}\right)=-17-129$$
$$f\left(\frac{43}{3}\right)=-146$$

Therefore, $f\left(g(5)\right)=-146$.

6. **Step 1**

Calculate the value of $g\left(\frac{1}{2}\right)$.

$$g(x)=\frac{2}{3}x+11$$
$$g\left(\frac{1}{2}\right)=\frac{2}{3}\left(\frac{1}{2}\right)+11$$
$$g\left(\frac{1}{2}\right)=\frac{1}{3}+11$$
$$g\left(\frac{1}{2}\right)=\frac{1}{3}+\frac{33}{3}$$
$$g\left(\frac{1}{2}\right)=\frac{34}{3}$$

Step 2

Calculate the value of $g\left(g\left(\frac{1}{2}\right)\right)$.

Since $g\left(\frac{1}{2}\right)=\frac{34}{3}$, find the value of $g\left(\frac{34}{3}\right)$.

$$g(x)=\frac{2}{3}x+11$$
$$g\left(\frac{34}{3}\right)=\frac{2}{3}\left(\frac{34}{3}\right)+11$$
$$g\left(\frac{34}{3}\right)=\frac{68}{9}+11$$
$$g\left(\frac{34}{3}\right)=\frac{68}{9}+\frac{99}{9}$$
$$g\left(\frac{34}{3}\right)=\frac{167}{9}$$

Therefore, $g\left(g\left(\frac{1}{2}\right)\right)=\frac{167}{9}$.

7. **Step 1**

Calculate the value of $g(-3)$.

$$g(x)=3x$$
$$g(-3)=3(-3)$$
$$g(-3)=-9$$

Step 2

Calculate the value of $h\left(g(-3)\right)$.

Since $g(-3)=-9$, find the value of $h(-9)$.

$$h(x)=\frac{1}{x^2+x}$$
$$h(-9)=\frac{1}{(-9)^2+(-9)}$$
$$h(-9)=\frac{1}{81-9}$$
$$h(-9)=\frac{1}{72}$$

Therefore, $h\left(g(-3)\right)=\frac{1}{72}$.

8. **Step 1**

Calculate the value of $h\left(\frac{1}{4}\right)$.

$$h(x)=\frac{1}{x^2+x}$$
$$h\left(\frac{1}{4}\right)=\frac{1}{\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)}$$
$$h\left(\frac{1}{4}\right)=\frac{1}{\frac{1}{16}+\frac{1}{4}}$$
$$h\left(\frac{1}{4}\right)=\frac{1}{\frac{1}{16}+\frac{4}{16}}$$
$$h\left(\frac{1}{4}\right)=\frac{1}{\frac{5}{16}}$$
$$h\left(\frac{1}{4}\right)=\frac{16}{5}$$

Step 2

Calculate the value of $h\left(h\left(\frac{1}{4}\right)\right)$.

Since $h\left(\frac{1}{4}\right)=\frac{16}{5}$, find the value of $h\left(\frac{16}{5}\right)$.

$$\begin{aligned}
h(x)&=\frac{1}{x^2+x}\\
h\left(\frac{16}{5}\right)&=\frac{1}{\left(\frac{16}{5}\right)^2+\left(\frac{16}{5}\right)}\\
h\left(\frac{16}{5}\right)&=\frac{1}{\frac{256}{25}-\frac{16}{5}}\\
h\left(\frac{16}{5}\right)&=\frac{1}{\frac{256}{25}-\frac{80}{25}}\\
h\left(\frac{16}{5}\right)&=\frac{1}{\frac{176}{25}}\\
h\left(\frac{16}{5}\right)&=\frac{25}{176}
\end{aligned}$$

Therefore, $h\left(h\left(\frac{1}{4}\right)\right)=\frac{25}{176}$.

Lesson 3—Graphing a Composition of Functions

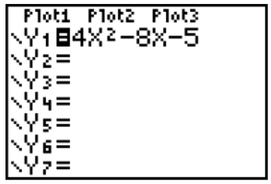
CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Find the composite function.

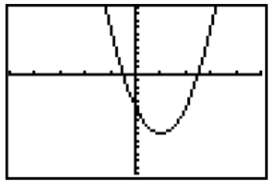
Replace variable *x* in $g(x)$ with $f(x)$, and simplify.

$$\begin{aligned}
g(x)&=x^2+6x\\
g(f(x))&=(2x-1)^2+6(2x-1)\\
g(f(x))&=(2x-1)(2x-1)+6(2x-1)\\
g(f(x))&=4x^2-2x-2x+1+12x-6\\
g(f(x))&=4x^2-8x-5
\end{aligned}$$

Step 2
Using a TI-83 graphing calculator, press $\boxed{\text{Y =}}$ and input the function as $Y_1=4X^2-8X-5$.



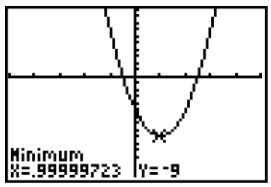
Step 3
Use the window settings of *x*: [–5,5,1] and *y*: [–15,10,1], and press $\boxed{\text{GRAPH}}$ to obtain this window.



Step 4
State the domain and range of the composite function.

The composite function is a parabola opening upward.

Determine the coordinates of the minimum by pressing $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$ and selecting 3:minimum.



Therefore, the domain is $x\in\mathrm{R}$, and the range is $y\geq-9$.

2. **Step 1**
Determine the composite function.
Replace variable *x* in $g(x)$ with $f(x)$.

$$\begin{aligned}
g(x)&=\frac{1}{3}x+2\\
g(f(x))&=\frac{1}{3}(x^2)+2\\
g(f(x))&=\frac{1}{3}x^2+2
\end{aligned}$$

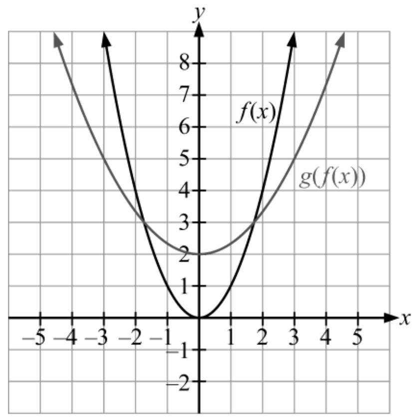
Step 2
Describe the transformations that occur to $f(x)$.

The graph of $f(x)=x^2$ is transformed to the graph of $g(f(x))=\frac{1}{3}x^2+2$. Thus, the graph of $g(f(x))$ is obtained from $f(x)$ by a vertical stretch factor of $\frac{1}{3}$ and a translation 2 units up.

Step 3

Sketch the graph of $g\left(f(x)\right)=\frac{1}{3}x^2+2$.

The graph of $g\left(f(x)\right)$ is the graph of $f(x)$ vertically stretched by a factor of $\frac{1}{3}$ and translated 2 units up.



PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. Step 1

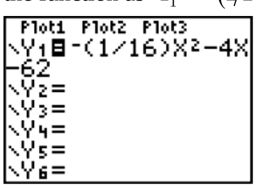
Find the composite function.

Replace variable x in $f(x)$ with $g(x)$, and simplify.

$$f(x)=-x^2+2x+1$$
$$f(g(x))=-\left(\frac{1}{4}x+9\right)^2+2\left(\frac{1}{4}x+9\right)+1$$
$$f(g(x))=-\left(\frac{1}{4}x+9\right)\left(\frac{1}{4}x+9\right)+2\left(\frac{1}{4}x+9\right)+1$$
$$f(g(x))=-\left(\frac{1}{16}x^2+\frac{9}{4}x+\frac{9}{4}x+81\right)+\frac{2}{4}x+18+1$$
$$f(g(x))=-\left(\frac{1}{16}x^2+\frac{18}{4}x+81\right)+\frac{2}{4}x+19$$
$$f(g(x))=-\frac{1}{16}x^2-\frac{18}{4}x-81+\frac{2}{4}x+19$$
$$f(g(x))=-\frac{1}{16}x^2-\frac{16}{4}x-62$$
$$f(g(x))=-\frac{1}{16}x^2-4x-62$$

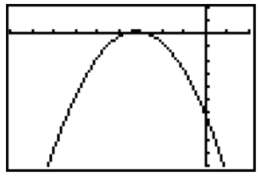
Step 2

Using a TI-83 graphing calculator, press $\boxed{\text{Y =}}$, and input the function as $\text{Y}_1=-(1/16)\text{X}^2-4\text{X}-62$.



Step 3

Use the window settings of x: [–90,20,10] and y: [–100,20,10], and then press $\boxed{\text{GRAPH}}$ to obtain this window.

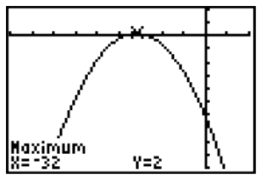


Step 4

State the domain and range of the composite function.

The composite function is a parabola opening downward.

Determine the coordinates of the minimum by pressing $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$ and selecting 4:maximum.



Therefore, the domain is $x\in\text{R}$, and the range is $y\le 2$.

2. Step 1

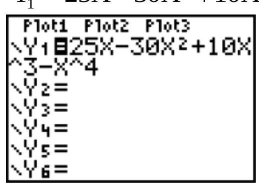
Find the composite function.

Replace variable x in $f(x)$ with $5x-x^2$, and simplify.

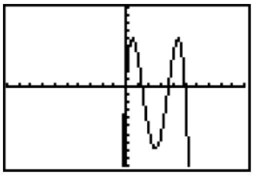
$$f(x)=5x-x^2$$
$$f\left(f(x)\right)=5\left(5x-x^2\right)-\left(5x-x^2\right)^2$$
$$f\left(f(x)\right)=5\left(5x-x^2\right)-\left(5x-x^2\right)\left(5x-x^2\right)$$
$$f\left(f(x)\right)=25x-5x^2-\left(25x^2-5x^3-5x^3+x^4\right)$$
$$f\left(f(x)\right)=25x-5x^2-\left(25x^2-10x^3+x^4\right)$$
$$f\left(f(x)\right)=25x-5x^2-25x^2+10x^3-x^4$$
$$f\left(f(x)\right)=25x-30x^2+10x^3-x^4$$

Step 2

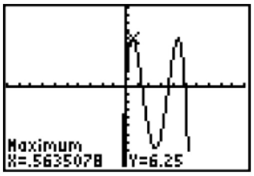
Using a TI-83 graphing calculator, press $\boxed{\text{Y =}}$, and input the function as $\text{Y}_1=25\text{X}-30\text{X}^2+10\text{X}\text{^}3-\text{X}\text{^}4$.

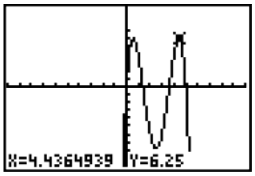


Step 3
Use the window settings of x: [–10,10,1] and y: [–10,10,1], and then press $\boxed{\text{GRAPH}}$ to obtain this window.



Step 4
State the domain and range of the composite function. Determine the coordinates of each maximum by pressing $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$ and selecting 4:maximum.





Therefore, the domain is $x \in \mathrm{R}$, and the range is $y \le 6.25$.

3. **Step 1**
Determine the composite function.

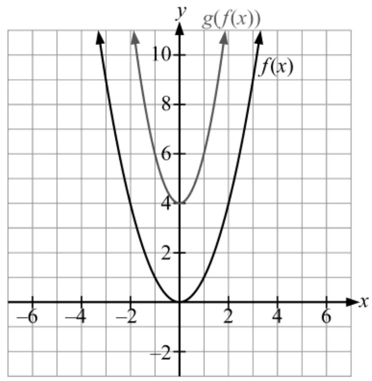
Replace variable x in $g(x)$ with $f(x)$.

$$g(x) = 2x + 4$$
$$g(f(x)) = 2(x^2) + 4$$
$$g(f(x)) = 2x^2 + 4$$

Step 2
Describe the transformations that occur to $f(x)$ to obtain $g(f(x))$. The graph of $f(x) = x^2$ is transformed to the graph of $g(f(x)) = 2x^2 + 4$. Thus, the graph of $g(f(x))$ is obtained from $f(x)$ by a vertical stretch factor of 2 and a translation 4 units up.

Step 3
Sketch the graph of $g(f(x)) = 2x^2 + 4$.

The graph of $g(f(x))$ is the graph of $f(x)$ vertically stretched by a factor of 2 and translated 4 units up.



4. **Step 1**
Determine the composite function.

Replace variable x in $f(x)$ with $g(x)$.

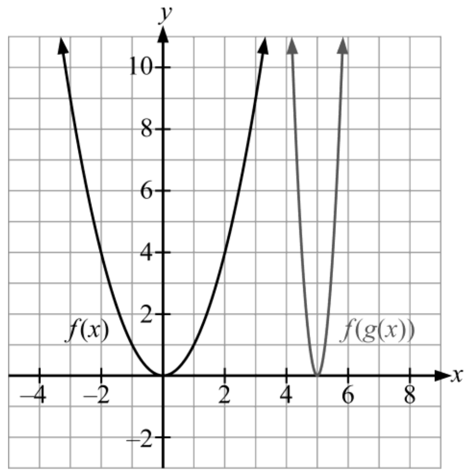
$$f(x) = x^2$$
$$f(g(x)) = (4(x-5))^2$$

Step 2
Describe the transformations that occur to $f(x)$ to obtain $f(g(x))$.

The graph of $f(x) = x^2$ is transformed to the graph of $f(g(x)) = (4(x-5))^2$. Thus, the graph of $f(g(x))$ is obtained from $f(x)$ by a horizontal stretch factor of $\frac{1}{4}$ and a translation 5 units to the right.

Step 3
Sketch the graph of $f\left(g\left(x\right)\right)=\left(4\left(x-5\right)\right)^2$.

The graph of $f\left(g\left(x\right)\right)$ is the graph of $f\left(x\right)$ stretched horizontally by a factor of $\frac{1}{4}$ and translated horizontally 5 units to the right.



4. **Step 1**
Determine the composite function.

Replace variable x in $f\left(x\right)$ with $g\left(x\right)$.

$$f\left(x\right)=x^2$$
$$f\left(g\left(x\right)\right)=\left(4\left(x-5\right)\right)^2$$

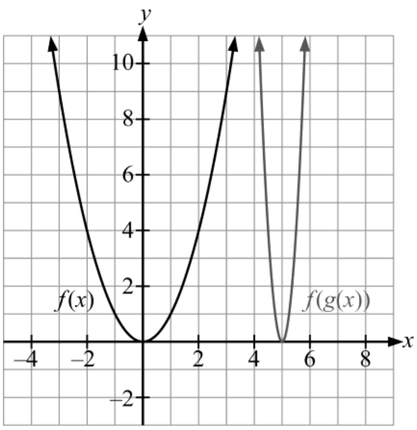
Step 2
Describe the transformations that occur to $f\left(x\right)$ to obtain $f\left(g\left(x\right)\right)$.

The graph of $f\left(x\right)=x^2$ is transformed to the graph of $f\left(g\left(x\right)\right)=\left(4\left(x-5\right)\right)^2$.

Thus, the graph of $f\left(g\left(x\right)\right)$ is obtained from $f\left(x\right)$ by a horizontal stretch factor of $\frac{1}{4}$ and a translation 5 units to the right.

Step 3
Sketch the graph of $f\left(g\left(x\right)\right)=\left(4\left(x-5\right)\right)^2$.

The graph of $f\left(g\left(x\right)\right)$ is the graph of $f\left(x\right)$ stretched horizontally by a factor of $\frac{1}{4}$ and translated horizontally 5 units to the right.



Practice Test

ANSWERS AND SOLUTIONS

1. **Step 1**
Substitute $3x+16$ for $f\left(x\right)$ and $3x^2+10x-32$ for $g\left(x\right)$ into the expression $f\left(x\right)+g\left(x\right)$.

$$f\left(x\right)+g\left(x\right)=\left(3x+16\right)+\left(3x^2+10x-32\right)$$

Step 2
Combine like terms, and simplify.

$$\begin{aligned}&f\left(x\right)+g\left(x\right)\\&=\left(3x+16\right)+\left(3x^2+10x-32\right)\\&=3x+16+3x^2+10x-32\\&=3x^2+13x-16\end{aligned}$$

2. **Step 1**
Substitute x^2+4 for $h\left(x\right)$ and $3x+16$ for $f\left(x\right)$ into the expression $h\left(x\right)-f\left(x\right)$.

$$h\left(x\right)-f\left(x\right)=\left(x^2+4\right)-\left(3x+16\right)$$

Step 2
Combine like terms, and simplify.

$$\begin{aligned}&h\left(x\right)-f\left(x\right)\\&=\left(x^2+4\right)-\left(3x+16\right)\\&=x^2+4-3x-16\\&=x^2-3x-12\end{aligned}$$

3. **Step 1**

Substitute $\dfrac{1}{x-2}$ for $k(x)$ and $3x^2+10x-32$ for $g(x)$ into the expression $k(x)g(x)$.

$$k(x)g(x)=\left(\frac{1}{x-2}\right)\left(3x^2+10x-32\right)$$

Step 2
State the non-permissible values.

$$\begin{aligned} x-2&\neq 0\\ x&\neq 2\end{aligned}$$

The non-permissible value of *x* is 2.

Step 3
Reduce if possible.

$$\begin{aligned}&\left(\frac{1}{x-2}\right)\left(3x^2+10x-32\right)\\ &=\frac{(3x+16)(x-2)}{x-2}\\ &=3x+16\end{aligned}$$

4. **Step 1**
Substitute $3x+16$ for $f(x)$ and $3x^2+10x-32$ for $g(x)$ into the expression $\dfrac{f(x)}{g(x)}$.

$$\frac{f(x)}{g(x)}=\frac{3x+16}{3x^2+10x-32}$$

Step 2
Factor the denominator of the rational expression.

$$\begin{aligned}&\frac{f(x)}{g(x)}\\ &=\frac{3x+16}{3x^2+10x-32}\\ &=\frac{3x+16}{(3x+16)(x-2)}\end{aligned}$$

Step 3
State the non-permissible values.

$$(3x+16)(x-2)\neq 0$$

$$\begin{aligned}3x+16&\neq 0 & \qquad x-2&\neq 0\\ 3x&\neq -16 & x&\neq 2\\ x&\neq -\frac{16}{3}\end{aligned}$$

The non-permissible values of *x* are $-\dfrac{16}{3}$ and 2.

Step 4
Reduce the rational expression.

$$\begin{aligned}&\frac{f(x)}{g(x)}\\ &=\frac{3x+16}{(3x+16)(x-2)}\\ &=\frac{1}{x-2}\end{aligned}$$

5. **Step 1**
Replace variable *x* in $k(x)$ with the function $f(x)$.

$$k(x)=\frac{1}{x-2}$$
$$k\circ f=\frac{1}{(3x+16)-2}$$

Step 2
State the non-permissible values.

$$\begin{aligned}(3x+16)-2&\neq 0\\ 3x+16&\neq 2\\ 3x&\neq -14\\ x&\neq -\frac{14}{3}\end{aligned}$$

The non-permissible value of *x* is $-\dfrac{14}{3}$.

Step 3
Simplify the expression.

$$k\circ f=\frac{1}{(3x+16)-2}$$
$$k\circ f=\frac{1}{3x+16-2}$$
$$k\circ f=\frac{1}{3x+14}$$

Therefore, $k\circ f=\dfrac{1}{3x+14}$, where $x\neq -\dfrac{14}{3}$.

6. **Step 1**
Replace variable *x* in $g(x)$ with the function $f(x)$.

$$g(x)=3x^2+10x-32$$
$$g\circ f=3(3x+16)^2+10(3x+16)-32$$

Step 2
Simplify the expression.

$$g\circ f=3(3x+16)^2+10(3x+16)-32$$
$$g\circ f=3(3x+16)(3x+16)+10(3x+16)-32$$
$$g\circ f=3\left(9x^2+96x+256\right)+30x+160-32$$
$$g\circ f=27x^2+288x+768+30x+128$$
$$g\circ f=27x^2+318x+896$$

7. Step 1

Replace variable *x* in $h(x)$ with the expression x^2+4.

$$h(x)=x^2+4$$
$$h(h(x))=(x^2+4)^2+4$$

Step 2

Simplify the expression.

$$h(h(x))=(x^2+4)^2+4$$
$$h(h(x))=(x^2+4)(x^2+4)+4$$
$$h(h(x))=x^4+4x^2+4x^2+16+4$$
$$h(h(x))=x^4+16x^2+20$$

8. Step 1

Replace variable *x* in $g(x)$ with the function $h(x)$.

$$g(x)=3x^2+10x-32$$
$$g(h(x))=3(x^2+4)^2+10(x^2+4)-32$$

Step 2

Simplify the expression.

$$g(h(x))=3(x^2+4)^2+10(x^2+4)-32$$
$$g(h(x))=3(x^2+4)(x^2+4)+10(x^2+4)-32$$
$$g(h(x))=3(x^4+4x^2+4x^2+16)+10x^2+40-32$$
$$g(h(x))=3(x^4+16x^2+16)+10x^2+8$$
$$g(h(x))=3x^4+48x^2+48+10x^2+8$$
$$g(h(x))=3x^4+58x^2+56$$

9. Step 1

Calculate the value of $f(6)$.

$$f(x)=x^2+6$$
$$f(6)=(6)^2+6$$
$$f(6)=36+6$$
$$f(6)=42$$

Step 2

Calculate the value of $g(f(6))$.

Since $f(6)=42$, find the value of $g(42)$.

$$g(x)=-\frac{12x}{7}$$
$$g(42)=-\frac{12(42)}{7}$$
$$g(42)=-\frac{504}{7}$$
$$g(42)=-72$$

Therefore, $g(42)=-72$.

10. Step 1

Calculate the value of $g(10)$.

$$g(x)=-\frac{1}{5x}$$
$$g(10)=-\frac{1}{5(10)}$$
$$g(10)=-\frac{1}{50}$$

Step 2

Calculate the value of $g(g(10))$.

Since $g(10)=-\frac{1}{50}$, find the value of $g(g(10))$.

$$g(x)=-\frac{1}{5x}$$
$$g\left(-\frac{1}{50}\right)=-\frac{1}{5\left(-\frac{1}{50}\right)}$$
$$g\left(-\frac{1}{50}\right)=\frac{1}{\frac{5}{50}}$$
$$g\left(-\frac{1}{50}\right)=\frac{50}{5}$$
$$g\left(-\frac{1}{50}\right)=10$$

Therefore, $g(g(10))=10$.

11. Step 1

Determine the composite function $g(f(x))$.

Replace variable *x* in $g(x)$ with $f(x)$, and simplify.

$$g(x)=3x-19$$
$$g(f(x))=3(x+1)-19$$
$$g(f(x))=3x+3-19$$
$$g(f(x))=3x-16$$

Step 2

Determine the value of *x* in the composite function.
Substitute 2 for $g(f(x))$, and solve for *x*.

$$g(f(x))=3x-16$$
$$2=3x-16$$
$$18=3x$$
$$6=x$$

Therefore, the value of *x* is 6.

12. Step 1

Replace variable x in $f(x)$ with $g(x)$.

$$f(x) = -6x^2 + x$$
$$f(g(x)) = -6(x+4)^2 + (x+4)$$

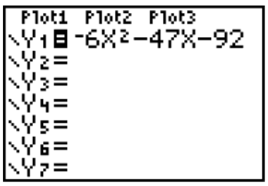
Step 2

Simplify the expression.

$$f(g(x)) = -6(x+4)^2 + (x+4)$$
$$f(g(x)) = -6(x+4)^2 + (x+4)$$
$$f(g(x)) = -6(x+4)(x+4) + (x+4)$$
$$f(g(x)) = -6(x^2 + 4x + 4x + 16) + x + 4$$
$$f(g(x)) = -6(x^2 + 8x + 16) + x + 4$$
$$f(g(x)) = -6x^2 - 48x - 96 + x + 4$$
$$f(g(x)) = -6x^2 - 47x - 92$$

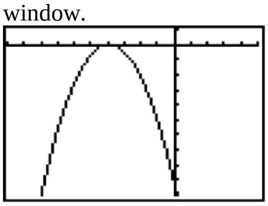
13. Step 1

Using a TI-83 graphing calculator, press $\boxed{\text{Y =}}$, and input the function as $Y_1 = -6X^2 - 47X - 92$.



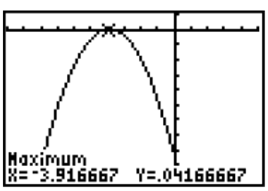
Step 2

Use the window settings x:[–10, 5, 1] a
nd y:[–100,10,10] and press $\boxed{\text{GRAPH}}$ to obtain this window.



14. The composite function is a parabola opening downward.

Determine the coordinates of the maximum by pressing $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$ and selecting 4:maximum.



Therefore, the domain is $x \in \mathbb{R}$, and the range is $y \le -0.0417$.

15. Step 1

Determine the composite function.

Replace variable x in $h(x)$ with $g(x)$.

$$h(x) = 2x - 10$$
$$h(g(x)) = 2(x^2) - 10$$
$$h(g(x)) = 2x^2 - 10$$

Step 2

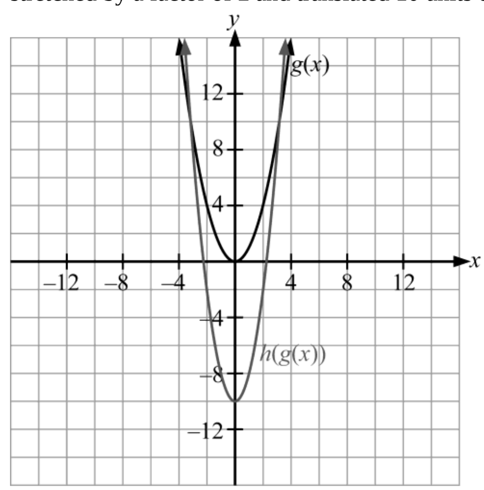
Describe the transformations that occur to $g(x) = x^2$ to obtain $h(g(x))$.

The graph of $g(x) = x^2$ is transformed to the graph of $h(g(x)) = 2x^2 - 10$. Thus, the graph of $h(g(x))$ is obtained from $g(x)$ by a vertical stretch factor of 2 and a translation 10 units down.

Step 3

Sketch the graph of $h(g(x)) = 2x^2 - 10$.

The graph of $h(g(x))$ is the graph of $g(x)$ vertically stretched by a factor of 2 and translated 10 units down.



INVERSE RELATIONS

Lesson 1—Graphs of Inverse Relations

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Step 1
Display points on the graph of $y = f(x)$ using a table of values.

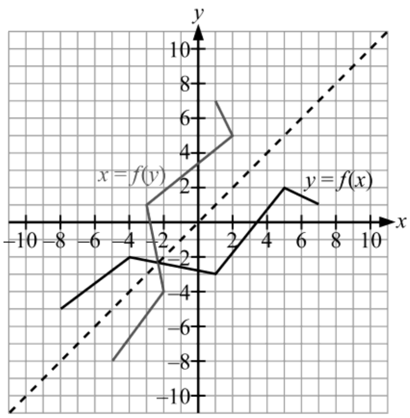
x	y
–8	–5
–4	–2
1	–3
5	2
7	1

Step 2
Reflect the points across the line $y = x$ by interchanging the *x*- and *y*-coordinates.

x	y
–5	–8
–2	–4
–3	1
2	5
1	7

Step 3
Plot and connect the reflected points.

The result is the graph of the inverse relation, which is a reflection of the graph of $y = f(x)$ through the line $y = x$.



2. Step 1
Determine the domain and range of $y = f(x)$.

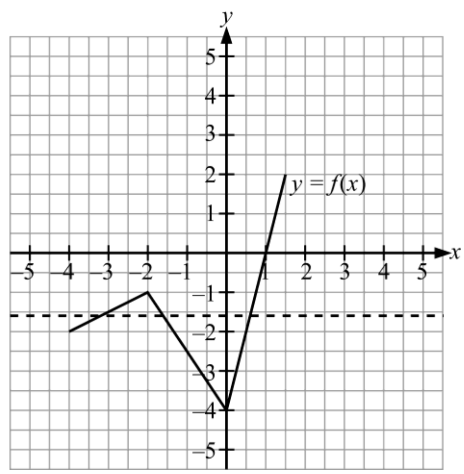
From the given graph, the domain is $x \geq -5$ and the range is $y \leq 2$.

Step 2
Determine the domain and range of the inverse relation.

The domain of the inverse relation is the range of $y = f(x)$. The range of the inverse relation is the domain of $y = f(x)$.

Therefore, the domain of the inverse relation is $x \leq 2$ and the range is $y \geq -5$.

3. Apply the horizontal line test to determine if the inverse of $y = f(x)$ is a function.



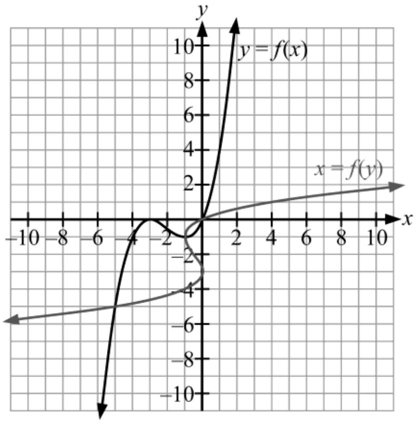
The graph of $y = f(x)$ does not pass the horizontal line test because the line passes through more than one point on the graph of the function.

Therefore, the inverse graph of $y = f(x)$ is not a function.

4. **Step 1**
Sketch the inverse of $y = f(x)$.

Points on the graph of $y = f(x)$ are (–5, –5), (–3, 0), (–2, –0.5), (–1, –1), and (1, 4). Therefore, points on the inverse graph are (–5, –5), (0, –3), (–0.5, –2), (–1, –1), and (4, 1).

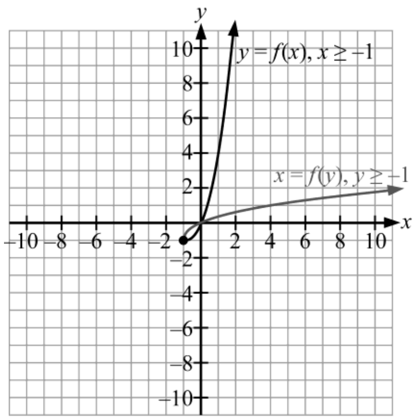
Plot and join the new points to obtain the inverse graph.



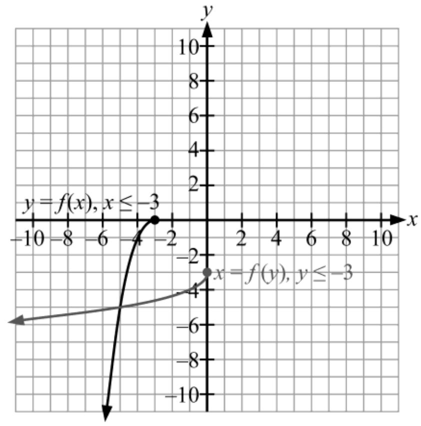
Step 2
Restrict the domain of $y = f(x)$ in order for its inverse to be a function.

For the inverse to be a function, the range should be $y \geq -1$ or $y \leq -3$. Therefore, the domain of $y = f(x)$ should be either $x \geq -1$ or $x \leq -3$ in order for its inverse to be a function.

When $y = f(x)$ has a domain of $x \geq -1$, the inverse graph has a range of $y \geq -1$.



When $y = f(x)$ has a domain of $x \leq -3$, the inverse graph has a range of $y \leq -3$.



PRACTICE EXERCISES ANSWERS AND SOLUTIONS

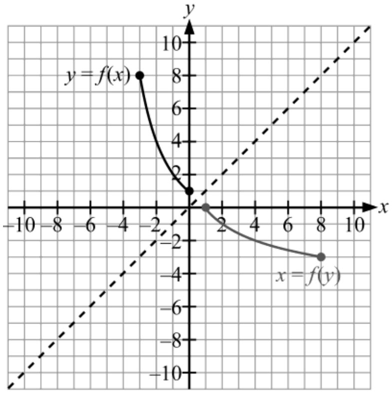
1. **Step 1**
Display points on the graph of $y = f(x)$ using a table of values.

x	y
–3	8
–2	4
–1	2
0	1

Step 2
Reflect the points across the line $y = x$ by interchanging the *x*- and *y*-coordinates.

x	y
8	–3
4	–2
2	–1
1	0

Step 3
Plot and connect the reflected points. The result is a reflection through the line $y = x$.



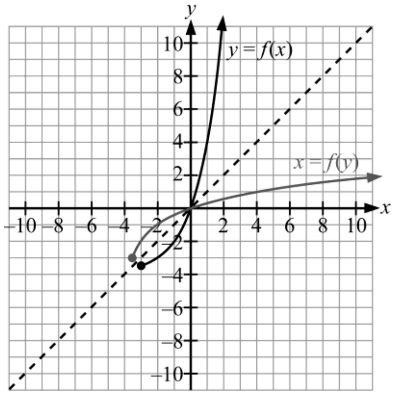
2. **Step 1**
Display points on the graph of $y = f(x)$ using a table of values.

x	y
–3	–3.5
–2	–3
–1	–2
0	0
1	4

Step 2
Reflect the points across the line $y = x$ by interchanging the x- and y-coordinates.

x	y
–3.5	–3
–3	–2
–2	–1
0	0
4	1

Step 3
Plot and connect the reflected points. The result is a reflection through the line $y = x$.



3. **Step 1**
Display points on the graph of $y = f(x)$ using a table of values.

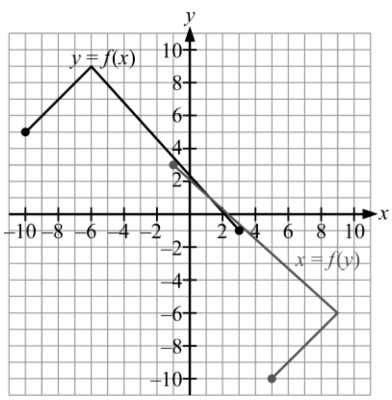
x	y
–10	5
–6	9
3	–1

Step 2
Reflect the points across the line $y = x$ by interchanging the x- and y-coordinates.

x	y
5	–10
9	–6
–1	3

Step 3
Plot and connect the reflected points.

The result is a reflection through the line $y = x$.

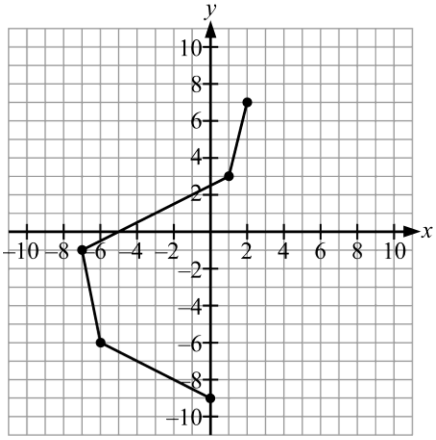


4. Step 1
Sketch the inverse of $y = f(x)$.

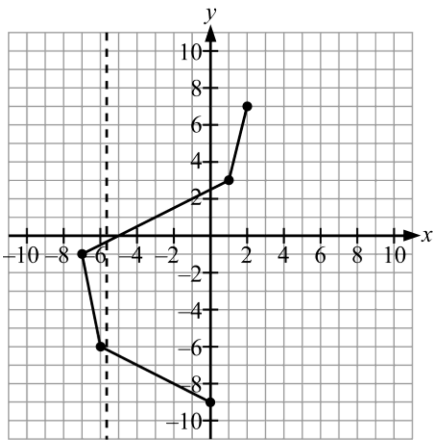
Points on the graph of $y = f(x)$ are
(–9, 0), (–6, –6), (–1, –7), (3, 1), and (7, 2).

Therefore, points on the inverse graph are
(0, –9), (–6, –6), (–7, –7), (1, 3), and (2, 7).

Plot and join the new points to obtain the inverse graph.



Step 2
Apply the vertical line test.



Since the inverse graph does not pass the vertical line test, it is not a function.

Step 3
Restrict the domain of $y = f(x)$ in order for its inverse to be a function.

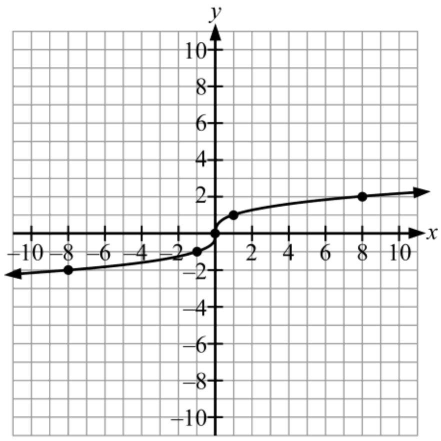
For the inverse to be a function, the range should be $y \geq -1$ or $-1 \leq y \leq -9$.

Therefore, the domain of $y = f(x)$ should be either $x \geq -1$ or $-1 \leq x \leq -9$ in order for its inverse to be a function.

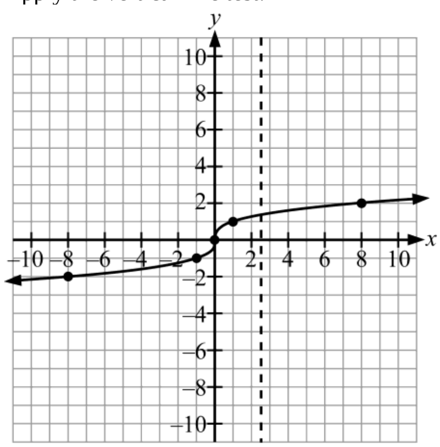
5. Step 1
Sketch the inverse of $y = f(x)$.

Points on the graph of $y = f(x)$ are (–2, –8), (–1, 1), (0, 0), (1, 1), and (2, 8). Therefore, points on the inverse graph are (–8, –2), (–1, –1), (0, 0), (1, 1), and (8, 2).

Plot and join the new points to obtain the inverse graph.



Step 2
Apply the vertical line test.

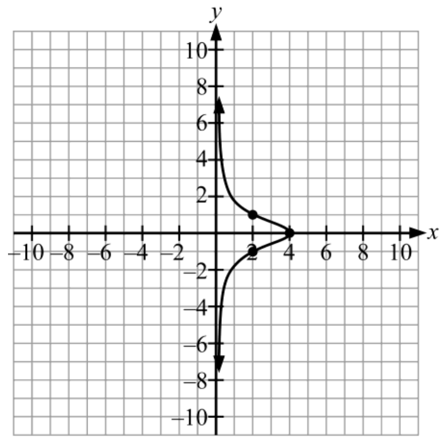


Since the inverse graph passes the vertical line test, it is a function.

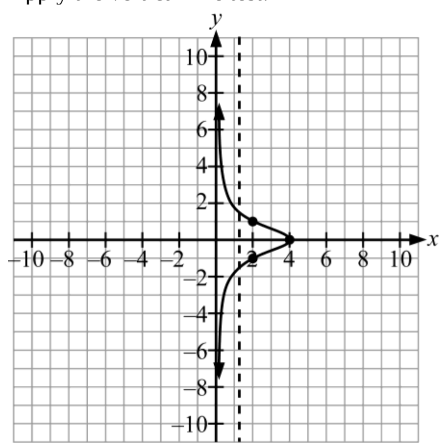
6. **Step 1**
Sketch the inverse of $y = f(x)$.

Points on the graph of $y = f(x)$ are (–1, 2), (0, 4), and (1, 2). Therefore, points on the inverse graph are (–2, –1), (4, 0), and (2, 1).

Plot and join the new points to obtain the inverse graph.



Step 2
Apply the vertical line test.



Since the inverse graph does not pass the vertical line test, it is not a function.

Step 3
Restrict the domain of $y = f(x)$ in order for its inverse to be a function.

For the inverse to be a function, the range should be $y \geq 0$ or $y \leq 0$. Therefore, the domain of $y = f(x)$ should be either $x \geq 0$ or $x \leq 0$ in order for its inverse to be a function.

Lesson 2—Equations of Inverse Relations

CLASS EXERCISES ANSWERS AND SOLUTION

1. **Step 1**
Replace $f(x)$ with *y*.

$$y = \frac{-10x - 8}{3}$$

Step 2
Interchange *x* and *y*.

$$x = \frac{-10y - 8}{3}$$

Step 3
Solve for *y*.

$$\begin{aligned} x &= \frac{-10y - 8}{3} \\ 3x &= -10y - 8 \\ 3x + 8 &= -10y \\ -\frac{3}{10}x - \frac{8}{10} &= y \\ y &= -\frac{3}{10}x - \frac{4}{5} \end{aligned}$$

Step 4
Replace *y* with $f^{-1}(x)$ if the inverse is a function.

Since $y = -\frac{3}{10}x - \frac{4}{5}$ is a linear function, the inverse is $f^{-1}(x) = -\frac{3}{10}x - \frac{4}{5}$.

2. **Step 1**
Replace $f(x)$ with *y*.

$$y = -3x - 6$$

Step 2
Determine the equation of the inverse relation.

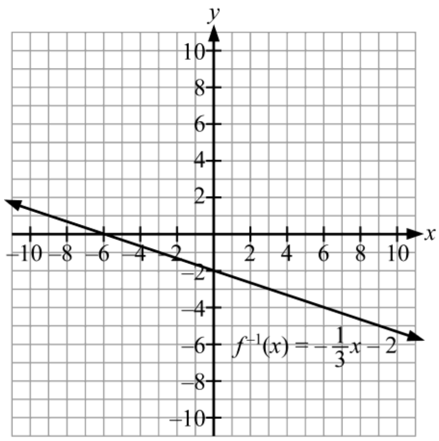
Interchange *x* and *y*, and solve for *y*.

$$\begin{aligned} y &= -3x - 6 \\ x &= -3y - 6 \\ x + 6 &= -3y \\ \frac{x+6}{-3} &= y \\ y &= -\frac{1}{3}x - 2 \end{aligned}$$

Step 3

Graph the equation $y = -\frac{1}{3}x - 2$.

The result is the graph of $f^{-1}(x) = -\frac{1}{3}x - 2$.



3. **Step 1**
Replace $f(x)$ with *y*.

$y = x^2 + 4x + 4$

Step 2
Interchange *x* and *y*.

$x = y^2 + 4y + 4$

Step 3
Solve for *y*.

$$x = y^2 + 4y + 4$$
$$x = (y+2)^2$$
$$\pm\sqrt{x} = y + 2$$
$$y = -2 \pm \sqrt{x}$$

Since the inverse is not a function, the equation of the inverse remains as $y = -2 \pm \sqrt{x}$.

4. **a)** **Step 1**
Replace $f(x)$ with *y*.

$y = (x-2)^2$

Step 2
Determine the equation of the inverse relation.

Interchange *x* and *y*, and solve for *y*.

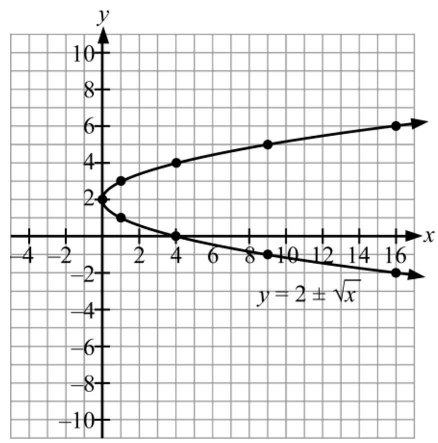
$$y = (x-2)^2$$
$$x = (y-2)^2$$
$$\pm\sqrt{x} = y - 2$$
$$2 \pm \sqrt{x} = y$$

Step 3
Determine a table of values for the inverse relation.

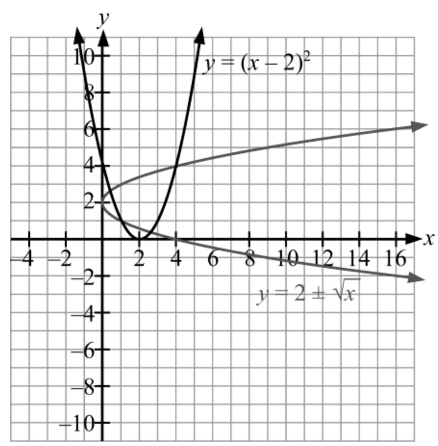
$y = 2+\sqrt{x}$		$y = 2-\sqrt{x}$	
x	*y*	*x*	*y*
0	2	0	2
1	3	1	1
4	4	4	0
9	5	9	–1
16	6	16	–2

Step 4
Plot and connect the points of the inverse relation.

The result is the graph of $y = 2 \pm \sqrt{x}$.



b) **Step 1**
Sketch the graph of $y = (x-2)^2$ and its inverse $y = 2 \pm \sqrt{x}$.

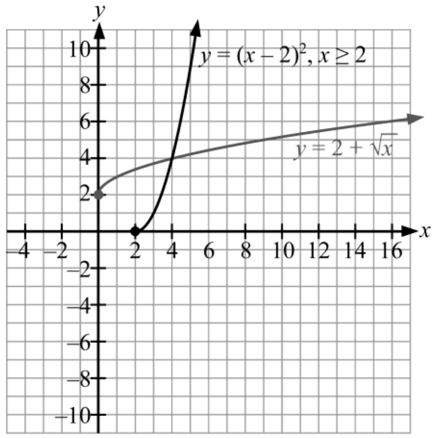


Step 2
Restrict the domain of $y=(x-2)^2$ so its inverse is a function.

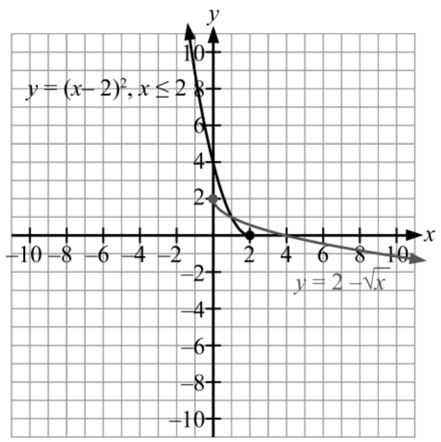
For the inverse to be a function, the range should be $y \geq 2$ or $y \leq 2$. Therefore, the domain of $y=x^2-2$ should be $x \geq 2$ or $x \leq 2$.

Step 3
Verify the restriction graphically.

When $y=(x-2)^2$ has a domain of $x \geq 2$, the inverse graph is defined by $y=2+\sqrt{x}$.



When $y=(x-2)^2$ has a domain of $x \leq 2$, the inverse graph is defined by $y=2-\sqrt{x}$.



Both inverse graphs pass the vertical line test.

5. **Step 1**
Replace $f(x)$ with *y*.

$y=7(x+2)^2-1$

Step 2
Interchange *x* and *y*, and solve for *y*.

$$\begin{aligned} y &= 7(x+2)^2-1 \\ x &= 7(y+2)^2-1 \\ \frac{x+1}{7} &= (y+2)^2 \\ \pm\sqrt{\frac{x+1}{7}} &= y+2 \\ y &= -2 \pm \sqrt{\frac{x+1}{7}} \end{aligned}$$

Step 3
Apply the restriction on the domain.

The domain of $f(x)$ is restricted to *x*-values on the left of the vertex.

Therefore, the inverse function is defined by the negative radical and can be stated as $f^{-1}(x)=-2-\sqrt{\frac{x+1}{7}}$.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Replace $f(x)$ with *y*.

$y=\frac{1}{2}x-13$

Step 2
Interchange *x* and *y*.

$x=\frac{1}{2}y-13$

Step 3
Solve for *y*.

$$\begin{aligned} x &= \frac{1}{2}y-13 \\ x+13 &= \frac{1}{2}y \\ y &= 2x+26 \end{aligned}$$

Step 4
Replace *y* with $f^{-1}(x)$ if the inverse is a function.

Since $y=2x+26$ is a linear function, the inverse is $f^{-1}(x)=2x+26$.

2. **Step 1**
Replace $f(x)$ with y.

$$y = -\frac{3}{2}x + 15$$

Step 2
Interchange x and y.

$$x = -\frac{3}{2}y + 15$$

Step 3
Solve for y.

$$\begin{aligned} x &= -\frac{3}{2}y + 15 \\ x - 15 &= -\frac{3}{2}y \\ 2x - 30 &= -3y \\ y &= -\frac{2}{3}x + 10 \end{aligned}$$

Step 4
Replace y with $f^{-1}(x)$ if the inverse is a function.

Since $y = -\frac{2}{3}x + 10$ is a linear function, the inverse is

$f^{-1}(x) = -\frac{2}{3}x + 10$.

3. **Step 1**
Replace $f(x)$ with y.

$$y = -5x + \frac{7}{2}$$

Step 2
Interchange x and y.

$$x = -5y + \frac{7}{2}$$

Step 3
Solve for y.

$$\begin{aligned} x &= -5y + \frac{7}{2} \\ 2x &= -10y + 7 \\ 2x - 7 &= -10y \\ \frac{2x - 7}{-10} &= y \\ y &= -\frac{1}{5}x + \frac{7}{10} \end{aligned}$$

Step 4
Replace y with $f^{-1}(x)$ if the inverse is a function.

Since $y = -\frac{1}{5}x + \frac{7}{10}$ is a linear function, the inverse is

$f^{-1}(x) = -\frac{1}{5}x + \frac{7}{10}$.

4. **Step 1**
Replace $f(x)$ with y.

$$y = x^2 + 14x + 49$$

Step 2
Interchange x and y.

$$x = y^2 + 14y + 49$$

Step 3
Solve for y.

$$\begin{aligned} x &= y^2 + 14y + 49 \\ x &= (y+7)(y+7) \\ x &= (y+7)^2 \\ \pm\sqrt{x} &= y + 7 \\ y &= -7 \pm \sqrt{x} \end{aligned}$$

Since the inverse is not a function, the equation of the inverse remains as $y = -7 \pm \sqrt{x}$.

5. **Step 1**
Replace $f(x)$ with y.

$$y = (x-3)^2 + 12$$

Step 2
Interchange x and y.

$$x = (y-3)^2 + 12$$

Step 3
Solve for y.

$$\begin{aligned} x &= (y-3)^2 + 12 \\ x - 12 &= (y-3)^2 \\ \pm\sqrt{x-12} &= y - 3 \\ y &= 3 \pm \sqrt{x-12} \end{aligned}$$

Since the inverse is not a function, the equation of the inverse remains as $y = 3 \pm \sqrt{x-12}$.

6. **Step 1**
Replace $f(x)$ with y.

$$y = (x+1)^2$$

Step 2
Interchange *x* and *y*.
$$x=(y+1)^2$$

Step 3
Solve for *y*.
$$\begin{aligned} x&=(y+1)^2 \\ \pm\sqrt{x}&=y+1 \\ y&=-1\pm\sqrt{x} \end{aligned}$$

Since the inverse is not a function, the equation of the inverse remains as $y=-1\pm\sqrt{x}$.

7. **Step 1**
Replace $f(x)$ with *y*.
$$y=\frac{-4x+3}{6}$$

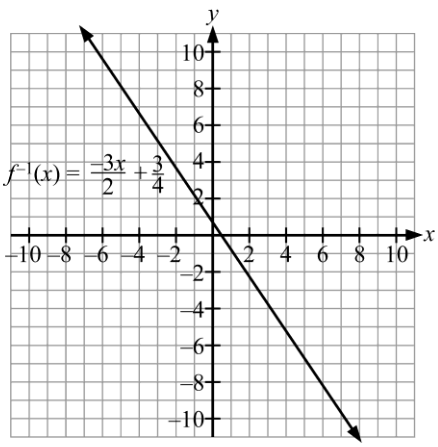
Step 2
Determine the equation of the inverse relation.

Interchange *x* and *y*, and solve for *y*.
$$\begin{aligned} y&=\frac{-4x+3}{6} \\ x&=\frac{-4y+3}{6} \\ 6x&=-4y+3 \\ 6x-3&=-4y \\ \frac{6x-3}{-4}&=y \\ y&=-\frac{3}{2}x+\frac{3}{4} \end{aligned}$$

Step 3

Graph the equation $y=-\frac{3}{2}x+\frac{3}{4}$.

The result is the graph of $f^{-1}(x)=-\frac{3}{2}x+\frac{3}{4}$.



8. **Step 1**
Replace $f(x)$ with *y*.
$$y=(x-5)^2$$

Step 2
Determine the equation of the inverse relation.

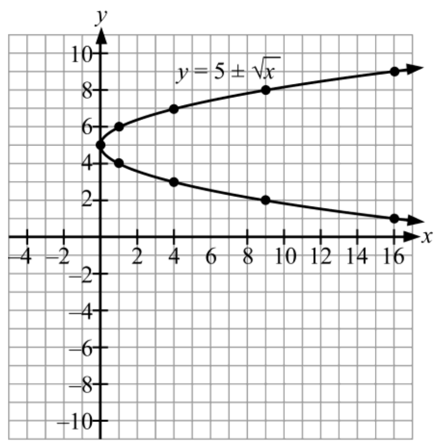
Interchange *x* and *y*, and solve for *y*.
$$\begin{aligned} y&=(x-5)^2 \\ x&=(y-5)^2 \\ \pm\sqrt{x}&=y-5 \\ y&=5\pm\sqrt{x} \end{aligned}$$

Step 3
Determine a table of values for the inverse relation.

$y=5+\sqrt{x}$		$y=5-\sqrt{x}$	
x	*y*	*x*	*y*
0	5	0	5
1	6	1	4
4	7	4	3
9	8	9	2
16	9	16	1

Step 4
Plot and connect the points of the inverse relation.

The result is the graph of $y=5\pm\sqrt{x}$.



9. **Step 1**
Replace $f(x)$ with *y*.
$$y=2(x-11)^2+12$$

Step 2
Interchange x and y, and solve for y.

$$\begin{aligned} y &= 2(x-11)^2+12 \\ x &= 2(y-11)^2+12 \\ x-12 &= 2(y-11)^2 \\ \frac{1}{2}x-6 &= (y-11)^2 \\ \pm\sqrt{\frac{1}{2}x-6} &= y-11 \\ y &= 11\pm\sqrt{\frac{1}{2}x-6} \end{aligned}$$

Step 3
Apply the restriction on the domain.

The domain of $f(x)$ is restricted to x-values on the left of the vertex.

Therefore, the inverse function is defined by the negative radical and can be stated as $f^{-1}(x)=11-\sqrt{\frac{1}{2}x-6}$.

10. **Step 1**
Replace $f(x)$ with y.

$$y=(x-7)^2$$

Step 2
Interchange x and y, and solve for y.

$$\begin{aligned} y &= (x-7)^2 \\ x &= (y-7)^2 \\ \pm\sqrt{x} &= y-7 \\ y &= 7\pm\sqrt{x} \end{aligned}$$

Step 3
Apply the restriction on the domain.

The domain of $f(x)$ is restricted to x-values on the right of the vertex.

Therefore, the inverse function is defined by the positive radical and can be stated as $f^{-1}(x)=7+\sqrt{x}$.

Lesson 3—Determining Whether Two Functions Are Inverses

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Compose the two functions as $g(h(x))$, and simplify.

$$\begin{aligned} g(x) &= 2x-4 \\ g(h(x)) &= 2\left(\frac{1}{2}x+8\right)-4 \\ g(h(x)) &= x+16-4 \\ g(h(x)) &= x+12 \end{aligned}$$

Since the composition of the two functions does not simplify to x, the functions are not inverses.

2. Compose the two functions as $f(g(x))$, and simplify.

$$\begin{aligned} f(x) &= 4x^2+7 \\ f(g(x)) &= 4\left(\sqrt{\frac{x-7}{4}}\right)^2+7 \\ f(g(x)) &= 4\left(\frac{x-7}{4}\right)+7 \\ f(g(x)) &= x-7+7 \\ f(g(x)) &= x \end{aligned}$$

Compose the two functions as $g(f(x))$, and simplify.

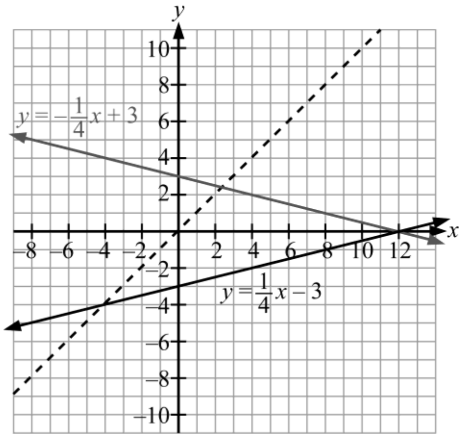
$$\begin{aligned} g(x) &= \sqrt{\frac{x-7}{4}} \\ g(f(x)) &= \sqrt{\frac{(4x^2+7)-7}{4}} \\ g(f(x)) &= \sqrt{\frac{4x^2}{4}} \\ g(f(x)) &= \sqrt{x^2} \end{aligned}$$

Since $x\ge 0$, then $\sqrt{x^2}=x$.

$$\begin{aligned} g(f(x)) &= \sqrt{x^2} \\ g(f(x)) &= x \end{aligned}$$

Since the result is x in both cases, the functions are inverses.

3. On the same axes, sketch the graphs of $y=\frac{1}{4}x-3$, $y=-\frac{1}{4}x+3$, and $y=x$.



The graphs of $y=\frac{1}{4}x-3$ and $y=-\frac{1}{4}x+3$ are not reflections of each other through the line $y=x$. Therefore, the functions are not inverses.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. Compose the two functions as $f(g(x))$, and simplify.

$$f(x)=2x-1$$
$$f(g(x))=2\left(\frac{1}{2}x+\frac{1}{2}\right)-1$$
$$f(g(x))=x+1-1$$
$$f(g(x))=x$$

Compose the two functions as $g(f(x))$, and simplify.

$$g(x)=\frac{1}{2}x+\frac{1}{2}$$
$$g(f(x))=\frac{1}{2}(2x-1)+\frac{1}{2}$$
$$g(f(x))=x-\frac{1}{2}+\frac{1}{2}$$
$$g(f(x))=x$$

Since the result is x in both cases, the functions are inverses.

2. Compose the two functions as $f(g(x))$, and simplify.

$$f(x)=\sqrt{36-x}$$
$$f(g(x))=\sqrt{36-(x^2-36)}$$
$$f(g(x))=\sqrt{36-x^2+36}$$
$$f(g(x))=\sqrt{-x^2}$$

Since the composition of the two functions does not simplify to x, the functions are not inverses.

3. Compose the two functions as $f(g(x))$, and simplify.

$$f(x)=-12x+10$$
$$f(g(x))=-12\left(-\frac{1}{12}x+\frac{5}{6}\right)+10$$
$$f(g(x))=x-10+10$$
$$f(g(x))=x$$

Compose the two functions as $g(f(x))$, and simplify.

$$g(x)=-\frac{1}{12}x+\frac{5}{6}$$
$$g(f(x))=-\frac{1}{12}(-12x+10)+\frac{5}{6}$$
$$g(f(x))=x-\frac{10}{12}+\frac{5}{6}$$
$$g(f(x))=x-\frac{5}{6}+\frac{5}{6}$$
$$g(f(x))=x$$

Since the result is x in both cases, the functions are inverses.

4. Compose the two functions as $f(g(x))$, and simplify.

$$f(x)=x^2+16x+64$$
$$f(g(x))=(-8+x)^2+16(-8+x)+64$$
$$f(g(x))=(-8+x)(-8+x)-128+16x+64$$
$$f(g(x))=64-16x+x^2-128-16x+64$$
$$f(g(x))=x^2$$

Since the composition of the two functions does not simplify to x, the functions are not inverses.

5. Compose the two functions as $f(g(x))$, and simplify.

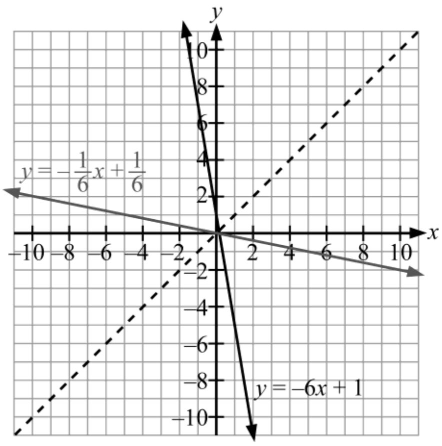
$$f(x)=2x+5$$
$$f(g(x))=2\left(\frac{x}{2}-\frac{5}{2}\right)+5$$
$$f(g(x))=x-5+5$$
$$f(g(x))=x$$

Compose the two functions as $g(f(x))$, and simplify.

$$g(x)=\frac{x}{2}-\frac{5}{2}$$
$$g(f(x))=\frac{(2x+5)}{2}-\frac{5}{2}$$
$$g(f(x))=\frac{2x+5-5}{2}$$
$$g(f(x))=\frac{2x}{2}$$
$$g(f(x))=x$$

Since the composition of these two functions simplifies to x, they are inverses.

6. On the same axes, sketch the graphs of $y=-6x+1$, $y=-\frac{1}{6}x+\frac{1}{6}$, and $y=x$.



The graphs of $y=-6x+1$ and $y=-\frac{1}{6}x+\frac{1}{6}$ are reflections of each other through the line $y=x$. Therefore, the functions are inverses.

Practice Test

ANSWERS AND SOLUTIONS

1. **Step 1**
Display points on the graph of $y=f(x)$ using a table of values.

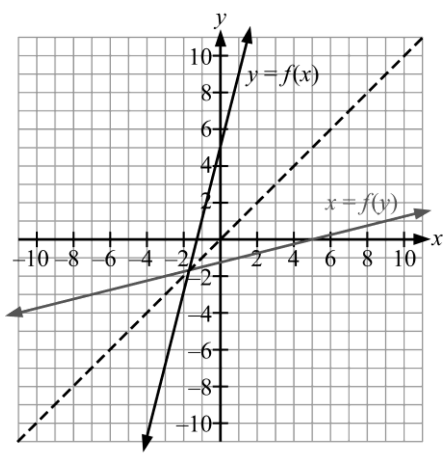
x	y
–3	–7
–2	–3
–1	1
0	5
1	9

Step 2
Reflect the points across the line $y=x$ by interchanging the *x*- and *y*-coordinates.

x	y
–7	–3
–3	–2
1	–1
5	0
9	1

Step 3
Plot and connect the reflected points.

The result is a reflection through the line $y=x$.



2. **Step 1**
Display points on the graph of $y = f(x)$ using a table of values.

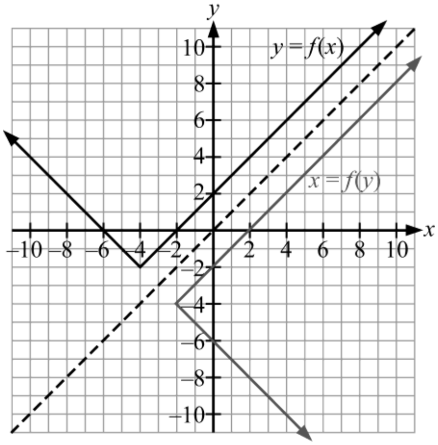
x	y
–8	2
–6	0
–4	–2
–2	0
0	2
2	4
4	6
6	8

Step 2
Reflect the points across the line $y = x$ by interchanging the *x*- and *y*-coordinates.

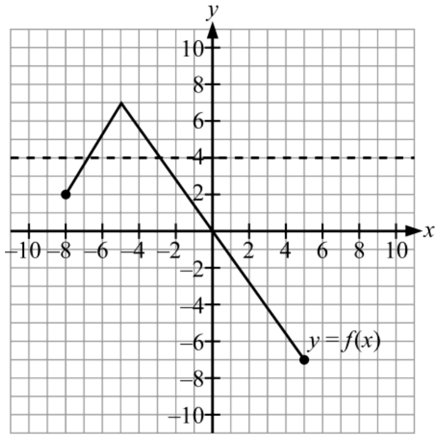
x	y
2	–8
0	–6
–2	–4
0	–2
2	0
4	2
6	4
8	6

Step 3
Plot and connect the reflected points.

The result is a reflection through the line $y = x$.



3. **Step 1**
Apply the horizontal line test to the graph of $y = f(x)$.



The graph of $y = f(x)$ does not pass the horizontal line test. Therefore, the inverse graph is not a function.

Step 2
Determine the domain and range of $y = f(x)$.

From the given graph, the domain is $-8 \le x \le 5$ and the range is $-7 \le y \le 7$.

Step 3
Determine the domain and range of the inverse relation.

The domain of the inverse relation is the range of $y = f(x)$. The range of the inverse relation is the domain of $y = f(x)$.

Therefore, the domain of the inverse relation is $-7 \le x \le 7$ and the range is $-8 \le y \le 5$.

4. **Step 1**
Replace $f(x)$ with *y*.

$$y = \frac{1}{4}x + 6$$

Step 2
Interchange *x* and *y*, and solve for *y*.

$$\begin{aligned} y &= \frac{1}{4}x + 6 \\ x &= \frac{1}{4}y + 6 \\ x - 6 &= \frac{1}{4}y \\ y &= 4x - 24 \end{aligned}$$

The inverse relation is defined as $y = 4x - 24$.

Step 3
Determine the x-intercept on the graph of $y = 4x - 24$.

Let $y = 0$, and solve for x.

$$\begin{aligned} y &= 4x - 24 \\ 0 &= 4x - 24 \\ 24 &= 4x \\ x &= 6 \end{aligned}$$

Therefore, the x-intercept on the graph of $y = 4x - 24$ is (6, 0).

5. Step 1
Replace $f(x)$ with y.

$$y = \frac{2}{3}x + 7$$

Step 2
Determine the equation of the inverse relation.

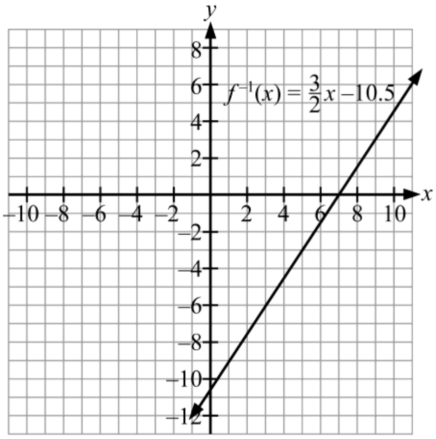
Interchange x and y, and solve for y.

$$\begin{aligned} x &= \frac{2}{3}y + 7 \\ x - 7 &= \frac{2}{3}y \\ 3x - 21 &= 2y \\ y &= \frac{3}{2}x - 10.5 \end{aligned}$$

Step 3

Graph the equation $y = \frac{3}{2}x - 10.5$.

The result is the graph of $f^{-1}(x) = \frac{3}{2}x - 10.5$.



6. Step 1
Replace $f(x)$ with y.

$$y = (x-5)^2 + 4$$

Step 2
Determine the equation of the inverse relation.

Interchange x and y, and solve for y.

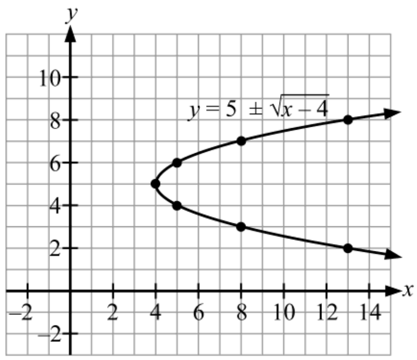
$$\begin{aligned} x &= (y-5)^2 + 4 \\ x - 4 &= (y-5)^2 \\ \pm\sqrt{x-4} &= y - 5 \\ y &= 5 \pm \sqrt{x-4} \end{aligned}$$

Step 3
Determine a table of values for the inverse relation.

$y = 5 + \sqrt{x-4}$		$y = 5 - \sqrt{x-4}$	
x	y	x	y
4	5	4	5
5	6	5	4
8	7	8	3
13	8	13	2

Step 4
Plot and connect the points of the inverse relation.

The result is the graph of $y = 5 \pm \sqrt{x-4}$.



Step 5
Restrict the domain of $f(x) = (x-5)^2 + 4$ so its inverse is a function.

For the inverse to be a function, the range should be $y \geq 5$ or $y \leq 5$.

Therefore, the domain of $f(x) = (x-5)^2 + 4$ should be $x \geq 5$ or $x \leq 5$.

7. Step 1
Replace $f(x)$ with y.

$$y = 5(x+10)^2 + 2$$

Step 2
Interchange *x* and *y*, and solve for *y*.

$$y = 5(x+10)^2 + 2$$
$$x = 5(y+10)^2 + 2$$
$$x-2 = 5(y+10)^2$$
$$\frac{(x-2)}{5} = (y+10)^2$$
$$\pm\sqrt{\frac{(x-2)}{5}} = y+10$$
$$y = -10 \pm \sqrt{\frac{(x-2)}{5}}$$

Step 3
Apply the restriction on the domain.

The domain of $f(x)$ is restricted to *x*-values on the left of the vertex.

Therefore, the inverse function is defined by the negative radical and is expressed as $f^{-1}(x) = -10 - \sqrt{\frac{(x-2)}{5}}$.

8. **Step 1**
Compose the two functions as $f(g(x))$, and simplify.

$$f(x) = 2(x-4)^2 + 5$$
$$f(g(x)) = 2\left(\left(4+\sqrt{\frac{x-5}{2}}\right)-4\right)^2 + 5$$
$$f(g(x)) = 2\left(\sqrt{\frac{x-5}{2}}\right)^2 + 5$$
$$f(g(x)) = 2\left(\frac{x-5}{2}\right) + 5$$
$$f(g(x)) = x-5+5$$
$$f(g(x)) = x$$

Step 2
Compose the two functions as $g(f(x))$, and simplify.

$$g(x) = 4+\sqrt{\frac{x-5}{2}}$$
$$g(f(x)) = 4+\sqrt{\frac{\left(2(x-4)^2+5\right)-5}{2}}$$
$$g(f(x)) = 4+\sqrt{\frac{2(x-4)^2}{2}}$$
$$g(f(x)) = 4+\sqrt{(x-4)^2}$$

Step 3
Since $x \ge 4$, then $\sqrt{(x-4)^2} = (x-4)$.

$$g(f(x)) = 4+(x-4)$$
$$g(f(x)) = 4+x-4$$
$$g(f(x)) = x$$

Since the composition of the two functions simplifies to *x*, the functions are inverses.

9. Compose the two functions as $f(g(x))$, and simplify.

$$f(x) = \frac{5}{2}x+11$$
$$f(g(x)) = \frac{5}{2}\left(11x - \frac{2}{5}\right)+11$$
$$f(g(x)) = \frac{55}{2}x - 1 + 11$$
$$f(g(x)) = \frac{55}{2}x - 10$$

Since the composition of the two functions does not simplify to *x*, the functions are not inverses.

10. Compose the two functions as $f(g(x))$, and simplify.

$$f(x) = 6(x+11)^2$$
$$f(g(x)) = 6\left(\left(-11+\sqrt{\frac{x}{6}}\right)+11\right)^2$$
$$f(g(x)) = 6\left(\sqrt{\frac{x}{6}}\right)^2$$
$$f(g(x)) = 6\left(\frac{x}{6}\right)$$
$$f(g(x)) = x$$

Compose the two functions as $g(f(x))$, and simplify.

$$g(x) = -11+\sqrt{\frac{x}{6}}$$
$$g(f(x)) = -11+\sqrt{\frac{6(x+11)^2}{6}}$$
$$g(f(x)) = -11+\sqrt{(x+11)^2}$$

Since $x \ge -11$, then $\sqrt{(x+11)^2} = x+11$.

$$g(f(x)) = -11+\sqrt{(x+11)^2}$$
$$g(f(x)) = -11+x+11$$
$$g(f(x)) = x$$

Since the composition of the two functions simplifies to *x*, the functions are inverses.

POLYNOMIAL FUNCTIONS

Lesson 1—Defining a Polynomial Function

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. The function $f(x)=3^{-x}+5x$ is not a polynomial function because it contains a term with a variable as an exponent.

The function $h(x)=5x-\sqrt{x}+1$ is not a polynomial function because it contains a term with an exponent that is not a whole number.

The function $g(x)=x^6-\frac{x}{9}+6$ is a polynomial function because the coefficients are real numbers, and the exponents are positive whole numbers.

2. The leading coefficient is $\sqrt{5}$, the degree is 6, and the constant term is 2.

3. Write the given function in general form.

$$f(x)=-6x(3x+1)+(9x^2-7)(x+5)+11$$
$$f(x)=-18x^2-6x+9x^3+45x^2-7x-35+11$$
$$f(x)=27x^2-13x+9x^3-24$$
$$f(x)=9x^3+27x^2-13x-24$$

The leading coefficient is 9, the degree is 3, and the constant term is –24.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. In the given polynomial, the coefficients are all real numbers, and the exponents are positive whole numbers. Therefore, the function is a polynomial function.

The leading coefficient is $-\frac{1}{5}$, the degree is 1, and the constant term is $\frac{39}{11}$.

2. The equivalent form of $f(x)=(4x^2-17)^{-2}$ is $f(x)=\frac{1}{(4x^2-17)^2}$. This function is not a polynomial because it contains a term with a variable in the denominator.

3. Express the function in general form.

$$n(x)=(-7x-1)(x+8)$$
$$n(x)=-7x^2-56x-x-8$$
$$n(x)=-7x^2-57x-8$$

The coefficients are all real numbers, and the exponents are positive whole numbers.
Therefore, the function is a polynomial function.

The leading coefficient is –7, the degree is 2, and the constant term is –8.

4. Express the function in general form.

$$g(x)=-(2x^3-3)\left(5x^{\frac{1}{3}}-3\right)+14$$
$$g(x)=-\left(10x-6x^3-15x^{\frac{1}{3}}+9\right)+14$$
$$g(x)=-10x+6x^3+15x^{\frac{1}{3}}-9+14$$
$$g(x)=6x^3-10x+15x^{\frac{1}{3}}+5$$

The function is not a polynomial function because it contains a term with an exponent that is not a whole number.

5. In the given polynomial, the coefficients are all real numbers, and the exponents are positive whole numbers. Therefore, the function is a polynomial function.

The leading coefficient is $-5\sqrt{7}$, the degree is 5, and the constant term is 87.

6. In the given polynomial, the coefficients are all real numbers, and the exponents are positive whole numbers. The function is a polynomial function.

The leading coefficient is 32, the degree is 0, and the constant term is 32.

7. The function $g(x)=2x^{-4}-3x^{-1}+7x^2+33$ is not a polynomial function because it contains two terms with negative exponents.

8. Express the function in general form.

$$f(x)=(-\sqrt{x}+1)(5\sqrt{x}-4x)$$
$$f(x)=-5x+4x^{\frac{3}{2}}+5\sqrt{x}-4x$$
$$f(x)=4x^{\frac{3}{2}}-9x+5\sqrt{x}$$

The function is not a polynomial function because it contains a term with an exponent that is not a whole number.

Lesson 2—Dividing a Polynomial by a Binomial

CLASS EXERCISES
ANSWERS AND SOLUTIONS

1. Set up the division expression as $x-5\overline{)3x^2-11x-20}$.

Step 1
Divide $3x^2$ by x to obtain the first term in the quotient.

$$\frac{3x^2}{x}=3x$$

The first term in the quotient is $3x$.

$$\begin{array}{r} 3x \\ x-5\overline{)3x^2-11x-20} \end{array}$$

Step 2
Multiply $3x$ by $(x-5)$, and align the product below the like terms in the dividend.

$$\begin{array}{r} 3x \\ x-5\overline{)3x^2-11x-20} \\ 3x^2-15x \end{array}$$

Step 3
Subtract $(3x^2-15x)$ from $(3x^2-11x)$, and bring down the next term (–20).

$$\begin{array}{r} 3x \\ x-5\overline{)3x^2-11x-20} \\ \underline{3x^2-15x} \\ 4x-20 \end{array}$$

Step 4
Divide $4x$ by x to obtain the second term in the quotient.

$$\frac{4x}{x}=4$$

The second term in the quotient is 4.

$$\begin{array}{r} 3x+4 \\ x-5\overline{)3x^2-11x-20} \\ \underline{3x^2-15x} \\ 4x-20 \end{array}$$

Step 5
Multiply 4 by $(x-5)$, and align the product below the like terms in the dividend.

$$\begin{array}{r} 3x+4 \\ x-5\overline{)3x^2-11x-20} \\ \underline{3x^2-15x} \\ 4x-20 \\ 4x-20 \end{array}$$

Step 6
Subtract $(4x-20)$ from $(4x-20)$ to determine the remainder.

$$\begin{array}{r} 3x+4 \\ x-5\overline{)3x^2-11x-20} \\ \underline{3x^2-15x} \\ 4x-20 \\ \underline{4x-20} \\ 0 \end{array}$$

The quotient is $3x+4$ with a remainder of 0.

The solution can also be written as

$$\frac{3x^2-11x-20}{x-5}=3x+4 \text{ or}$$

$3x^2-11x-20=(x-5)(3x+4)$.

2. Set up the division expression as
$x+7\overline{)8x^3+54x^2-9x+29}$.

Step 1
Divide $8x^3$ by x to obtain the first term in the quotient.

$$\frac{8x^3}{x}=8x^2$$

The first term in the quotient is $8x^2$.

$$\begin{array}{r} 8x^2 \\ x+7\overline{)8x^3+54x^2-9x+29} \end{array}$$

Step 2
Multiply $8x^2$ by $(x + 7)$, and align the product below the like terms in the dividend.

$$\begin{array}{r} 8x^2 \\ x+7\overline{)8x^3+54x^2-9x+29} \\ 8x^3+56x^2 \end{array}$$

Step 3
Subtract $(8x^3+56x^2)$ from $(8x^3+54x^2)$, and bring down the next term $(-9x)$.

$$\begin{array}{r} 8x^2 \\ x+7\overline{)8x^3+54x^2-9x+29} \\ \underline{8x^3+56x^2} \\ -2x^2-9x \end{array}$$

Step 4
Divide $-2x^2$ by x to obtain the second term in the quotient.

$$\frac{-2x^2}{x}=-2x$$

The second term in the quotient is $-2x$.

$$\begin{array}{r} 8x^2-2x \\ x+7\overline{)8x^3+54x^2-9x+29} \\ \underline{8x^3+56x^2} \\ -2x^2-9x \end{array}$$

Step 5
Multiply $-2x$ by $(x + 7)$, and align the product below the like terms in the dividend.

$$\begin{array}{r} 8x^2-2x \\ x+7\overline{\big)8x^3+54x^2-9x+29} \\ \underline{8x^3+56x^2} \\ -2x^2-9x \\ -2x^2-14x \end{array}$$

Step 6
Subtract $\left(-2x^2-14x\right)$ from $\left(-2x^2-9x\right)$, and bring down the next term (29).

$$\begin{array}{r} 8x^2-2x \\ x+7\overline{\big)8x^3+54x^2-9x+29} \\ \underline{8x^3+56x^2} \\ -2x^2-9x \\ \underline{-2x^2-14x} \\ 5x+29 \end{array}$$

Step 7
Divide $5x$ by x to obtain the third term in the quotient.

$$\frac{5x}{x}=5$$

The third term in the quotient is 5.

$$\begin{array}{r} 8x^2-2x+5 \\ x+7\overline{\big)8x^3+54x^2-9x+29} \\ \underline{8x^3+56x^2} \\ -2x^2-9x \\ \underline{-2x^2-14x} \\ 5x+29 \end{array}$$

Step 8
Multiply 5 by $(x+7)$, and align the product below the like terms in the dividend.

$$\begin{array}{r} 8x^2-2x+5 \\ x+7\overline{\big)8x^3+54x^2-9x+29} \\ \underline{8x^3+56x^2} \\ -2x^2-9x \\ \underline{-2x^2-14x} \\ 5x+29 \\ 5x+35 \end{array}$$

Step 9
Subtract $(5x+35)$ from $(5x+29)$ to determine the remainder.

$$\begin{array}{r} 8x^2-2x+5 \\ x+7\overline{\big)8x^3+54x^2-9x+29} \\ \underline{8x^3+56x^2} \\ -2x^2-9x \\ \underline{-2x^2-14x} \\ 5x+29 \\ \underline{5x+35} \\ -6 \end{array}$$

The quotient is $8x^2-2x+5$ with a remainder of -6.

The solution can also be written as

$$\frac{8x^3+54x^2-9x+29}{x+7}=8x^2-2x+3-\frac{6}{x+7}$$ or

$$8x^3+54x^2-9x+29=(x+7)\left(8x^2-2x+3\right)-6.$$

3. The expression $\left(12x+9x^2\right)$ is equivalent to $\left(9x^2+12x+0\right)$. The division expression is $x-11\overline{\big)9x^2+12x+0}$.

The complete division process is as shown:

$$\begin{array}{r} 9x+111 \\ x-11\overline{\big)9x^2+12x+0} \\ \underline{9x^2-99x} \\ 111x+0 \\ \underline{111x-1\,221} \\ 1\,221 \end{array}$$

The quotient is $9x+111$ with a remainder of 1 221.

The solution can also be written as

$$\frac{9x^2+12}{x-11}=9x+9+\frac{1\,221}{x-11}$$ or

$$9x^2+12=(x-11)(9x+9)+1\,221.$$

4. The root of $(x-6)$ is 6.

Step 1
Express the divisor and dividend in synthetic division form, and bring down the first coefficient (5).

$$\begin{array}{r|rrrrrr} 6 & 5 & -30 & -3 & 18 & 1 & -6 \\ & \downarrow & & & & & \\ \hline & 5 & & & & & \end{array}$$

Step 2
Multiply 6 by 5 to get 30. Write 30 under –30, and add to get 0.

$$\begin{array}{r|rrrrrr} 6 & 5 & -30 & -3 & 18 & 1 & -6 \\ & \downarrow & 30 & & & & \\ \hline & 5 & 0 & & & & \end{array}$$

Step 3
Multiply 6 by 0 to get 0. Write 0 under –3, and add to get –3.

$$\begin{array}{r|rrrrrr} 6 & 5 & -30 & -3 & 18 & 1 & -6 \\ & \downarrow & 30 & 0 & & & \\ \hline & 5 & 0 & -3 & & & \end{array}$$

Step 4
Multiply 6 by –3 to get –18. Write –18 under 18, and add to get 0.

$$\begin{array}{c|cccccc} 6 & 5 & -30 & -3 & 18 & 1 & -6 \\ & \downarrow & 30 & 0 & -18 & & \\ \hline & 5 & 0 & -3 & 0 & & \end{array}$$

Step 5
Multiply 6 by 0 to get 0. Write 0 under 1, and add to get 1.

$$\begin{array}{c|cccccc} 6 & 5 & -30 & -3 & 18 & 1 & -6 \\ & \downarrow & 30 & 0 & -18 & 0 & \\ \hline & 5 & 0 & -3 & 0 & 1 & \end{array}$$

Step 6
Multiply 6 by 1 to get 6. Write 6 under –6, and add to get 0.

$$\begin{array}{c|cccccc} 6 & 5 & -30 & -3 & 18 & 1 & -6 \\ & \downarrow & 30 & 0 & -18 & 0 & 6 \\ \hline & 5 & 0 & -3 & 0 & 1 & 0 \end{array}$$

Step 7
Express the solution in the form
$P(x)=D(x)Q(x)+R(x)$.

The quotient is $5x^4-3x^2+1$ with a remainder of 0.
The result can be stated as

$$\begin{aligned}&5x^5-30x^4-3x^3+18x^2+x-6\\&=(x+6)\left(5x^4-3x^2+1\right)\end{aligned}.$$

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. The division expression is $x+3\overline{\big)x^2-\frac{2}{3}x-7}$.

The complete division process is as shown:

$$\begin{array}{r l} & x-\frac{11}{3} \\ \cline{2-2} x+3 \big) & x^2-\frac{2}{3}x-7 \\ & \underline{x^2+3x} \\ & \quad -\frac{11}{3}x-7 \\ & \quad \underline{-\frac{11}{3}x-11} \\ & \qquad\qquad 4 \end{array}$$

The quotient is $x-\frac{11}{3}$ with a remainder of 4.

Therefore, the solution is

$x^2-\frac{2}{3}x-7=(x+3)\left(x-\frac{11}{3}\right)+4$.

2. The missing x-term must be added to the polynomial. Therefore, the division expression is

$x-3\overline{\big)2x^4-7x^3-5x^2+0x-2}$.

The complete division process is as shown:

$$\begin{array}{r l} & 2x^3-x^2-8x-24 \\ \cline{2-2} x-3 \big) & 2x^4-7x^3-5x^2+0x-2 \\ & \underline{2x^4-6x^3} \\ & \quad -x^3-5x^2 \\ & \quad \underline{-x^3+3x^2} \\ & \qquad -8x^2+0x \\ & \qquad \underline{-8x^2+24x} \\ & \qquad\quad -24x-2 \\ & \qquad\quad \underline{-24x+72} \\ & \qquad\qquad -74 \end{array}$$

The quotient is $2x^3-x^2-8x-24$ with a remainder of –74.

Therefore, the solution is

$$\begin{aligned}&2x^4-7x^3-5x^2+0x-2\\&=(x-3)\left(2x^3-x^2-8x-24\right)-74\end{aligned}$$

3. The expression $4x^3-5-4x$ is equivalent to $4x^3 - 4x - 5$. The missing x^2-term must be added to the polynomial. Therefore, the division expression is

$x-6\overline{\big)4x^3-0x^2-4x-5}$.

The complete division process is as shown:

$$\begin{array}{r l} & 4x^2+24x+140 \\ \cline{2-2} x-6 \big) & 4x^3-0x^2-4x-5 \\ & \underline{4x^3-24x^2} \\ & \quad 24x^2-4x \\ & \quad \underline{24x^3-144x} \\ & \qquad 140x-5 \\ & \qquad \underline{140x-840} \\ & \qquad\qquad 835 \end{array}$$

The quotient is $4x^2 + 24x + 140$ with a remainder of 835.

Therefore, the solution is

$4x^3-5-4x=(x-6)\left(4x^2+24x+140\right)+835$.

4. The expression $8x^3 – 5 – 7x – 2x^2$ is equivalent to $8x^3 – 2x^2 – 7x – 5$.

The synthetic division process is as shown:

$$\begin{array}{r|rrrr} 2 & 8 & -2 & -7 & -5 \\ & \downarrow & 16 & 28 & 42 \\ \hline & 8 & 14 & 21 & 37 \end{array}$$

The quotient is $8x^2 + 14x + 21$ with a remainder of 37.

Therefore, the solution is

$$\frac{8x^3 - 2x^2 - 7x - 5}{x-2} = 8x^2 + 14x + 21 + \frac{37}{x-2}.$$

5. The expression $(x^4 – 1)$ is equivalent to $\left(x^4 + 0x^3 + 0x^2 + 0x - 1\right)$, and the root of $(x – 1)$ is 1.

The synthetic division process is as shown:

$$\begin{array}{r|rrrrr} 1 & 1 & 0 & 0 & 0 & -1 \\ & \downarrow & 1 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 1 & 0 \end{array}$$

The quotient is $x^3 + x^2 + x$ 1 with a remainder of 0.

Therefore, the solution is $\frac{x^4 - 1}{x - 1} = x^3 + x^2 + x + 1$.

6. The expression $\left(3x^4 - 5x^2 + 12x\right)$ is equivalent to $\left(3x^4 + 0x^3 - 5x^2 + 12x + 0\right)$, and the root of $\left(x + 1\right)$ is –1.

The synthetic division process is as shown:

$$\begin{array}{r|rrrrr} -1 & 3 & 0 & -5 & 12 & 0 \\ & \downarrow & -3 & 3 & 2 & -14 \\ \hline & 3 & -3 & -2 & 14 & -14 \end{array}$$

The quotient is $3x^3 - 3x^2 - 2x + 14$ with a remainder of –14.

Therefore, the solution is

$$\frac{3x^4 - 5x^2 + 12x}{x+1} = 3x^3 - 3x^2 - 2x + 14 - \frac{14}{x+1}.$$

Lesson 3—The Remainder Theorem

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Apply the remainder theorem.

When $P(x)$ is divided by $\left(x - 2\right)$, the remainder is $P\left(2\right)$.

Determine the value of $P\left(2\right)$.

$$\begin{aligned} P(x) &= x^3 - 4x^2 + 2x + 5 \\ P(2) &= (2)^3 - 4(2)^2 + 2(2) + 5 \\ P(2) &= 8 - 16 + 4 + 5 \\ P(2) &= 1 \end{aligned}$$

The remainder is 1.

Step 2
Verify the solution by dividing $P(x)$ by $\left(x - 2\right)$ using synthetic division.

$$\begin{array}{r|rrrr} 2 & 1 & -4 & 2 & 5 \\ & \downarrow & 2 & -4 & -4 \\ \hline & 1 & -2 & -2 & 1 \end{array}$$

The remainder using synthetic division is 1.

2. **Step 1**
Apply the remainder theorem.
When $P(x)$ is divided by $\left(x + 2\right)$ (or $\left(x - \left(-2\right)\right)$), the remainder is equal to $P\left(-2\right)$.

Since the remainder is –18 when $P(x)$ is divided by $\left(x + 2\right)$, $P\left(-2\right) = -18$.

Step 2
Substitute –2 for x and –18 for $P(x)$, and solve for k.

$$\begin{aligned} P(x) &= x^5 - 7x^2 + kx + 10 \\ -18 &= (-2)^5 - 7(-2)^2 + k(-2) + 10 \\ -18 &= -32 - 28 - 2k + 10 \\ -18 &= -50 - 2k \\ 32 &= -2k \\ k &= -16 \end{aligned}$$

The value of k is –16.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Apply the remainder theorem.

When $P(x)=\frac{1}{2}x^2+5x-13$ is divided by $(x-4)$, the remainder is $P(4)$.

Determine the value of $P(4)$.

$$P(x)=\frac{1}{2}x^2+5x-13$$
$$P(4)=\frac{1}{2}(4)^2+5(4)-13$$
$$P(4)=8+20-13$$
$$P(4)=15$$

The remainder is 15.

Step 2

Verify the solution by dividing $\left(\frac{1}{2}x^2+5x-13\right)$ by $(x-2)$ using synthetic division.

$$\begin{array}{r|rrr} 4 & \frac{1}{2} & 5 & -13 \\ & \downarrow & 2 & 28 \\ \hline & \frac{1}{2} & 7 & 15 \end{array}$$

The remainder using synthetic division is 15.

2. **Step 1**
Apply the remainder theorem.
When $P(x)=15x^4-4x^5-7$ is divided by $(x+1)$, the remainder is $P(-1)$

Determine the value of $P(-1)$.

$$P(x)=15x^4-4x^5-7$$
$$P(-1)=15(-1)^4-4(-1)^5-7$$
$$P(-1)=15+4-7$$
$$P(-1)=12$$

The remainder is 12.

Step 2

Verify the solution by dividing $\left(15x^4-4x^5-7\right)$ by $(x+1)$ using synthetic division.

$$\begin{array}{r|rrrrrr} -1 & -4 & 15 & 0 & 0 & 0 & -7 \\ & \downarrow & 4 & -19 & 19 & -19 & 19 \\ \hline & -4 & 19 & -19 & 19 & -19 & 12 \end{array}$$

The remainder using synthetic division is 12.

3. **Step 1**
Apply the remainder theorem.

When $P(x)=-2x^3-16x^2+\frac{1}{3}x-19$ is divided by $(x-9)$, the remainder is $P(9)$.

Determine the value of $P(9)$.

$$P(x)=-2x^3-16x^2+\frac{1}{3}x-19$$
$$P(9)=-2(9)^3-16(9)^2+\frac{1}{3}(9)-19$$
$$P(9)=-1458-1296+3-19$$
$$P(9)=-2770$$

The remainder is –2770.

Step 2
Verify the solution by dividing $\left(-2x^3-16x^2+\frac{1}{3}x-19\right)$ by $(x-9)$ using synthetic division.

$$\begin{array}{r|rrrr} 9 & -2 & -16 & \frac{1}{3} & -19 \\ & \downarrow & -18 & -306 & -2\,751 \\ \hline & -2 & -34 & \frac{-917}{3} & -2\,770 \end{array}$$

The remainder using synthetic division is –2 770.

4. **Step 1**
Apply the remainder theorem.
When $P(x)$ is divided by $(x-4)$, the remainder is equal to $P(4)$.

Since the remainder is –7 when $P(x)$ is divided by $(x-4)$, $P(4)=-7$.

Step 2
Substitute 4 for x and –7 for $P(x)$, and solve for m.

$$P(x)=\frac{1}{2}x^3-mx^2+14$$
$$-7=\frac{1}{2}(4)^3-m(4)^2+14$$
$$-7=32-16m+14$$
$$-53=-16m$$
$$m=\frac{53}{16}$$

The value of m is $\frac{53}{16}$.

5. **Step 1**
Apply the remainder theorem.
Since dividing $f(x)$ by $(x-1)$ gives a remainder of –24, $f(1) = -24$.

Substitute 1 for x and –24 for $f(x)$, and simplify.

$$\begin{aligned} f(x) &= x^3 + mx^2 + nx - 30 \\ -24 &= (1)^3 + m(1)^2 + n(1) - 30 \\ -24 &= 1 + m + n - 30 \\ -24 &= -29 + m + n \\ 5 &= m + n \end{aligned}$$

Since dividing $f(x)$ by $(x+2)$ gives a remainder of –12, $f(-2) = -12$.

Substitute –2 for x and –12 for $f(x)$, and simplify.

$$\begin{aligned} f(x) &= x^3 + mx^2 + nx - 30 \\ -12 &= (-2)^3 + m(-2)^2 + n(-2) - 30 \\ -12 &= -8 + 4m - 2n - 30 \\ -12 &= -38 + 4m - 2n \\ 26 &= 4m - 2n \\ 13 &= 2m - n \end{aligned}$$

Step 2
Set up a system of equations.
Let $5 = m + n$ represent equation ① and $13 = 2m - n$ represent equation ②.

① $5 = m + n$
② $13 = 2m - n$

Step 3
Solve for m and n using the method of elimination.
Add equations ① and ②.

$$\begin{array}{r} 5 = m + n \\ \underline{13 = 2m - n} \\ 18 = 3m \end{array}$$

Solve for m.

$$\begin{aligned} 18 &= 3m \\ m &= 6 \end{aligned}$$

Substitute 6 for m in equation ①, and solve for n.

$$\begin{aligned} 5 &= m + n \\ 5 &= 6 + n \\ n &= -1 \end{aligned}$$

Therefore, $m = 6$ and $n = -1$.

Lesson 4—The Factor Theorem

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Apply the factor theorem.

If $P(-4) = 0$, then $(x+4)$ is a factor of $P(x)$.

Evaluate $P(-4)$.

$$\begin{aligned} P(x) &= x^4 + 9x^3 - 2x + 5 \\ P(-4) &= (-4)^4 + 9(-4)^3 - 2(-4) + 5 \\ P(-4) &= 256 - 576 + 8 + 5 \\ P(-4) &= -307 \end{aligned}$$

Since $P(-4) \neq 0$, $(x+4)$ is not a factor of $P(x) = x^4 + 9x^3 - 2x + 5$.

2. **Step 1**
Apply the factor theorem.

If $P(a) = 0$, then $x - a$ is a factor of $P(x)$.

Since $P(-5) = 0$, $(x+5)$ is a factor of $P(x)$.

Step 2
Using synthetic division, find another factor of $P(x)$ by dividing it by the known factor $(x+5)$.

$$\begin{array}{r|rrrr} -5 & 3 & -13 & -131 & 45 \\ & \downarrow & -15 & 140 & -45 \\ \hline & 3 & -28 & 9 & 0 \end{array}$$

The other factor is $3x^2 - 28x + 9$, and $P(x)$ can be expressed as $P(x) = (x+5)(3x^2 - 28x + 9)$.

Step 3
Factor the trinomial $3x^2 - 28x + 9$.

The factor $3x^2 - 28x + 9$ is a quadratic trinomial. It can be factored as $(x-9)(3x-1)$.

The complete factorization is
$P(x) = (x+5)(x-9)(3x-1)$.

3. Apply the factor theorem.

If $(x-10)$ is a factor of $P(x)$, then $P(10)=0$.

Substitute 10 for x and 0 for $P(x)$, and solve for k.

$$\begin{aligned} P(x) &= kx^3+14x^2-41x+10 \\ 0 &= k(10)^3+14(10)^2-41(10)+10 \\ 0 &= 1000k+1400-410+10 \\ 0 &= 1000k+1000 \\ -1000 &= 1000k \\ k &= -1 \end{aligned}$$

The value of k is –1.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. Apply the factor theorem.

If $P(6)=0$, then $(x-6)$ is a factor of $P(x)$.

Evaluate $P(6)$.

$$\begin{aligned} P(x) &= 3x^3-10x^2+27x-450 \\ P(6) &= 3(6)^3-10(6)^2+27(6)-450 \\ P(6) &= 648-360+162-450 \\ P(6) &= 0 \end{aligned}$$

Since $P(6)=0$, $(x-6)$ is a factor of $P(x)=3x^3-10x^2+27x-450$.

2. Apply the factor theorem.
If $P(-1)=0$, then $x+1$ is a factor of $P(x)$.

Evaluate $P(-1)$.

$$\begin{aligned} P(x) &= 2x^5+5x^4-x^3-14x+1 \\ P(-1) &= 2(-1)^5+5(-1)^4-(-1)^3-14(-1)+1 \\ P(-1) &= -2+5+1+14+1 \\ P(-1) &= 19 \end{aligned}$$

Since $P(-1)\neq 0$, $(x+1)$ is not a factor of $P(x)=2x^5+5x^4-x^3-14x+1$.

3. **Step 1**
Apply the factor theorem.
If $P(a)=0$, then $x-a$ is a factor of $P(x)$.

Since $P(-1)=0$, $x+1$ is a factor of $P(x)$.

Step 2
Using synthetic division, find another factor of $P(x)$ by dividing it by the known factor $(x+1)$.

$$\begin{array}{r|rrrr} -1 & 1 & -4 & -131 & -126 \\ & \downarrow & -1 & 5 & 126 \\ \hline & 1 & -5 & -126 & 0 \end{array}$$

The other factor is $x^2-5x-126$, and $P(x)$ can be expressed as $P(x)=(x+1)(x^2-5x-126)$.

Step 3
Factor the trinomial $x^2-5x-126$.
The factor $x^2-5x-126$ is a quadratic trinomial. It can be factored as $(x-14)(x+9)$.

The complete factorization is
$P(x)=(x+1)(x-14)(x+9)$.

4. **Step 1**
Apply the factor theorem.
If $P(a)=0$, then $x-a$ is a factor of $P(x)$.

Since $P(-4)=0$, $(x+4)$ is a factor of $P(x)$.

Step 2
Using synthetic division, find another factor of $P(x)$ by dividing it by the known factor $(x+4)$.

$$\begin{array}{r|rrrr} -4 & 12 & 48 & 1 & 4 \\ & \downarrow & -48 & 0 & -4 \\ \hline & 12 & 0 & 1 & 0 \end{array}$$

The other factor is $12x^2+1$, and $P(x)$ can be expressed as $P(x)=(x+4)(12x^2+1)$.

Step 3
Factor the binomial $12x^2+1$.
The factor $12x^2+1$ is a quadratic binomial that cannot be factored any further.

The complete factorization is
$P(x)=(x+4)(12x^2+1)$.

5. **Step 1**
Apply the factor theorem.
If $(x+11)$ is a factor of $P(x)$, then $P(-11)=0$.

Step 2
Determine the value of k.
Substitute –11 for x and 0 for $P(x)$, and solve for k.

$$\begin{aligned} P(x) &= x^4 - x^3 + 2x^2 + kx - 22 \\ 0 &= (-11)^4 - (-11)^3 + 2(-11)^2 + k(-11) - 22 \\ 0 &= 14641 + 1331 + 242 - 11k - 22 \\ 0 &= 16192 - 11k \\ 11k &= 16192 \\ k &= 1472 \end{aligned}$$

The value of k is 1 472.

6. Apply the factor theorem.
If $(x+2)$ and $(x-1)$ are factors of $P(x) = 2x^3 - mx + n$, $P(-2) = 0$ and $P(1) = 0$.

Step 1
Since $P(-2) = 0$, substitute −2 for x and 0 for $P(x)$ in $P(x) = 2x^3 - mx + n$, and simplify.

$$\begin{aligned} P(x) &= 2x^3 - mx + n \\ 0 &= 2(-2)^3 - m(-2) + n \\ 2m + n &= 16 \end{aligned}$$

Step 2
Since $P(1) = 0$, substitute 1 for x and 0 for $P(x)$ in $P(x) = 2x^3 - mx + n$, and simplify.

$$\begin{aligned} P(x) &= 2x^3 - mx + n \\ 0 &= 2(1)^3 - m(1) + n \\ 0 &= 2 - m + n \\ m - n &= 2 \end{aligned}$$

Step 3
Set up a system of equations using the simplified equations.
① $2m + n = 16$
② $m - n = 2$

Step 4
Solve the system of equations using the method of elimination.

Add equations ① and ② to eliminate n.
① $2m + n = 16$
② $m - n = 2$

$3m = 18$

Solve for m.

$$\begin{aligned} 3m &= 18 \\ m &= 6 \end{aligned}$$

Step 5
Substitute $m = 6$ into equation ①.

$$\begin{aligned} 2m + n &= 16 \\ 2(6) + n &= 16 \\ 12 + n &= 16 \\ n &= 4 \end{aligned}$$

Therefore, $m = 6$ and $n = 4$.

Lesson 5—The Integral Zero Theorem

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Identify the potential integral zeros of $P(x)$.
The potential integral zeros are the factors of –15.
These factors are ±1, ±3, ±5, and ±15.

Step 2
Determine a factor using the factor theorem.

Since –1 is a potential zero, determine if $(x + 1)$ is a factor of $P(x)$.

$$\begin{aligned} P(x) &= x^3 - 9x^2 + 23x - 15 \\ P(-1) &= (-1)^3 - 9(-1)^2 + 23(-1) - 15 \\ P(-1) &= -48 \end{aligned}$$

Therefore, $(x + 1)$ is not a factor.

Since 1 is a potential zero, determine if $(x-1)$ is a factor of $P(x)$.

$$\begin{aligned} P(x) &= x^3 - 9x^2 + 23x - 15 \\ P(1) &= (1)^3 - 9(1)^2 + 23(1) - 15 \\ P(1) &= 0 \end{aligned}$$

Therefore, $(x – 1)$ is a factor.

Step 3
Find another factor by dividing $P(x)$ by $(x-1)$.
Divide $P(x)$ by $(x-1)$ using synthetic division.

$$\begin{array}{r|rrrr} 1 & 1 & -9 & 23 & -15 \\ & \downarrow & 1 & -8 & 15 \\ \hline & 1 & -8 & 15 & 0 \end{array}$$

Another factor is $x^2 - 8x + 15$, and $P(x)$ can be expressed as $P(x) = (x-1)(x^2 - 8x + 15)$.

Step 4
Factor the trinomial $\left(x^2-8x+15\right)$.

The factor $\left(x^2-8x+15\right)$ can be factored as $(x-3)(x-5)$.

The complete factorization is
$P(x)=(x+2)(x-3)(x-5)$.

2. **Step 1**
Identify the potential integral zeros of $P(x)$.
The potential integral zeros are the factors of 35.
These factors are ±1, ±5, ±7 and ±35.

Step 2
Determine a factor of $P(x)$ using the factor theorem.

Since 1 is a potential zero, determine if $(x-1)$ is a factor of $P(x)$.
$P(x)=x^4+10x^3+12x^2-58x+35$
$P(1)=(1)^4+10(1)^3+12(1)^2-58(1)+35$
$P(1)=0$

Therefore, $(x-1)$ is a factor.

Step 3
Divide $P(x)$ by $(x-1)$ using synthetic division.

$$\begin{array}{r|rrrrr} 1 & 1 & 10 & 12 & -58 & 35 \\ & \downarrow & 1 & 11 & 23 & -35 \\ \hline & 1 & 11 & 23 & -35 & 0 \end{array}$$

Another factor is $\left(x^3+11x^2+23x-35\right)$, and $P(x)$ can be expressed as $P(x)=(x-1)\left(x^3+11x^2+23x-35\right)$.

Step 4
Apply the factor and integral zero theorems to factor $\left(x^3+11x^2+23x-35\right)$.

The potential zeros of $\left(x^3+11x^2+23x-35\right)$ are ±1, ±5, ±7, and ±35.

By inspection, 1 is a zero, and $(x-1)$ is a factor of $\left(x^3+11x^2+23x-35\right)$.

Divide $\left(x^3+11x^2+23x-35\right)$ by $(x-1)$ using synthetic division.

$$\begin{array}{r|rrrr} 1 & 1 & 11 & 23 & -35 \\ & \downarrow & 1 & 12 & 35 \\ \hline & 1 & 12 & 35 & 0 \end{array}$$

Another factor is $\left(x^2+12x+35\right)$, and $P(x)$ can be expressed as $P(x)=(x-1)(x-1)\left(x^2+12x+35\right)$.

Step 5
Complete the factoring process by factoring $\left(x^2+12x+35\right)$.

The factor $\left(x^2+12x+35\right)$ is a quadratic trinomial and can be factored as $(x+7)(x+5)$.

Therefore, the complete factorization is
$P(x)=(x-1)(x-1)(x+7)(x+5)$ or
$P(x)=(x-1)^2(x+7)(x+5)$.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Identify the potential integral zeros of $P(x)$.

The potential integral zeros are the factors of –6.
These factors are ±1, ±2, ±3, and ±6.

Step 2
Determine a factor using the factor theorem.
Since 1 is a potential zero, determine if $(x-1)$ is a factor of $P(x)$.
$P(x)=x^3+4x^2+x-6$
$P(1)=(1)^3+4(1)^2+(1)-6$
$P(1)=0$

Therefore, $(x-1)$ is a factor.

Step 3
Using synthetic division, find another factor by dividing $P(x)$ by $(x-1)$.

$$\begin{array}{r|rrrr} 1 & 1 & 4 & 1 & -6 \\ & \downarrow & 1 & 5 & 6 \\ \hline & 1 & 5 & 6 & 0 \end{array}$$

Another factor is (x^2+5x+6), and $P(x)$ can be expressed as $P(x)=(x-1)\left(x^2+5x+6\right)$.

Step 4
Factor the trinomial $\left(x^2+5x+6\right)$.

The factor $\left(x^2+5x+6\right)$ can be factored as $(x+2)(x+3)$.

The complete factorization is
$P(x)=(x-1)(x+2)(x+3)$.

2. **Step 1**
Identify the potential integral zeros of $P(x)$.
The potential integral zeros are the factors of –3.
These factors are ±1 and ±3.

Step 2
Determine a factor using the factor theorem.
Since –1 is a potential zero, determine if $(x + 1)$ is a factor of $P(x)$.

$$P(x)=2x^3+7x^2+2x-3$$
$$P(-1)=2(-1)^3+7(-1)^2+2(-1)-3$$
$$P(-1)=0$$

Therefore, $(x + 1)$ is a factor.

Step 3
Find another factor by dividing $P(x)$ by $(x + 1)$.
Divide $P(x)$ by $(x + 1)$ using synthetic division.

$$\begin{array}{r|rrrr} -1 & 2 & 7 & 2 & -3 \\ & \downarrow & -2 & -5 & 3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

Another factor is $(2x^2+5x-3)$, and $P(x)$ can be expressed as $P(x)=(x+1)(2x^2+5x-3)$.

Step 4
Factor the trinomial $(2x^2+5x-3)$.

The factor $(2x^2+5x-3)$ can be factored as $(2x-1)(x+3)$.

The complete factorization is
$P(x)=(x+1)(2x-1)(x+3)$.

3. **Step 1**
Identify the potential integral zeros of $P(x)$.
The potential integral zeros are ±1, ±2, ±3, ±4, ±6, ±8, ±12, and ±24.

Step 2
Determine a factor of $P(x)$ using the factor theorem.

Since 2 is a potential zero, determine if $(x-2)$ is a factor of $P(x)$.

$$P(x)=x^4-3x^3-6x^2+28x-24$$
$$P(2)=(2)^4-3(2)^3-6(2)^2+28(2)-24$$
$$P(2)=0$$

Therefore, $(x-2)$ is a factor.

Step 3
Divide $P(x)$ by $(x-2)$ using synthetic division.

$$\begin{array}{r|rrrrr} 2 & 1 & -3 & -6 & 28 & -24 \\ & \downarrow & 2 & -2 & -16 & 24 \\ \hline & 1 & -1 & -8 & 12 & 0 \end{array}$$

Another factor is $(x^3-x^2-8x+12)$, and $P(x)$ can be expressed as $P(x)=(x-2)(x^3-x^2-8x+12)$.

Step 4
Apply the factor and integral zero theorems to factor $(x^3-x^2-8x+12)$.

The potential zeros of $(x^3-x^2-8x+12)$ are ±1, ±2, ±3, ±4, ±6, and ±12.

By inspection, 2 is a zero, and $(x-2)$ is a factor of $(x^3-x^2-8x+12)$.

Divide $(x^3-x^2-8x+12)$ by $(x-2)$ using synthetic division.

$$\begin{array}{r|rrrr} 2 & 1 & -1 & -8 & 12 \\ & \downarrow & 2 & 2 & -12 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

Another factor is (x^2+x-6), and $P(x)$ can be expressed as $P(x)=(x-2)(x-2)(x^2+x-6)$.

Step 5
Complete the factoring process by factoring (x^2+x-6).

The factor (x^2+x-6) is a quadratic trinomial and can be factored as $(x+3)(x-2)$.

Therefore, the complete factorization is
$P(x)=(x-2)(x-2)(x+3)(x-2)$ or
$P(x)=(x-2)^3(x+3)$.

4. **Step 1**
Identify the potential integral zeros of $P(x)$.
The potential integral zeros are ±1, ±2, ±3, and ±6.

Step 2
Determine a factor of $P(x)$ using the factor theorem.

Since 1 is a potential zero, determine if $(x – 1)$ is a factor of $P(x)$.

$$P(x)=8x^4+33x^3+12x^2-47x-6$$
$$P(x)=8(1)^4+33(1)^3+12(1)^2-47(1)-6$$
$$P(x)=0$$

Therefore, $(x – 1)$ is a factor.

Step 3
Divide $P(x)$ by $(x – 1)$ using synthetic division.

1	8	33	12	–47	–6
	↓	8	41	53	6
	8	41	53	6	0

Another factor is $\left(8x^3+41x^2+53x+6\right)$, and $P(x)$ can be expressed as $P(x)=(x-1)\left(8x^3+41x^2+53x+6\right)$.

Step 4
Apply the factor and zero theorems to factor $\left(8x^3+41x^2+53x+6\right)$.

The potential integral zeros of $\left(8x^3+41x^2+53x+6\right)$ are ±1, ±2, ±3, and ±6.

By inspection, –3 is a zero, and $(x + 3)$ is a factor of $\left(8x^3+41x^2+53x+6\right)$.

Divide $\left(8x^3+41x^2+53x+6\right)$ by $(x + 3)$ using synthetic division.

–3	8	41	53	6
	↓	–24	–51	–6
	8	17	2	0

Another factor is $(8x^2 + 17x + 2)$, and $P(x)$ can be expressed as $P(x)=(x-1)(x+3)\left(8x^2+17x+2\right)$.

Step 5
Complete the factoring process by factoring $(8x^2 + 17x + 2)$.
The factor $(8x^2 + 17x + 2)$ is a quadratic trinomial that can be factored as $(8x+1)(x+2)$.

The complete factorization is
$P(x)=(x-1)(x+3)(8x+1)(x+2)$.

Lesson 6—Graphs of Polynomial Functions

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. When a polynomial function is in factored form $f(x)=(x-r_1)(x-r_2)(x-r_3)$, the zeros will be equal to r_1, r_2, and r_3.

Step 1
Determine the zeros of the polynomial function $f(x)=(x+7)(x-2)(x+3)$.

Write the polynomial function in the form $f(x)=(x-r_1)(x-r_2)(x-r_3)$.
$$f(x)=(x+7)(x-2)(x+3)$$
$$f(x)=(x-(-7))(x-2)(x-(-3))$$

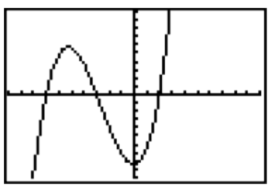
The zeros of the polynomial function are $x = –7$, $x = –3$, and $x = 2$.

Step 2
Confirm the solution by graphing $y = (x + 7)(x – 2)(x + 3)$.

Press $\boxed{\text{Y=}}$, enter Y_1 = (X + 7)(X – 2)(X + 3), and press $\boxed{\text{GRAPH}}$.

An appropriate window setting is x: [–10, 10, 1] and y: [–50, 50, 5].

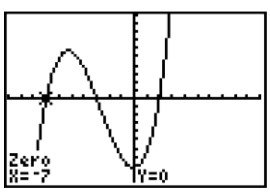
The resulting graph is as shown.



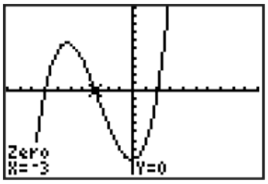
Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 2:zero.

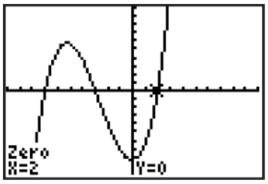
When asked for a left bound, move the cursor to the left of the first zero, and press $\boxed{\text{ENTER}}$. When asked for a right bound, move the cursor to the right of the first zero, and press $\boxed{\text{ENTER}}$.

Press $\boxed{\text{ENTER}}$ after the "Guess?" prompt.



Repeat the process for the second and third zeros.





The solution is confirmed, since the graphing calculator shows that the x-intercepts are at $x = -7$, $x = -3$, and $x = 2$.

2. **Step 1**
Determine the zeros of the function
$f(x) = x(x+2)^3(x-4)^2$.

The zeros of the function are 0, –2, and 4.

Step 2
Determine the multiplicity of each of the zeros.

- Since factor x has an exponent of 1, the zero 0 has a multiplicity of 1.
- Since the factor $(x+2)$ is cubed, the zero –2 has a multiplicity of 3.
- Since the factor $(x-4)$ is squared, the zero 4 has a multiplicity of 2.

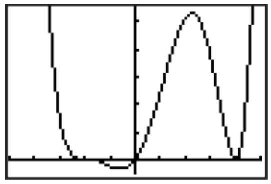
Step 3
Determine the behaviour of the graph at the x-intercepts.

- Because the multiplicity of 0 is 1, the graph cuts the x-axis at $x = 0$.
- Because the multiplicity of –2 is an odd number greater than 1, the graph is tangent to the x-axis at $x = -2$ and crosses it.
- Because the multiplicity of the zero 4 is an even number, the graph touches, or is tangent to, the x-axis at $x = 4$.

Step 4
Graph the function $f(x) = x(x+2)^3(x-4)^2$ using a TI-83 or similar calculator.

Press $\boxed{\text{Y=}}$, and input the function
$\text{Y}_1 = \text{X}(\text{X}+2)\wedge 3(\text{X}-4)\wedge 2$.

Use the window settings of x: [− 5, 5, 1] and y: [− 50, 550, 100], and press $\boxed{\text{GRAPH}}$ to obtain this window.



The graph of $f(x) = x(x+2)^3(x-4)^2$ confirms the behaviour in step 3.

3. The zeros of the function $f(x) = (x+6)^3(x+2)(x-2)$ are $x = -6$, $x = -2$, and $x = 2$.

The zero –6 has a multiplicity of 3, and the zeros –2 and 2 each have a multiplicity of 1.

Therefore, the degree of the function is 3 + 1 + 1 = 5.

4. As $x \to \pm\infty$, the y-values of $f(x)$ will be extremely large and positive. Also, the extreme ends of the graph are in quadrants I and II.

Therefore, the degree of $f(x)$ is even.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. Write the polynomial function to have the form $f(x) = (x-r_1)(x-r_2)(x-r_3)$.

$f(x) = (x+6)(x-8)(x-1)$
$f(x) = (x-(-6))(x-8)(x-1)$

The zeros of the polynomial function are $x = -6$, $x = 8$, and $x = 1$.

2. Write the polynomial function to have the form $f(x) = a(x-r_1)(x-r_2)(x-r_3)(x-r_4)$.

$f(x) = (2x+16)(x-9)(x)(x+3)$
$f(x) = 2(x+8)(x-9)(x-0)(x+3)$
$f(x) = 2(x-(-8))(x-9)(x-0)(x-(-3))$

The zeros of the polynomial function are $x = -8$, $x = 9$, $x = 0$, and $x = -3$.

3. The zeros of the function $f(x) = (x-7)^5(x+9)$ are 7 and –9. Since the factor $(x-7)$ has a power of 5, the multiplicity of the zero 7 is 5. Likewise, the multiplicity of the zero –9 is 1.

4. The zeros of the function $f(x)=x(3x+9)^2\left(x-\frac{1}{2}\right)^2$ are 0, –3, and $\frac{1}{2}$. Since the factors $(3x+9)$ and $\left(x-\frac{1}{2}\right)$ are squared, the multiplicity of each of the zeros –3 and $\frac{1}{2}$ is 2. Likewise, the multiplicity of the zero 0 is 1.

5. a) Step 1
Determine the zeros of the function
$f(x)=-(x-5)^4(x+4)^5$.

The zeros of the function are 5 and –4.

Step 2
Determine the multiplicity of each of the zeros.
- Since the factor (x – 5) has an exponent of 4, the zero 5 has a multiplicity of 4.
- Since the factor (x + 4) has an exponent of 5, the zero –4 has a multiplicity of 5.

Step 3
Determine the behaviour of the graph at the x-intercepts.
- Because the multiplicity of the zero 5 is an even number, the graph is tangent to the x-axis at x = 5, but it does not cross it.
- Because the multiplicity of the zero –4 is an odd number greater than 1, the graph crosses the x-axis at x = –4.

b) Step 1
Determine the degree and leading coefficient of the polynomial.

The degree is 9, and the leading coefficient is –1.

Step 2
Describe the end behaviour of the graph of the function.

The degree of $f(x)=-(x-5)^4(x+4)^5$ is odd, and the value of a is negative. Therefore, as $x \to -\infty$, the y-values of $f(x)$ will be extremely large and positive. Also, as $x \to \infty$, the y-values of $f(x)$ will be extremely large and negative.

The extreme ends of the graph are in quadrants II and IV.

6. a) Step 1
Determine the zeros of the function
$f(x)=2x(x-2)^3(x+10)^2$.
The zeros of the function are 0, 2, and –10.

Step 2
Determine the multiplicity of each of the zeros.
- Since the factor (x) has an exponent of 1, the zero 0 has a multiplicity of 1.
- Since the factor (x – 2) is cubed, the zero 2 has a multiplicity of 3.
- Since the factor (x + 10) is squared, the zero –10 has a multiplicity of 2.

Step 3
Determine the behaviour of the graph at the x-intercepts.
- Because the multiplicity of the zero 0 is 1, the graph passes through the x-axis at x = 1.
- Because the multiplicity of the zero 2 is an odd number, the graph is tangent to the x-axis at x = 2 and crosses it.
- Because the multiplicity of the zero –10 is an even number, the graph touches, or is tangent to, the x-axis at x = –10.

b) Step 1
Determine the degree and leading coefficient of the polynomial.

The degree is 6, and the leading coefficient is 2.

Step 2
Describe the end behaviour of the graph of the function.

The degree of $f(x)=2x(x-2)^3(x+10)^2$ is even, and the value of a is positive. Therefore, as $x \to -\infty$, the y-values of $f(x)$ will be extremely large and positive. Also, as $x \to \infty$, the y-values of $f(x)$ will be extremely large and positive.

The extreme ends of the graph are in quadrants I and II.

7. Step 1
Determine the zeros of the graph.
The zeros of the graph are –5, 0, and 4.

Step 2
Determine the multiplicity of each of the zeros.

- Since the graph is tangent at $x = –5$ and crosses it, the multiplicity of the zero –5 is odd and at least 3.
- Since the graph is tangent to the x-axis at $x = 0$ and $x = 4$ but does not cross it, the multiplicity of the zeros 0 and 4 is even and at least 2.

Step 3
Determine the minimum degree of the polynomial function represented by the given sketch.

The degree of a polynomial function is equal to the sum of the multiplicities of its zeros.

Therefore, the minimum degree of the polynomial modelled by the given sketch is $3+2+2=7$.

8. **Step 1**
Determine the zeros of the graph.
The zeros of the graph are –6, –4, –1, and 0.

Step 2
Determine the multiplicity of each of the zeros.

- Since the graph passes straight through the x-axis at $x=-6$, $x=-1$, and $x=0$, the multiplicity of each zero is 1.
- Since the graph is tangent to the x-axis at $x=-4$ and does not cross it, the multiplicity of the zero –4 is even and at least 2.

Step 3
Determine the minimum degree of the polynomial function represented by the given sketch.

The degree of a polynomial function is equal to the sum of the multiplicities of its zeros.

Therefore, the minimum degree of the polynomial modelled by the given graph is $1+1+1+2=5$.

Lesson 7—Sketching Polynomial Functions

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Determine the x- and y-intercepts of the graph of the function.

According to the factored form, the x-intercepts of the graph are 2 and 5.

The y-intercept is $y=-(0-2)^2(0-5)^2=-100$.

Step 2
Determine the behaviour of the graph at the x-intercepts.

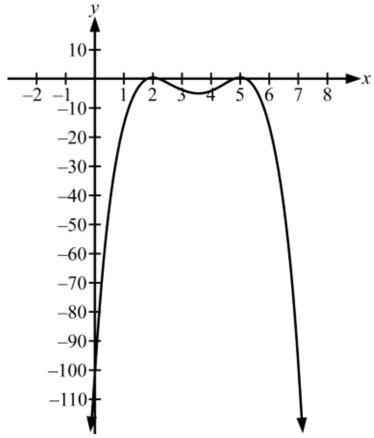
Since the multiplicity of 2 and 5 is 2, the graph does not cross but is tangent to the x-axis at $x = 2$ and $x = 5$.

Step 3
Determine the end behaviour.

The degree of $f(x)=-(x-2)^2(x-5)^2$ is an even number (4), and the leading coefficient is a negative number (–1). As $x \to -\infty$, the y-values of $f(x)$ will be extremely large and negative. Also, as $x \to \infty$, the y-values of $f(x)$ will be extremely large and negative. Therefore, the extreme ends of the graph are in quadrants III and IV.

Step 4
Sketch the graph of the function.

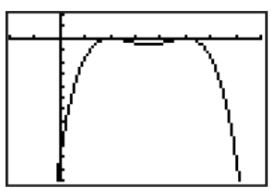
Plot the intercepts, and use the multiplicities of the zeros and end behaviour to draw a smooth curve through the points.



Step 5
Verify the shape of the graph using a TI-83 or similar calculator.

Press $\boxed{\text{Y=}}$, and input the function
$Y_1=-(X-2)^2(X-5)^2$.

Use the window settings of x: [–2, 8, 1] and y: [−120, 20, 10], and press $\boxed{\text{GRAPH}}$ to obtain this window.



The graph obtained on the calculator verifies the shape of the graph.

2. **Step 1**
Apply the factor theorem.

If $P(a)=0$, then $x-a$ is a factor of $P(x)$.
Since $f(3)=0$, $(x-3)$ is a factor of $f(x)$.

Step 2
Using synthetic division, completely factor $f(x)=-2x^3+8x^2+6x-36$ using the known factor $(x-3)$.

$$\begin{array}{r|rrrr} 3 & -2 & 8 & 6 & -36 \\ & \downarrow & -6 & 6 & 36 \\ \hline & -2 & 2 & 12 & 0 \end{array}$$

Another factor is $-2x^2+2x+12$, which can be factored as $-2(x-3)(x+2)$. Therefore, the complete factorization is $f(x)=-2(x-3)(x-3)(x+2)$ or $f(x)=-2(x-3)^2(x+2)$.

Step 3
Determine the x- and y-intercepts of the graph of the function.

According to the factored form, the x-intercepts of the graph are 3 and –2.

According to the general form, the y-intercept is –36.

Step 4
Determine the behaviour of the graph at the x-intercepts.

Since the multiplicity of 3 is 2, the graph is tangent to the x-axis at $x=3$, but it does not cross it.

Since the multiplicity of –2 is 1, the graph passes through the x-axis at $x=-2$.

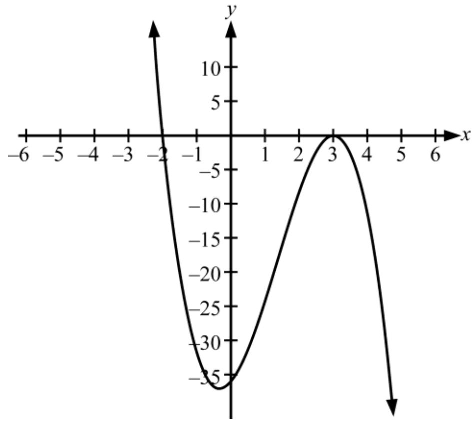
Step 5
Determine the end behaviour.

The degree of $f(x)=-2x^3+8x^2+6x-36$ is an odd number (3), and the leading coefficient is a negative number (–2). As $x\to-\infty$, the y-values of $f(x)$ will be extremely large and positive. Also, as $x\to\infty$, the y-values of $f(x)$ will be extremely large and negative.

Therefore, the extreme ends of the graph are in quadrants II and IV.

Step 6
Sketch the graph of the function.

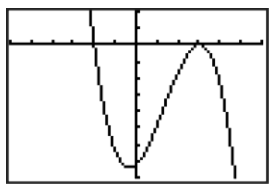
Plot the intercepts, and use the multiplicities of the zeros and end behaviour to draw a smooth curve through the points.



Step 7
Verify the shape of the graph using a TI-83 or similar calculator.

Press $\boxed{\text{Y=}}$, and input the function $Y_1=-2X\wedge3+8X\wedge2+6X-36$.

Use the window settings of x: [–6, 6, 1] and y: [−40, 10, 5], and press $\boxed{\text{GRAPH}}$ to obtain this window.



The graph obtained on the calculator verifies the shape of the graph.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. The factored form of $f(x)=x^3(x-5)$ is $f(x)=(x-0)^3(x-5)$.

Step 1
Determine the x- and y-intercepts of the graph of the function.

According to the factored form, the x-intercepts of the graph are 0 and 5.

The y-intercept is $y=(0-0)^3(0-5)=0$.

Step 2
Determine the behaviour of the graph at the *x*-intercepts.

Since the multiplicity of 0 is 3, the graph is tangent to the *x*-axis at $x = 0$ and crosses it.

Since the multiplicity of 5 is 1, the graph passes straight through the *x*-axis at $x = 5$.

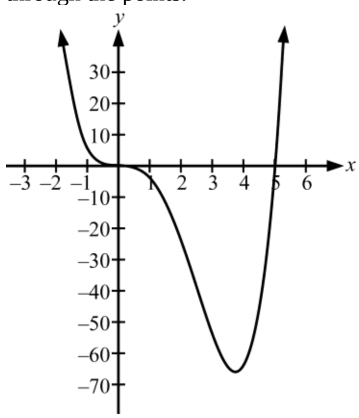
Step 3
Determine the end behaviour.

The degree of $f(x) = x^3(x-5)$ is an even number (4), and the leading coefficient is a positive number (1).
As $x \to -\infty$, the *y*-values of $f(x)$ will be extremely large and positive. Also, as $x \to \infty$, the *y*-values of $f(x)$ will be extremely large and positive.

Therefore, the extreme ends of the graph are in quadrants I and II.

Step 4
Sketch the graph of the function.

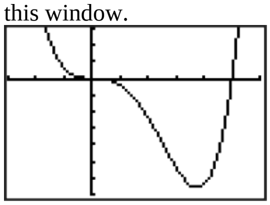
Plot the intercepts, and use the multiplicities of the zeros and end behaviour to draw a smooth curve through the points.



Step 5
Verify the shape of the graph using a TI-83 or similar calculator.

Press $\boxed{\text{Y=}}$, and input the function Y_1 = X^3(X–5).

Use the window settings of *x*: [–3, 6, 1] and *y*: [−70, 30, 10], and press $\boxed{\text{GRAPH}}$ to obtain this window.



The graph obtained on the calculator verifies the shape of the graph.

2. **Step 1**
Determine the *x*- and *y*-intercepts of the graph of the function.

According to the factored form, the *x*-intercepts of the graph are –4 and $\frac{1}{2}$.

The *y*-intercept is $y = -(0+4)^2\left(0-\frac{1}{2}\right)^3 = 2$.

Step 2
Determine the behaviour of the graph at the *x*-intercepts.

Since the multiplicity of –4 is 2, the graph is tangent to the *x*-axis at $x = -4$, but it does not cross it.

Since the multiplicity of $\frac{1}{2}$ is 3, the graph is tangent to the *x*-axis at $x = \frac{1}{2}$ and crosses it.

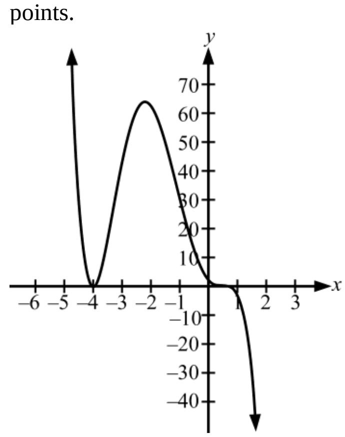
Step 3
Determine the end behaviour.

The degree of $f(x) = -(x+4)^2\left(x-\frac{1}{2}\right)^3$ is an odd number (5), and the leading coefficient is a negative number (–1). As $x \to -\infty$, the *y*-values of $f(x)$ will be extremely large and positive.
Also, as $x \to \infty$, the *y*-values of $f(x)$ will be extremely large and negative.

Therefore, the extreme ends of the graph are in quadrants II and IV.

Step 4
Sketch the graph of the function.

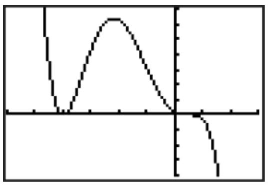
Plot the intercepts, and use the multiplicities of the zeros and end behaviour to draw a smooth curve through the points.



Step 5
Verify the shape of the graph using a TI-83 or similar calculator.

Press $\boxed{\text{Y=}}$, and input the function
$Y_1 = -(X+4)\wedge 2(X-(1/2))\wedge 3$.

Use the window settings of *x*: [–6, 3, 1]
and *y*: [−40, 70, 10], and press $\boxed{\text{GRAPH}}$ to obtain this window.



The graph obtained on the calculator verifies the shape of the graph.

3. **Step 1**
Determine the *x*- and *y*-intercepts of the graph of the function.

According to the factored form, the *x*-intercepts of the graph are –6 and –1.

The *y*-intercept is $y = \frac{1}{4}(0+6)^2(0+1)^2 = 9$.

Step 2
Determine the behaviour of the graph at the *x*-intercepts.

Since the multiplicity for both –6 and –1 is 2, the graph is tangent to the *x*-axis at $x = -6$ and $x = -1$, but it does not cross at these points.

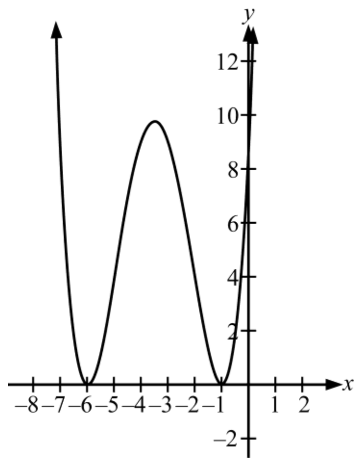
Step 3
Determine the end behaviour.

The degree of $f(x) = \frac{1}{4}(x+6)^2(x+1)^2$ is an even number (4), and the leading coefficient is a positive number $\left(\frac{1}{4}\right)$. As $x \to -\infty$, the *y*-values of *f*(*x*) will be extremely large and positive. Also, as $x \to \infty$, the *y*-values of *f*(*x*) will be extremely large and positive.

Therefore, the extreme ends of the graph are in quadrants I and II.

Step 4
Sketch the graph of the function.

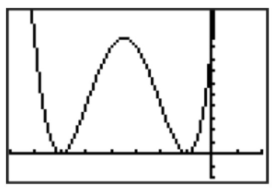
Plot the intercepts, and use the multiplicities of the zeros and end behaviour to draw a smooth curve through the points.



Step 5
Verify the shape of the graph using a TI-83 or similar calculator.

Press $\boxed{\text{Y=}}$, and input the function
$Y_1 = (1/4)(X+6)\wedge 2(X+1)\wedge 2$.

Use the window settings of *x*: [–8, 2, 1]
and *y*: [−2, 12, 1], and press $\boxed{\text{GRAPH}}$ to obtain this window.



The graph obtained on the calculator verifies the shape of the graph.

4. **Step 1**
Apply the factor theorem.
If $P(a) = 0$, then $x - a$ is a factor of *P*(*x*).
Since $f(-1) = 0$, $(x+1)$ is a factor of *f*(*x*).

Step 2
Using synthetic division, completely factor $f(x) = x^3 + 11x^2 + 35x + 25$ using the known factor $(x+1)$.

$$\begin{array}{r|rrrr} -1 & 1 & 11 & 35 & 25 \\ & \downarrow & -1 & -10 & -25 \\ \hline & 1 & 10 & 25 & 0 \end{array}$$

Another factor is $x^2+10x+25$, which can be factored as $(x+5)(x+5)$ or $(x+5)^2$.

Therefore, the factored form of the function is $f(x)=(x+1)(x+5)^2$.

Step 3
Determine the *x*- and *y*-intercepts of the graph of the function.

According to the factored form, the *x*-intercepts of the graph are –1 and –5.

The *y*-intercept is 25.

Step 4
Determine the behaviour of the graph at the *x*-intercepts.

Since the multiplicity of –1 is 1, the graph passes through the *x*-axis at $x=1$.

Since the multiplicity of –5 is 2, the graph is tangent to the *x*-axis at $x=-5$.

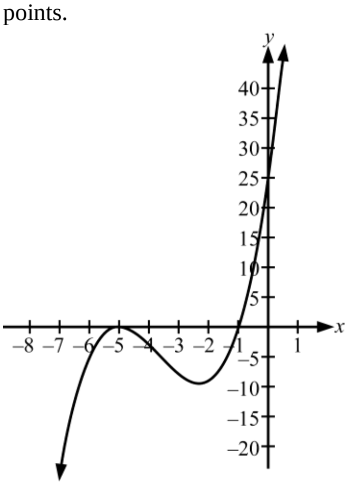
Step 5
Determine the end behaviour.

The degree of $f(x)=(x+1)(x+5)^2$ is an odd number (3), and the leading coefficient is a positive number (1). As $x\to-\infty$, the *y*-values of *f*(*x*) will be extremely large and negative. Also, as $x\to\infty$, the *y*-values of *f*(*x*) will be extremely large and positive.

Therefore, the extreme ends of the graph are in quadrants I and III.

Step 6
Sketch the graph of the function.

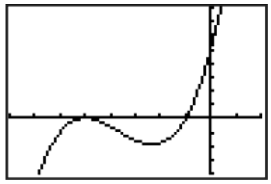
Plot the intercepts, and use the multiplicities of the zeros and end behaviour to draw a smooth curve through the points.



Step 7
Verify the shape of the graph using a TI-83 or similar calculator.

Press $\boxed{\text{Y=}}$, and input the function $\text{Y}_1=\text{X}^\wedge3+11\text{X}^\wedge2+35\text{X}+25$.

Use the window settings of *x*: [–8, 2, 1] and *y*: [−20, 40, 5], and press $\boxed{\text{GRAPH}}$ to obtain this window.



The graph obtained on the calculator verifies the shape of the graph.

5. **Step 1**
Apply the factor theorem.
If $P(a)=0$, then $x-a$ is a factor of *P*(*x*).
Since $f(1)=0$, $(x-1)$ is a factor of *f*(*x*).

Step 2
Using synthetic division, completely factor $f(x)=4x^3+8x^2-7x-5$ using the known factor $(x-1)$.

$$\begin{array}{r|rrrr} 1 & 4 & 8 & -7 & -5 \\ & \downarrow & 4 & 12 & 5 \\ \hline & 4 & 12 & 5 & 0 \end{array}$$

Another factor is $4x^2+12x+5$, which can be factored as $(2x+5)(2x+1)$ or $4\left(x+\frac{5}{2}\right)\left(x+\frac{1}{2}\right)$.

Therefore, the factored form of the function is $f(x)=4(x-1)\left(x+\frac{5}{2}\right)\left(x+\frac{1}{2}\right)$.

Step 3
Determine the *x*- and *y*-intercepts of the graph of the function.

According to the factored form, the *x*-intercepts of the graph are 1, $-\frac{5}{2}$, and $-\frac{1}{2}$.

The *y*-intercept is $y=4(0-1)\left(0+\frac{5}{2}\right)\left(0+\frac{1}{2}\right)=5$.

Step 4
Determine the behaviour of the graph at the x-intercepts.

Since the multiplicity of each zero is 1, the graph passes through the x-axis at $x=1$, $x=\frac{5}{2}$, and $x=\frac{1}{2}$.

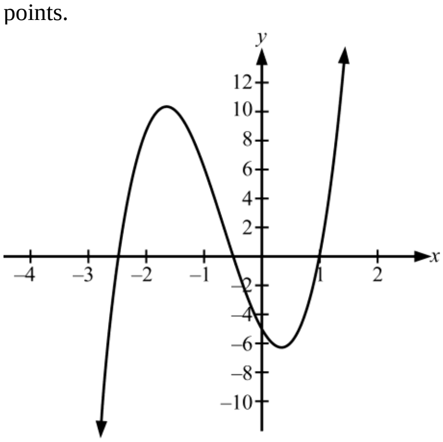
Step 5
Determine the end behaviour.

The degree of $f(x)=4(x-1)\left(x+\frac{5}{2}\right)\left(x+\frac{1}{2}\right)$ is an odd number (3), and the leading coefficient is a positive number (4). As $x\to-\infty$, the y-values of $f(x)$ will be extremely large and negative. Also, as $x\to\infty$, the y-values of $f(x)$ will be extremely large and positive.

Therefore, the extreme ends of the graph are in quadrants I and III.

Step 6
Sketch the graph of the function.

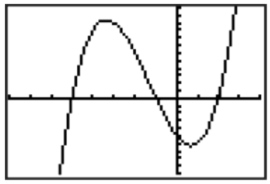
Plot the intercepts, and use the multiplicities of the zeros and end behaviour to draw a smooth curve through the points.



Step 7
Verify the shape of the graph using a TI-83 or similar calculator.

Press $\boxed{\text{Y=}}$, and input the function
$\text{Y}_1=4\text{X}^\wedge 3+8\text{X}^\wedge 2-7\text{X}-5$.

Use the window settings of x: [–4, 2, 0.5] and y: [−10, 12, 5], and press $\boxed{\text{GRAPH}}$ to obtain this window.



The graph obtained on the calculator verifies the shape of the graph.

Lesson 8—Solving Problems with Polynomial Functions

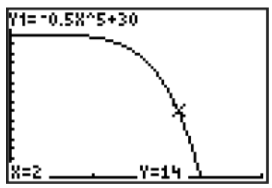
CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **a)** Since the function $h(t)=-\frac{1}{2}t^5+30$ is in general form, the y-intercept corresponds to the constant (30).

Therefore, the ledge is 30 m tall.

b) Determine the height when $t=2$.

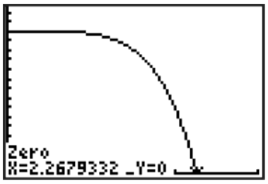
Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 1:value. Input 2, and press $\boxed{\text{ENTER}}$ to obtain this window.



Therefore, the height of the arrow after 2 s is 14.0 m.

c) The time at which the arrow hits the ground corresponds to when $h(t)=0$. The point at which $h(t)=0$ corresponds to the x-intercept on the graph of $y=-\frac{1}{2}x^5+30$. Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 2:zero.

When asked for a left bound, move the cursor to the left of the zero, and press $\boxed{\text{ENTER}}$. When asked for a right bound, move the cursor to the right of the zero, and press $\boxed{\text{ENTER}}$. Press $\boxed{\text{ENTER}}$ after the "Guess?" prompt.



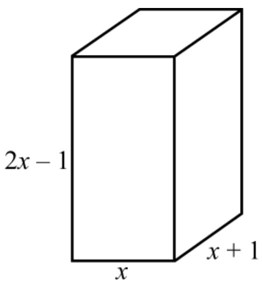
Therefore, the arrow hits the ground approximately 2.27 s after its release.

2. a) Step 1
Determine the independent and dependent variables.

The independent variable, *x*, is the width, and the dependent variable, *V*(*x*), is the volume of the bin.

Step 2
Draw a diagram.

Let *x* equal the width of the bin.



Step 3
Determine a function that models the volume of the storage bin.

$$\text{volume} = \text{length} \times \text{width} \times \text{height}$$
$$V(x) = (x+1)(x)(2x-1)$$

Therefore, the function that represents the volume of the storage bin is $V(x) = (x)(x+1)(2x-1)$, where *x* represents the width of the bin.

b) Since the volume of the bin is 7.5 m^3,
$7.5 = (x)(x+1)(2x-1)$.

Solve the equation $7.5 = (x)(x+1)(2x-1)$ using a TI-83 or similar calculator.

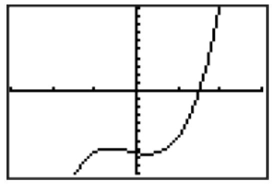
Step 1
Subtract 7.5 from both sides of the equation.

$$7.5 = (x)(x+1)(2x-1)$$
$$0 = (x)(x+1)(2x-1) - 7.5$$

Step 2
Graph the related function on a TI-83 or similar calculator.

Press $\boxed{\text{Y=}}$, and input the equation as

Use the window settings of *x*: [–3, 3, 1] and *y*: [–10, 10, 1], and press $\boxed{\text{GRAPH}}$ to obtain this window.



The width, *x*, corresponds to the positive *x*-intercept.

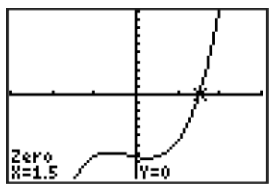
Step 3
Determine the zeros of the graph.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 2:zero.

When asked for a left bound, move the cursor to the left of the first zero, and press $\boxed{\text{ENTER}}$.

When asked for a right bound, move the cursor to the right of the first zero, and press $\boxed{\text{ENTER}}$.

Press $\boxed{\text{ENTER}}$ after the "Guess?" prompt.



Step 4
Determine the dimensions of the storage bin.

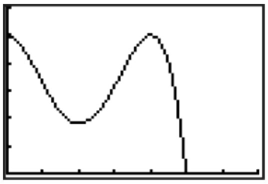
Since the zero of the graph represents the width, the width of the storage bin is 1.5 m, the length is $1.5 + 1 = 2.5\text{ m}$, and the height is $2(1.5) - 1 = 2\text{ m}$.

PRACTICE EXERCISES
ANSWERS AND SOLUTIONS

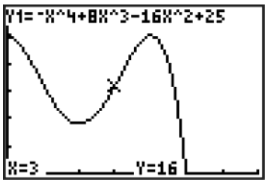
1. Using a TI-83 or similar calculator, determine the value of $h(t)$ when $t = 3$.

Press Y=, and input the function as $Y_1 = -X \wedge 4 + 8X \wedge 3 - 16X\wedge 2 + 25$.

Use the window settings of *x*: [0, 7, 1] and *y*: [0, 30, 5], and press GRAPH to obtain this window.



Press 2nd TRACE, and select 1:value. Input 3, and press ENTER to obtain this window.

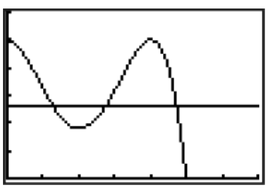


Therefore, the height of the toy airplane after 3 s is 16 m.

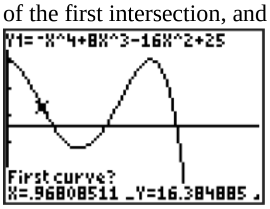
2. Since the height is 13 m, $13 = -t^4 + 8t^3 - 16t^2 + 25$.

Graph the line $y = 13$ in the same window as $h(t) = -t^4 + 8t^3 - 16t^2 + 25$.

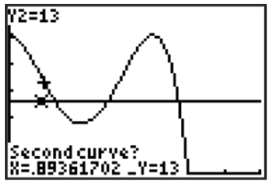
Press Y=, and input the equation as $Y_2 = 13$. Press GRAPH to obtain this window.



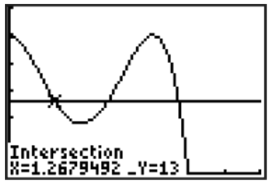
Press 2nd TRACE, and select 5:intersect.
When asked for a first curve, move the cursor to the left of the first intersection, and press ENTER.



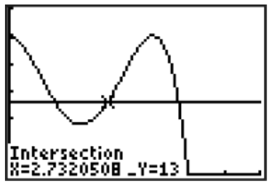
When asked for a second curve, move the cursor to the left of the first intersection point, and press ENTER.

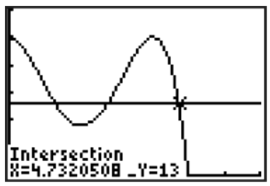


Press ENTER after the "Guess?" prompt.



Repeat this process for the other two intersection points.



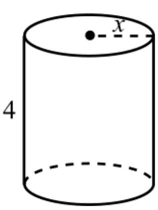


Therefore, the toy airplane is 13 m above the ground at 1.27 s, 2.73 s, and 4.73 s.

3. **Step 1**
Determine the independent and dependent variables.

The independent variable, *x*, is the radius, and the dependent variable, $V(x)$, is the volume of the jewellery container.

Step 2
Draw a diagram.
Variable *x* is the radius of the container.

3*x* – 4


Step 3
Determine a function that models the volume of the jewellery container.

$$V = \pi \times (\text{radius})^2 \times \text{height}$$
$$V(x) = \pi x^2 (3x - 4)$$

Therefore, the function that represents the volume of the jewellery container is $V(x) = \pi x^2 (3x-4)$, where x represents the radius of the container.

4. Since the volume is $402\ \text{cm}^3$, $402 = \pi x^2 (3x-4)$.
Solve the equation $402 = \pi x^2 (3x-4)$ using a TI-83 or similar calculator.

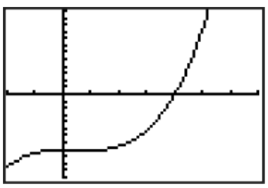
Step 1
Subtract 402 from both sides of the equation.

$$402 = \pi x^2 (3x-4)$$
$$0 = \pi x^2 (3x-4) - 402$$

Step 2
Graph the related function using a TI-83 or similar calculator.

Press $\boxed{\text{Y=}}$, and input the equation as $Y_1 = 3.14X \wedge 2(3X-4) - 402$.

Use the window settings of x: [–2, 7, 1] and y: [–600, 600, 50], and press $\boxed{\text{GRAPH}}$ to obtain this window.



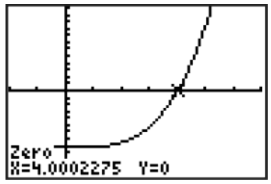
Step 3
Using the graph, determine the radius of the container.

The radius corresponds to the positive x-intercept. Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 2:zero.

When asked for a left bound, move the cursor to the left of the first zero, and press $\boxed{\text{ENTER}}$.

When asked for a right bound, move the cursor to the right of the first zero, and press $\boxed{\text{ENTER}}$.

Press $\boxed{\text{ENTER}}$ after the "Guess?" prompt.

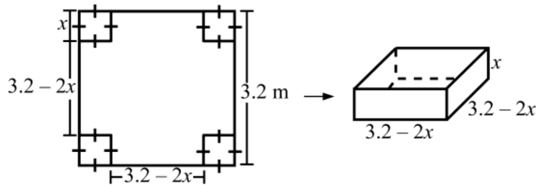


Therefore, the radius of the jewellery container is 4 cm.

5. **Step 1**
Determine the independent and dependent variables.

The independent variable, x, is the side length of the corner square, which is equivalent to the height of the box. The dependent variable, $V(x)$, is the volume of the box.

Step 2
Draw a diagram.
Variable x is equal to the height of the box.



Step 3
Determine a function that models the volume of the cardboard box.

$$\text{volume} = \text{length} \times \text{width} \times \text{height}$$
$$V(x) = (3.2 - 2x)(3.2 - 2x)x$$

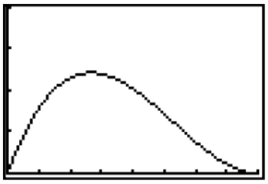
Therefore, the function that represents the volume of the box is $V(x) = (3.2-2x)(3.2-2x)x$, where x represents the height of the box.

6. The maximum volume corresponds to the maximum value of the function $V(x) = (3.2-2x)(3.2-2x)x$, where $0 < x < 1.6$.

Step 1
Graph the function using a TI-83 or similar calculator.

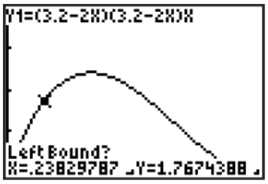
Press $\boxed{\text{Y=}}$, and input the function as $Y_1 = (3.2 - 2X)(3.2 - 2X)X$.

Use the window settings of x: [0, 1.6, 0.2] and y: [0, 4, 1], and press $\boxed{\text{GRAPH}}$ to obtain this window.

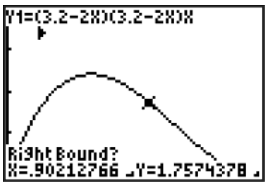


Step 2
Determine the maximum of the function.

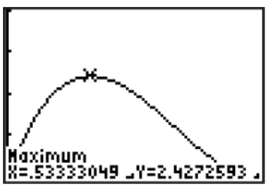
Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 4:maximum.
For "Left Bound?", position the cursor just left of the maximum, and press $\boxed{\text{ENTER}}$.



For "Right Bound?", position the cursor just right of the maximum, and press $\boxed{\text{ENTER}}$.



For "Guess?", press $\boxed{\text{ENTER}}$. The results are the coordinates of the maximum of the function.



The maximum of the function is approximately (0.53, 2.43).

Step 3
Determine the dimensions of the box that result in a maximum volume.

The maximum of the function is (0.53, 2.43).

In order to maximize the volume of the box, the height must be approximately 0.53 m and the length and width must both be $3.2-2(0.53)\approx 2.14\,\text{m}$.

Practice Test

ANSWERS AND SOLUTIONS

1. Step 1
Divide each term by 3.

$$f(x)=\frac{24x-12x^5+15x^2+8}{3}$$

$$f(x)=8x-4x^5+5x^2+\frac{8}{3}$$

Step 2
Rewrite the terms in order from highest degree to lowest degree.

$$f(x)=-4x^5+5x^2+8x+\frac{8}{3}$$

The leading coefficient is –4 because it is the coefficient of the term of highest degree. The degree is 5 because it is the largest exponent on a variable. The constant term is $\frac{8}{3}$.

2. The division expression is

$$x-8\overline{\big)x^5-12x^4+32x^3+3x^2-13x-8}\,.$$

The complete division process is as shown:

$$\begin{array}{r}
x^4\quad -4x^3\quad +0\quad +3x+11\\
x-8\overline{\big)x^5-12x^4+32x^3+3x^2-13x-8}\\
\underline{x^5-8x^4}\qquad\qquad\qquad\qquad\qquad\\
-4x^4+32x^3\qquad\qquad\qquad\quad\\
\underline{-4x^4+32x^3}\qquad\qquad\qquad\quad\\
0+3x^2\qquad\qquad\\
\underline{0+0}\qquad\qquad\\
3x^2-13x\qquad\\
\underline{3x^2-24x}\qquad\\
11x-8\\
\underline{11x-88}\\
80
\end{array}$$

The quotient is $x^4-4x^3+3x+11$ with a remainder of 80.

The solution is written as

$$\begin{aligned}&\left(x^5-12x^4+32x^3+3x^2-13x-8\right)\\&=(x-8)\left(x^4-4x^3+3x+11\right)+80\end{aligned}.$$

3. Since $P(-3)=44$, according to the remainder theorem, when $P(x)$ is divided by $(x+3)$, the remainder is 44.

Substitute –3 for x and 44 for $P(x)$, and solve for k.

$$\begin{aligned} P(x) &= 5x^3+kx^2-9x+8 \\ 44 &= 5(-3)^3+k(-3)^2-9(-3)+8 \\ 44 &= -135+9k+27+8 \\ 44 &= -100+9k \\ 144 &= 9k \\ k &= 16 \end{aligned}$$

Therefore, the value of k is 16.

4. **Step 1**

Apply the remainder theorem.

Since dividing $f(x)$ by $(x-2)$ gives a remainder of 6, $f(2)=6$.

Substitute 2 for x and 6 for $f(x)$, and simplify.

$$\begin{aligned} f(x) &= x^3+ax^2-3x+c \\ 6 &= (2)^3+a(2)^2-3(2)+c \\ 6 &= 8+4a-6+c \\ 6 &= 2+4a+c \\ 4 &= 4a+c \end{aligned}$$

Since $f(1)=8$, substitute 1 for x and 8 for $f(x)$, and simplify.

$$\begin{aligned} f(x) &= x^3+ax^2-3x+c \\ 8 &= (1)^3+a(1)^2-3(1)+c \\ 8 &= 1+a-3+c \\ 8 &= -2+a+c \\ 10 &= a+c \end{aligned}$$

Step 2

Set up a system of equations.

Let $4=4a+c$ represent equation ① and $10=a+c$ represent equation ②.

① $4=4a+c$

② $10=a+c$

Step 3

Solve for a and c using the method of elimination.

Subtract equation ② from equation ①.

$$\begin{aligned} 4 &= 4a+c \\ \underline{10} &\underline{= a+c} \\ -6 &= 3a \end{aligned}$$

Solve for a.

$$\begin{aligned} -6 &= 3a \\ a &= -2 \end{aligned}$$

Substitute –2 for a in equation ①, and solve for c.

$$\begin{aligned} 4 &= 4a+c \\ 4 &= 4(-2)+c \\ 4 &= -8+c \\ c &= 12 \end{aligned}$$

Therefore, $a=-2$, and $c=12$.

5. **a)** **Step 1**

Apply the factor theorem.

Since $f(-9)=0$, $(x+9)$ is a factor of $f(x)$.

Step 2

Using synthetic division, find another factor of $f(x)$ by dividing it by the known factor $(x+9)$.

$$\begin{array}{r|rrrr} -9 & 1 & 10 & \frac{37}{4} & \frac{9}{4} \\ & \downarrow & -9 & -9 & -\frac{9}{4} \\ \hline & 1 & 1 & \frac{1}{4} & 0 \end{array}$$

The other factor is $x^2+x+\frac{1}{4}$, and $f(x)$ can be expressed as $f(x)=(x+9)\left(x^2+x+\frac{1}{4}\right)$.

Step 3

Factor the trinomial $x^2+x+\frac{1}{4}$.

The factor $x^2+x+\frac{1}{4}$ is a quadratic trinomial and can be factored as $\left(x+\frac{1}{2}\right)\left(x+\frac{1}{2}\right)$ or $\left(x+\frac{1}{2}\right)^2$.

The complete factorization is

$$f(x)=(x+9)\left(x+\frac{1}{2}\right)^2.$$

Step 4
Determine the *x*-intercepts of the graph.
The *x*-intercepts of the graph are $x = -9$ and $x = -\frac{1}{2}$.

b) **Step 1**
Determine the multiplicity of each of the zeros.

Since the factor $(x+9)$ has an exponent of 1, the zero –9 has a multiplicity of 1.

Since the factor $\left(x+\frac{1}{2}\right)$ is squared, the zero $-\frac{1}{2}$ has a multiplicity of 2.

Step 2
Determine the behaviour of the graph at the *x*-intercepts.

Because the multiplicity of –9 is 1, the graph cuts through the *x*-axis at $x = -9$.

Because the multiplicity of $-\frac{1}{2}$ is an even number, the graph is tangent to the *x*-axis at $x = -\frac{1}{2}$, but it does not cross it.

c) **Step 1**
Determine the degree and leading coefficient of the polynomial.

The leading coefficient is 1.

The zero –9 has a multiplicity of 1, and the zero $-\frac{1}{2}$ has a multiplicity of 2.

Therefore, the degree of the function is 1 + 2 + 3.

Step 2
Describe the end behaviour of the graph of the function.

The degree of $f(x) = (x+9)\left(x+\frac{1}{2}\right)^2$ is an odd number, and the value of *a* is a positive number. Therefore, as $x \to -\infty$, the *y*-values of *f*(*x*) will be extremely large and negative. Also, as $x \to \infty$, the *y*-values of *f*(*x*) will be extremely large and positive.

The extreme ends of the graph are in quadrants I and III.

6. **Step 1**
Apply the factor theorem.
If $P(a) = 0$, then $x - a$ is a factor of *P*(*x*).
Since $f(4) = 0$, $(x-4)$ is a factor of *f*(*x*).

Step 2
Using synthetic division, completely factor $f(x) = x^3 - 5x^2 - 2x + 24$ using the known factor (*x* – 4).

$$\begin{array}{r|rrrr} 4 & 1 & -5 & -2 & 24 \\ & \downarrow & 4 & -4 & -24 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

Another factor is $x^2 - x - 6$, which can be factored as $(x+2)(x-3)$.

Therefore, the complete factorization is $f(x) = (x-4)(x+2)(x-3)$.

Step 3
Determine the *x*- and *y*-intercepts of the graph of the function.

According to the factored form, the *x*-intercepts of the graph are –2, 3, and 4.
The *y*-intercept is 24.

Step 4
Determine the behaviour of the graph at the *x*-intercepts.

Since the multiplicity of each zero is 1, the graph passes through the *x*-axis at $x = -2$, $x = 3$, and $x = 4$.

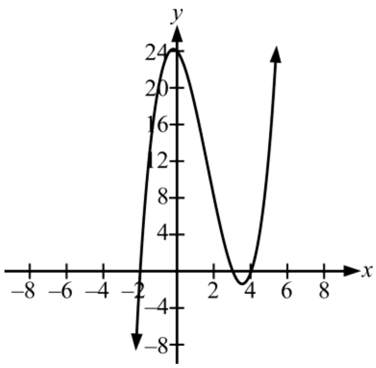
Step 5
Determine the end behaviour.

The degree of $f(x) = (x-4)(x+2)(x-3)$ is an odd number (3), and the leading coefficient is a positive number (1). As $x \to -\infty$, the *y*-values of *f*(*x*) will be extremely large and negative. Also, as $x \to \infty$, the *y*-values of *f*(*x*) will be extremely large and positive.

Therefore, the extreme ends of the graph are in quadrants I and III.

Step 6
Sketch the graph of the function.

Plot the intercepts, and use the multiplicities of the zeros and end behaviour to draw a smooth curve through the points.



7. Use the given diagram and the formula $\text{volume} = \frac{(\text{width})(\text{height})}{2} \times \text{length}$ to determine the function, $V(x)$.

$$\text{volume} = \frac{(\text{width})(\text{height})}{2} \times \text{length}$$
$$V(x) = \frac{x(0.5+x)}{2} \times 2x$$
$$V(x) = x^2(0.5+x)$$
$$V(x) = 0.5x^2 + x^3$$

Therefore, the function that represents the volume of the tent is $V(x) = 0.5x^2 + x^3$, where x represents the length of the base.

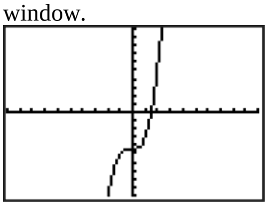
8. Since the volume is $4.5\ \text{m}^3$, $4.5 = 0.5x^2 + x^3$.

Solve the equation $4.5 = 0.5x^2 + x^3$ using a TI-83 or similar calculator.

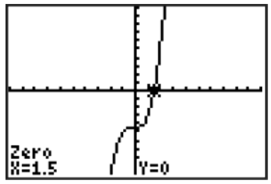
Subtract 4.5 from both sides of the equation.
$$4.5 = 0.5x^2 + x^3$$
$$0 = 0.5x^2 + x^3 - 4.5$$

Press $\boxed{\text{Y=}}$, and input the equation as $\text{Y}_1 = 0.5\text{X}^\wedge 2 + \text{X}^\wedge 3 - 4.5$.

Use the window settings of x: [–10, 10, 1] and y: [–10, 10, 1], and press $\boxed{\text{GRAPH}}$ to obtain this window.



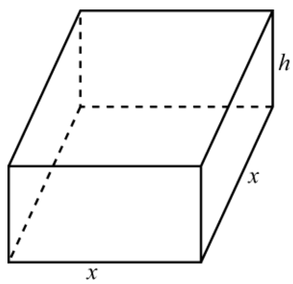
Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 2:zero. When asked for a left bound, move the cursor to the left of the first zero, and press $\boxed{\text{ENTER}}$. When asked for a right bound, move the cursor to the right of the first zero, and press $\boxed{\text{ENTER}}$. Press $\boxed{\text{ENTER}}$ after the “Guess?” prompt.



Therefore, the length of the base is 1.5 m.
The height of the base is $0.5 + 1.5 = 2\ \text{m}$, and the height of the prism is $2(1.5) = 3\ \text{m}$.

9. **Step 1**
Draw a diagram.

Let x represent the side length of the square base and h represent the height of the box.



Step 2
Determine an equation that represents the surface area.
$$SA = (x)(x) + 4(xh)$$
$$40 = x^2 + 4xh$$

Therefore, the equation that represents the surface area of the box is $40 = x^2 + 4xh$, where x represents the side length of the base, and h represents the height of the box.

10. The equation that represents the surface area of the box is $40 = x^2 + 4xh$.

Step 1
Isolate variable h.
$$40 = x^2 + 4xh$$
$$40 - x^2 = 4xh$$
$$\frac{40 - x^2}{4x} = h$$
$$h = \frac{10}{x} - \frac{1}{4}x$$

Step 2
Determine the function that represents the volume as a function of side length, *x*.

$$\text{volume} = \text{length} \times \text{width} \times \text{height}$$
$$V = (x)(x)(h)$$
$$V = x^2 h$$

Replace *h* with the expression $\left(\frac{10}{x} - \frac{1}{4}x\right)$, and simplify.

$$V = x^2 h$$
$$V = x^2\left(\frac{10}{x} - \frac{1}{4}x\right)$$
$$V = 10x - \frac{1}{4}x^3$$

Therefore, the function that represents the volume of the box is $V = 10x - \frac{1}{4}x^3$ or $V(x) = 10x - \frac{1}{4}x^3$, where *x* represents the side length of the base.

11. The maximum volume corresponds to the maximum value of the function $V(x) = 10x - \frac{1}{4}x^3$.

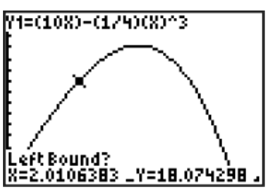
Step 1
Graph the function using a TI-83 or similar calculator.

Press $\boxed{\text{Y=}}$, and input the function as $\text{Y}_1 = (10\text{X}) - (1/4)(\text{X})\text{^}3$.

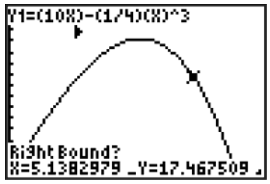
Use the window settings of *x*: [0, 7, 1] and *y*: [0, 30, 2], and press $\boxed{\text{GRAPH}}$ to obtain this window.

Step 2
Determine the maximum of the function.
Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 4:maximum.

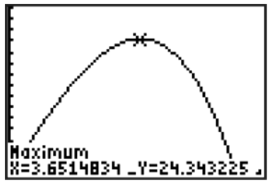
For "Left Bound?", position the cursor just left of the maximum, and press $\boxed{\text{ENTER}}$.



For "Right Bound?", position the cursor just right of the maximum, and press $\boxed{\text{ENTER}}$.



For "Guess?", press $\boxed{\text{ENTER}}$. The results are the coordinates of the maximum of the function.



The maximum of the function is approximately (3.65, 24.34).

Step 3
Determine the dimensions of the box that result in a maximum volume.

The maximum of the function is approximately (3.65, 24.34), so to maximize volume, the side length must be approximately 3.65 m.
Use the formula $V = x^2 h$ to determine the height.

$$V = x^2 h$$
$$24.3 \approx (3.65)^2 h$$
$$\frac{24.3}{(3.65)^2} \approx h$$
$$1.83 \approx h$$

Therefore, in order to maximize the volume of the box, the side length of the base must be approximately 3.65 m, and the height must be approximately 1.83 m.

RADICAL FUNCTIONS

Lesson 1—Properties of $y = \sqrt{x}$

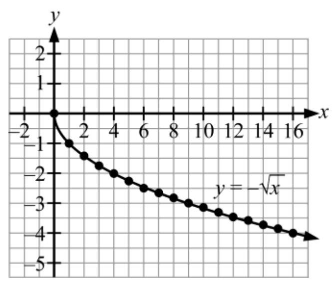
CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Build a table of values.

Only positive values of x need to be considered, since the square root of a negative number does not exist.

$y = -\sqrt{x}$	
x	***y***
0	0
1	–1
2	–1.4141…
3	–1.7320…
4	–2
5	–2.2360…
6	–2.4494…
7	–2.6457…
8	–2.8284…
9	–3
10	–3.1622…
11	–3.3166…
12	–3.4641…
13	–3.6055…
14	–3.7416…
15	–3.8729…
16	–4

Step 2
Plot the points, and sketch the graph of the function $y = -\sqrt{x}$.



Step 3
State the domain and range.

Since the square root of a negative number does not exist, the domain of $y = -\sqrt{x}$ is $x \geq 0$. The graph extends downward, so the range is $y \leq 0$.

Step 4
Determine the intercepts.

The graph of the function $y = -\sqrt{x}$ has x- and y-intercepts located at (0, 0).

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Determine the inverse relation of the function $y = x^2$.

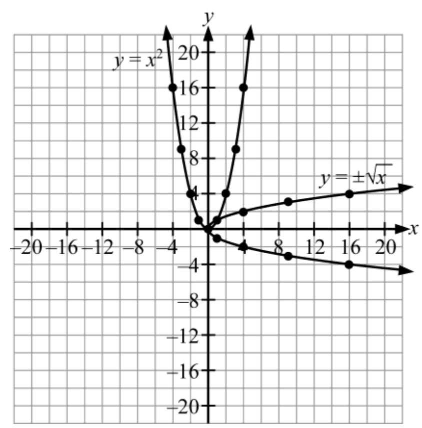
Find the inverse of the relation $y = x^2$ by interchanging x and y and then solving for y.

$y = x^2$
$x = y^2$
$y = \pm\sqrt{x}$

The inverse relation is $y = \pm\sqrt{x}$.

Step 2
Graph $y = x^2$ and $y = \pm\sqrt{x}$ on the same Cartesian plane.

The graphs of the function and its inverse are as shown.



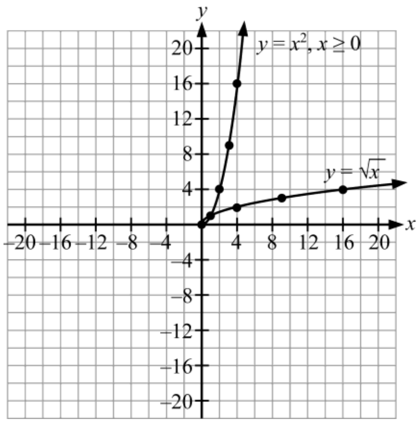
Step 3
By looking at the graphs, describe how the inverse of $y = x^2$ is not a function.

You can determine whether the graph of $y = \pm\sqrt{x}$ represents a function by using the vertical line test. The vertical line test implies that if a vertical line drawn on the grid touches the graph at more than one point, then the graph is not a function.

The inverse relation $y = \pm\sqrt{x}$ is a sideways parabola that does not pass the vertical line test. Therefore, the inverse relation is not a function.

2. If the domain of $y = x^2$ is restricted to only positive x-values $(x \geq 0)$, then the inverse relation is restricted to only positive y-values.

The graphs of $y = x^2$, where $x \geq 0$, and its inverse are shown.



The resulting inverse graph is the graph of the radical function $y = \sqrt{x}$, which has the shape of half of a parabola that has been reflected about the $y = x$ line. Therefore, when the domain of $y = x^2$ is restricted to $x \geq 0$, its inverse is the radical function $y = \sqrt{x}$.

Lesson 2—Graphs of Radical Functions

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Step 1
Determine the equation of the transformed function.

Since the graph of $y = \sqrt{x}$ is reflected in the x-axis and stretched about the x-axis by a factor of 4, replace y with $-\frac{1}{4}y$.

$$y = \sqrt{x}$$
$$-\frac{1}{4}y = \sqrt{x}$$

Solve for y.

$$-\frac{1}{4}y = \sqrt{x}$$
$$-y = 4\sqrt{x}$$
$$y = -4\sqrt{x}$$

Therefore, the equation of the transformed graph is $y = -4\sqrt{x}$.

Step 2
State the domain and range of the function $y = -4\sqrt{x}$.

Since only non-negative numbers have square roots, the domain is $x \geq 0$.

Since $a < 0$ in the equation $y = -4\sqrt{x}$, the range is $y \leq 0$.

2. Step 1
Determine how to obtain the graph of $y = \sqrt{(-6x)}$ from the graph of $y = \sqrt{x}$.

The equation $y = \sqrt{(-6x)}$ is of the form $y = \sqrt{bx}$, where $b = -6$.

Therefore, the graph of $y = \sqrt{(-6x)}$ is obtained from $y = \sqrt{x}$ by a reflection in the y-axis and a horizontal stretch by a factor of $\frac{1}{6}$.

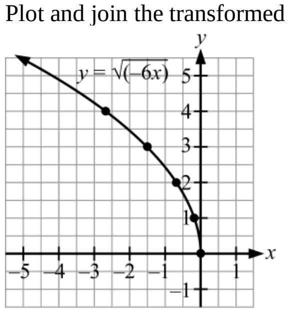
Step 2
Transform the points on the graph of $y = \sqrt{x}$.

Points on the graph of $y = \sqrt{x}$ are (0, 0), (1, 1), (4, 2), (9, 3), and (16, 4).

Multiply the x-coordinates by $-\frac{1}{6}$.

$$\left(0 \times \left(-\frac{1}{6}\right), 0\right) \rightarrow (0,0)$$
$$\left(1\left(-\frac{1}{6}\right), 1\right) \rightarrow \left(-\frac{1}{6}, 1\right)$$
$$\left(4\left(-\frac{1}{6}\right), 2\right) \rightarrow \left(-\frac{2}{3}, 2\right)$$
$$\left(9\left(-\frac{1}{6}\right), 3\right) \rightarrow \left(-\frac{3}{2}, 3\right)$$
$$\left(16\left(-\frac{1}{6}\right), 4\right) \rightarrow \left(-\frac{8}{3}, 4\right)$$

Step 3
Plot and join the transformed points.



Step 4
State the domain and range.

The domain is $x \leq 0$, and the range is $y \geq 0$.

3. **Step 1**
Determine the equation of the transformed graph.

If the graph of $y=\sqrt{x}$ is horizontally translated 9 units to the left, the value of h is –9, and the new equation is $y=\sqrt{(x-(-9))}$ or $y=\sqrt{(x+9)}$.

Step 2
Find the new intercepts.

Since the x-coordinates will decrease by 9 units, the x-intercept of (0, 0) on the graph of $y=\sqrt{x}$ will become (–9, 0) on the graph of $y=\sqrt{(x+9)}$.

To find the y-intercept of the graph of $y=\sqrt{(x+9)}$, substitute $x=0$, and solve for y.

$$\begin{aligned} y&=\sqrt{(x+9)}\\ y&=\sqrt{(0+9)}\\ y&=\sqrt{9}\\ y&=3 \end{aligned}$$

The y-intercept is (0, 3).

4. **Step 1**
Apply the vertical translation.

Let k represent the vertical translation. The equation of the radical function after the vertical translation will be $y=\sqrt{x}+k$.

The graph passes through the point (9, 10). Substitute 9 for x and 10 for y in the equation $y=\sqrt{x}+k$, and solve for k.

$$\begin{aligned} y&=\sqrt{x}+k\\ 10&=\sqrt{9}+k\\ 10&=3+k\\ k&=7 \end{aligned}$$

Step 2
Write the equation of the transformed function.

After a vertical translation of 7 units up, the graph of the function $y=\sqrt{x}$ is transformed to the graph of the function $y=\sqrt{x}+7$.

5. **a)** **Step 1**
Apply the vertical reflection.
Since the graph of $y=\sqrt{x}$ is reflected in the x-axis, replace y with $-y$ in the equation $y=\sqrt{x}$.

$$\begin{aligned} y&=\sqrt{x}\\ -y&=\sqrt{x} \end{aligned}$$

Step 2
Apply the horizontal stretch.

Since the graph of $y=\sqrt{x}$ is horizontally stretched by a factor of $\frac{7}{2}$, replace x with $\frac{2}{7}x$ in the equation $-y=\sqrt{x}$.

$$\begin{aligned} -y&=\sqrt{x}\\ -y&=\sqrt{\left(\frac{2}{7}x\right)} \end{aligned}$$

Step 3
Apply the vertical translation.

Since the graph of $y=\sqrt{x}$ is translated 3 units down, replace y with $y-(-3)$ or $y+3$ in the equation $-y=\sqrt{\left(\frac{2}{7}x\right)}$.

$$\begin{aligned} -y&=\sqrt{\left(\frac{2}{7}x\right)}\\ -(y+3)&=\sqrt{\left(\frac{2}{7}x\right)} \end{aligned}$$

Step 4

Solve for y in the equation $-(y+3)=\sqrt{\left(\frac{2}{7}x\right)}$.

$$\begin{aligned} -(y+3)&=\sqrt{\left(\frac{2}{7}x\right)}\\ y+3&=-\sqrt{\left(\frac{2}{7}x\right)}\\ y&=-\sqrt{\left(\frac{2}{7}x\right)}-3 \end{aligned}$$

Therefore, the equation of the transformed function is $y=-\sqrt{\left(\frac{2}{7}x\right)}-3$.

b) The equation $y=-\sqrt{\left(\frac{2}{7}x\right)}-3$ is of the form $y=a\sqrt{b(x-h)}+k$, where $a=-1$, $b=\frac{2}{7}$, $h=0$, and $k=-3$.

Step 1

Determine the domain of $y=-\sqrt{\left(\frac{2}{7}x\right)}-3$.

Parameters *b* and *h* have an effect on the domain. Since $b>0$ and $h=0$, the domain is $x \geq 0$.

Step 2

Determine the range of $y=-\sqrt{\left(\frac{2}{7}x\right)}-3$.

Parameters *a* and *k* have an effect on the range. Since $a<0$ and $k=-3$, the range is $y \leq -3$.

6. **Step 1**
Determine how to obtain the graph of $y=-\sqrt{-\frac{1}{3}(x+8)}-1$ from the graph of $y=\sqrt{x}$.

The equation $y=-\sqrt{-\frac{1}{3}(x+8)}-1$ is of the form $y=a\sqrt{b(x-h)}+k$, where $a=-1$, $b=-\frac{1}{3}$, $h=-8$, and $k=-1$. Therefore, the graph of $y=-\sqrt{-\frac{1}{3}(x+8)}-1$ is obtained from $y=\sqrt{x}$ by a vertical reflection in the *x*-axis, a horizontal reflection in the *y*-axis, a translation 8 units left, and a translation 1 unit down.

Step 2
Apply the vertical and horizontal reflections.

Some points on the graph of $y=\sqrt{x}$ are (0, 0), (1, 1), (4, 2), (9, 3), and (16, 4).

Multiply the *x*- and *y*-coordinates by –1.

$$(0\times -1, 0\times -1)\rightarrow(0,0)$$
$$(1\times -1, 1\times -1)\rightarrow(-1,-1)$$
$$(4\times -1, 2\times -1)\rightarrow(-4,-2)$$
$$(9\times -1, 3\times -1)\rightarrow(-9,-3)$$
$$(16\times -1, 4\times -1)\rightarrow(-16,-4)$$

Step 3
Apply the horizontal stretch to the points obtained in step 2.

Multiply the *x*-coordinates by 3.

$$(0\times 3, 0)\rightarrow(0,0)$$
$$(-1\times 3, -1)\rightarrow(-3,-1)$$
$$(-4\times 3, -2)\rightarrow(-12,-2)$$
$$(-9\times 3, -3)\rightarrow(-27,-3)$$
$$(-16\times 3, -4)\rightarrow(-48,-4)$$

Step 4
Apply the horizontal translation to the points obtained in step 3.

Decrease the *x*-coordinates by 8 units.

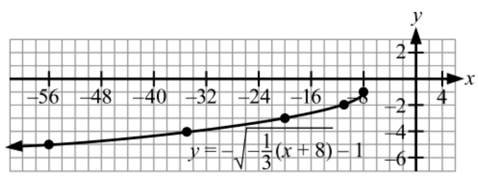
$$(0-8, 0)\rightarrow(-8,0)$$
$$(-3-8, -1)\rightarrow(-11,-1)$$
$$(-12-8, -2)\rightarrow(-20,-2)$$
$$(-27-8, -3)\rightarrow(-35,-3)$$
$$(-48-8, -4)\rightarrow(-56,-4)$$

Step 5
Apply the vertical translation to the points obtained in step 4.

Decrease the *y*-coordinates by 1 unit.

$$(-8, 0-1)\rightarrow(-8,-1)$$
$$(-11, -1-1)\rightarrow(-11,-2)$$
$$(-20, -2-1)\rightarrow(-20,-3)$$
$$(-35, -3-1)\rightarrow(-35,-4)$$
$$(-56, -4-1)\rightarrow(-56,-5)$$

Step 6
Plot and join the transformed points.



Step 7
State the domain and range.

The domain is $x \leq -8$, and the range is $y \leq -1$.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. If the graph of $y=\sqrt{x}$ is vertically stretched by a factor of 4, then any point (*x*, *y*) on the graph of $y=\sqrt{x}$ becomes point (*x*, 4*y*) on the graph of the transformed function. Since (144, 12) is on the graph of $y=\sqrt{x}$, it follows that point (144, 12 × 4), or (144, 48), is on the graph of the transformed function.

2. **Step 1**
Determine how to obtain the graph of $y=\sqrt{(x+4)}$ from the graph of $y=\sqrt{x}$.

The equation $y=\sqrt{(x+4)}$ is of the form $y=\sqrt{(x-h)}$, where $h=-4$. Therefore, the graph of $y=\sqrt{(x-h)}$ is formed from $y=\sqrt{x}$ by a translation 4 units to the left.

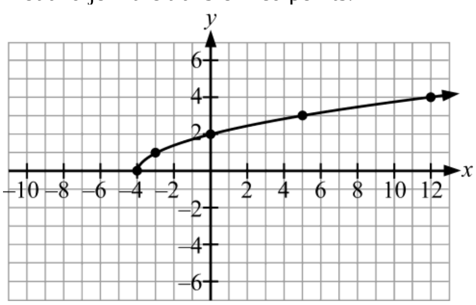
Step 2
Transform points on the graph of $y=\sqrt{x}$ to obtain points on the graph of $y=f(x+4)$.

Some points on the graph of $y=\sqrt{x}$ are (0, 0), (1, 1), (4, 2), (9, 3), and (16, 4).

Decrease the *x*-coordinates of $y=\sqrt{x}$ by 4 units.

$$(0-4,0)\rightarrow(-4,0)$$
$$(1-4,1)\rightarrow(-3,1)$$
$$(4-4,2)\rightarrow(0,2)$$
$$(9-4,3)\rightarrow(5,3)$$
$$(16-4,4)\rightarrow(12,4)$$

Step 3
Plot and join the transformed points.



Step 4
State the domain and range.

The domain is $x\geq-4$, and the range is $y\geq0$.

3. **a)** **Step 1**
Apply the vertical stretch.
Since the graph of $y=\sqrt{x}$ is vertically stretched by a factor of 4, replace *y* with $\frac{1}{4}y$ in the equation $y=\sqrt{x}$.

$$y=\sqrt{x}$$
$$\frac{1}{4}y=\sqrt{x}$$

Step 2
Apply the horizontal stretch.
Since the graph of $y=\sqrt{x}$ is horizontally stretched by a factor of 8, replace *x* with $\frac{1}{8}x$ in the equation $\frac{1}{4}y=\sqrt{x}$.

$$\frac{1}{4}y=\sqrt{x}$$
$$\frac{1}{4}y=\sqrt{\left(\frac{1}{8}x\right)}$$

Step 3
Apply the horizontal translation.
Since the graph of $y=\sqrt{x}$ is translated 7 units left, replace *x* with $x-(-7)$ or $x+7$ in the equation $\frac{1}{4}y=\sqrt{\left(\frac{1}{8}x\right)}$.

$$\frac{1}{4}y=\sqrt{\left(\frac{1}{8}x\right)}$$
$$\frac{1}{4}y=\sqrt{\left(\frac{1}{8}(x+7)\right)}$$

Step 4

Solve for *y* in the equation $\frac{1}{4}y=\sqrt{\left(\frac{1}{8}(x+7)\right)}$.

$$\frac{1}{4}y=\sqrt{\left(\frac{1}{8}(x+7)\right)}$$
$$y=4\sqrt{\left(\frac{1}{8}(x+7)\right)}$$

Therefore, the equation of the transformed function is $y=4\sqrt{\left(\frac{1}{8}(x+7)\right)}$.

b) The equation $y=4\sqrt{\left(\frac{1}{8}(x+7)\right)}$ is of the form $y=a\sqrt{b(x-h)}+k$, where $a=4$, $b=\frac{1}{8}$, $h=-7$, and $k=0$.

Step 1

Determine the domain of $y=4\sqrt{\left(\frac{1}{8}(x+7)\right)}$.

Parameters *b* and *h* have an effect on the domain. Since $b>0$ and $h=-7$, the domain is $x\geq-7$.

Step 2

Determine the range of $y = 4\sqrt{\left(\frac{1}{8}(x+7)\right)}$.

Parameters a and k have an effect on the range. Since $a > 0$ and $k = 0$, the range is $y \geq 0$.

4. **Step 1**
Apply the horizontal stretch.

Let b represent the horizontal stretch. The equation of the radical function after the horizontal stretch will be $y = \sqrt{bx}$.

Step 2
Apply the horizontal translation.

Let h represent the horizontal translation.
The equation of the radical function after the horizontal translation will be $y = \sqrt{b(x-h)}$.

Step 3
Set up a system of equations.
The graph passes through the points (11, 2) and (14, 4).

Substitute 11 for x and 2 for y in the equation $y = \sqrt{b(x-h)}$.

$$y = \sqrt{b(x-h)}$$
$$2 = \sqrt{b(11-h)}$$

Substitute 14 for x and 4 for y in the equation $y = \sqrt{b(x-h)}$.

$$y = \sqrt{b(x-h)}$$
$$4 = \sqrt{b(14-h)}$$

Therefore, let ① represent the equation $2 = \sqrt{b(11-h)}$ and ② represent the equation $4 = \sqrt{b(14-h)}$.

Step 4
Solve for b and h using the method of substitution.
Isolate b in equation ①.

$$2 = \sqrt{b(11-h)}$$
$$4 = b(11-h)$$
$$\frac{4}{11-h} = b$$

Substitute $\frac{4}{11-h}$ for b in equation ②, and solve for h.

$$4 = \sqrt{b(14-h)}$$
$$4 = \sqrt{\left(\frac{4}{11-h}\right)(14-h)}$$
$$16 = \left(\frac{4}{11-h}\right)(14-h)$$
$$16(11-h) = 4(14-h)$$
$$4(11-h) = 14-h$$
$$44-4h = 14-h$$
$$-3h = -30$$
$$h = 10$$

Substitute 10 for h in equation ①, and solve for b.

$$2 = \sqrt{b(11-h)}$$
$$2 = \sqrt{b(11-10)}$$
$$2 = \sqrt{b}$$
$$b = 4$$

Step 5
Write the equation of the transformed function.

After a horizontal stretch by a factor of $\frac{1}{4}$ and a translation of 10 units to the right, the graph of the function $y = \sqrt{x}$ is transformed to the graph of the function $y = \sqrt{4(x-10)}$.

5. **Step 1**
Determine how to obtain the graph of $y = -5\sqrt{x-4}$ from the graph of $y = \sqrt{x}$.

The equation $y = -5\sqrt{x-4}$ is of the form $y = a\sqrt{x-h}$, where $a = -5$ and $h = 4$.
Therefore, the graph of $y = -5\sqrt{x-4}$ is obtained from $y = \sqrt{x}$ by a vertical reflection in the x-axis, a vertical stretch factor of 5, and a translation 4 units right.

Step 2
Apply the vertical reflection.

Some points on the graph of $y = \sqrt{x}$ are (0, 0), (1, 1), (4, 2), (9, 3), and (16, 4).

Multiply the *y*-coordinates by –1.

$$(0, 0\times(-1)) \to (0, 0)$$
$$(1, 1\times(-1)) \to (1, -1)$$
$$(4, 2\times(-1)) \to (4, -2)$$
$$(9, 3\times(-1)) \to (9, -3)$$
$$(16, 4\times(-1)) \to (16, -4)$$

Step 3
Apply the vertical stretch to the points obtained in step 2.

Multiply the *y*-coordinates by 5.

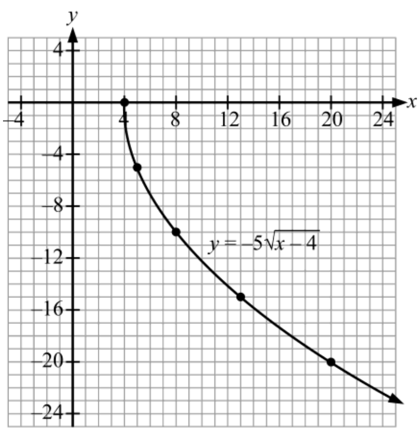
$$(0, 0\times 5) \to (0, 0)$$
$$(1, -1\times 5) \to (1, -5)$$
$$(4, -2\times 5) \to (4, -10)$$
$$(9, -3\times 5) \to (9, -15)$$
$$(16, -4\times 5) \to (16, -20)$$

Step 4
Apply the horizontal translation to the points obtained in step 3.

Increase the *x*-coordinates by 4 units.

$$(0+4, 0) \to (4, 0)$$
$$(1+4, -5) \to (5, -5)$$
$$(4+4, -10) \to (8, -10)$$
$$(9+4, -15) \to (13, -15)$$
$$(16+4, -20) \to (20, -20)$$

Step 5
Plot and join the transformed points.



6. The function $y = 2\sqrt{(-3x+15)} - 8$ is equivalent to $y = 2\sqrt{-3(x-5)} - 8$.

Step 1
Determine how to obtain the graph of $y = 2\sqrt{-3(x-5)} - 8$ from the graph of $y = \sqrt{x}$.

The equation $y = 2\sqrt{-3(x-5)} - 8$ is of the form $y = a\sqrt{b(x-h)} + k$, where $a = 2$, $b = -3$, $h = 5$, and $k = -8$. Therefore, the graph of $y = 2\sqrt{-3(x-5)} - 8$ is obtained from $y = \sqrt{x}$ by a vertical stretch factor of 2, a horizontal reflection in the *y*-axis, a horizontal stretch factor of $\frac{1}{3}$, a translation 8 units down, and a translation
5 units right .

Step 2
Apply the vertical stretch.
Some points on the graph of $y = \sqrt{x}$ are (0, 0), (1, 1), (4, 2), (9, 3), and (16, 4).

Multiply the *y*-coordinates by 2.

$$(0, 0\times 2) \to (0, 0)$$
$$(1, 1\times 2) \to (1, 2)$$
$$(4, 2\times 2) \to (4, 4)$$
$$(9, 3\times 2) \to (9, 6)$$
$$(16, 4\times 2) \to (16, 8)$$

Step 3
Apply the horizontal reflection to the points obtained in step 2.

Multiply the *x*-coordinates by –1.

$$(0\times(-1), 0) \to (0, 0)$$
$$(1\times(-1), 2) \to (-1, 2)$$
$$(4\times(-1), 4) \to (-4, 4)$$
$$(9\times(-1), 6) \to (-9, 6)$$
$$(16\times(-1), 8) \to (-16, 8)$$

Step 4
Apply the horizontal stretch to the points obtained in step 3.

Multiply the *x*-coordinates by $\frac{1}{3}$.

$$\left(0\times\frac{1}{3},0\right)\rightarrow(0,0)$$
$$\left(-1\times\frac{1}{3},2\right)\rightarrow\left(-\frac{1}{3},2\right)$$
$$\left(-4\times\frac{1}{3},4\right)\rightarrow\left(-\frac{4}{3},4\right)$$
$$\left(-9\times\frac{1}{3},6\right)\rightarrow(-3,6)$$
$$\left(-16\times\frac{1}{3},8\right)\rightarrow\left(-\frac{16}{3},8\right)$$

Step 5
Apply the horizontal translation to the points obtained in step 4.
Increase the *x*-coordinates by 5 units.

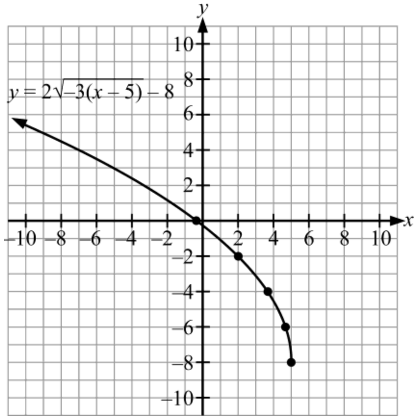
$$(0+5,0)\rightarrow(5,0)$$
$$\left(-\frac{1}{3}+5,2\right)\rightarrow\left(\frac{14}{3},2\right)$$
$$\left(-\frac{4}{3}+5,4\right)\rightarrow\left(\frac{11}{3},4\right)$$
$$(-3+5,6)\rightarrow(2,6)$$
$$\left(-\frac{16}{3}+5,8\right)\rightarrow\left(-\frac{1}{3},8\right)$$

Step 6
Apply the vertical translation to the points obtained in step 5.

Decrease the *y*-coordinates by 8 units.

$$(5,0-8)\rightarrow(5,-8)$$
$$\left(\frac{14}{3},2-8\right)\rightarrow\left(\frac{14}{3},-6\right)$$
$$\left(\frac{11}{3},4-8\right)\rightarrow\left(\frac{11}{3},-4\right)$$
$$(2,6-8)\rightarrow(2,-2)$$
$$\left(-\frac{1}{3},8-8\right)\rightarrow\left(-\frac{1}{3},0\right)$$

Step 7
Plot and join the transformed points.

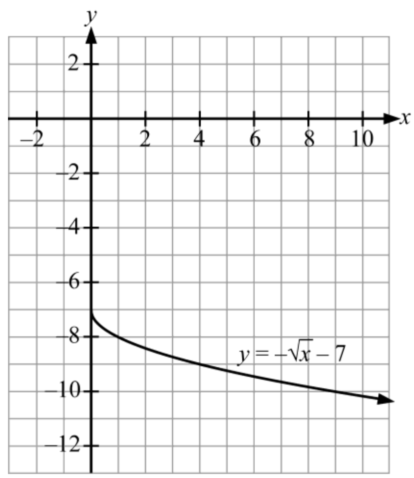


Lesson 3—Solving Radical Equations Graphically

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Bring all terms to one side of the equation $2=-\sqrt{x}-5$.
$$2=-\sqrt{x}-5$$
$$0=-\sqrt{x}-7$$

Step 2
Graph the related function $y=-\sqrt{x}-7$.



Step 3
Determine the solution to the equation $2=-\sqrt{x}-5$.

The graph does not cross the *x*-axis.

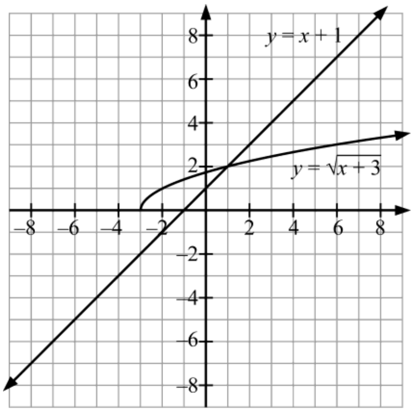
Therefore, the solution set is { } or $\varnothing$.

2. Step 1

Graph each side of the equation $x+1=\sqrt{x+3}$.
The equation $y=x+1$ represents a linear graph with a slope of 1 and a y-intercept of (0, 1).

The graph of $y=\sqrt{x+3}$ is obtained from the graph of $y=\sqrt{x}$ by a horizontal transformation 3 units left.

The graphs of $y=x+1$ and $y=\sqrt{x+3}$ are as shown.



Step 2

Determine the solution to the equation $x+1=\sqrt{x+3}$.

The point of intersection between the two graphs is located at (1, 2).

Therefore, the solution set is {1}.

3. Step 1

Bring all terms to one side of the equation $\sqrt{\left(x^2+2x+6\right)}=3x^2-5$.

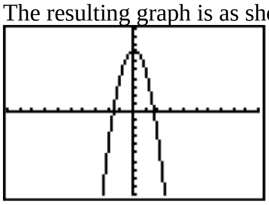
$$\sqrt{\left(x^2+2x+6\right)}=3x^2-5$$
$$\sqrt{\left(x^2+2x+6\right)}-3x^2=-5$$
$$\sqrt{\left(x^2+2x+6\right)}-3x^2+5=0$$

Step 2

Graph the related function using a TI-83 or similar graphing calculator.

Press $\boxed{\text{Y=}}$, enter the equation as $\text{Y}_1=\sqrt{\left(\text{X^2}+2\text{X}+6\right)}-3\text{X}^2+5$, and press $\boxed{\text{GRAPH}}$.
An appropriate window setting is
x: [–10, 10, 1] and y: [–10, 10, 1].

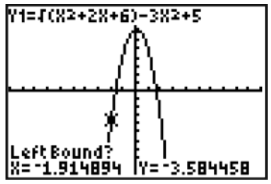
The resulting graph is as shown.



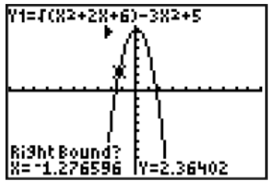
Step 3

Determine the zeros of the function.
Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 2:zero.

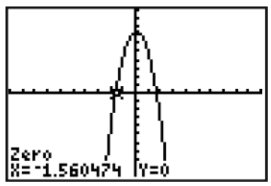
For "Left Bound?", position the cursor just left of one zero, and press $\boxed{\text{ENTER}}$.



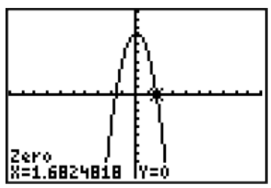
For "Right Bound?", position the cursor just right of the same zero, and press $\boxed{\text{ENTER}}$.



For "Guess?", press $\boxed{\text{ENTER}}$. The result is one of the zeros of the function.



Repeat the process to find the second zero.



The zeros of the function are –1.56 and 1.68.

Therefore, the solution set is {–1.56, 1.68}.

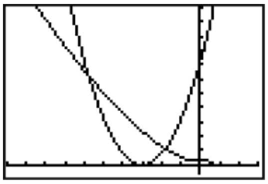
4. Step 1

Graph each side of the equation $\sqrt{-5x^3+2}=4\left(x+3\right)^2$ using a TI-83 or similar graphing calculator.

Press $\boxed{\text{Y=}}$, enter the equation as $\text{Y}_1=\sqrt{\left(-5\text{X^3}+2\right)}$ and $\text{Y}_2=4\left(\text{X}+3\right)\text{^}2$ and press $\boxed{\text{GRAPH}}$.

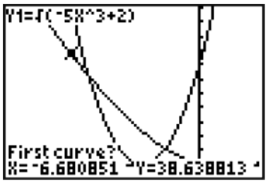
An appropriate window setting is x: [–10, 3, 1] and y: [–3, 55, 5].

The resulting graph is as shown.

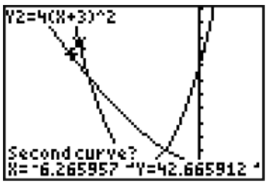


Step 2
Find the points of intersection.
Press 2nd TRACE, and select 5:intersect.

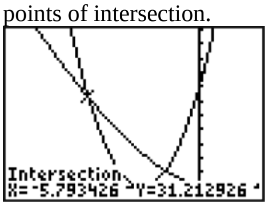
For "First curve?", position the cursor just left or right of the first intersection point, and press ENTER.



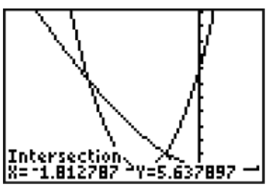
For "Second curve?", position the cursor just left or right of the first intersection point, and press ENTER.



For "Guess?", press ENTER. The result is one of the points of intersection.



Repeat the process for the second point of intersection.



The points of intersection are (–5.79, 31.21) and (–1.81, 5.64).

Therefore, the solution set is {–5.79, –1.81}.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

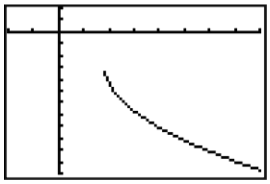
1. **Step 1**
Bring all terms to one side of the equation $-8\sqrt{x-9}=14$.

$$-8\sqrt{x-9}=14$$
$$-8\sqrt{x-9}-14=0$$

Step 2
Graph the related function using a TI-83 or similar graphing calculator.

Press Y=, enter the equation as $Y_1=-8\sqrt{(X-9)}-14$, and press GRAPH. An appropriate window setting is x: [–10, 40, 5] and y: [–60, 10, 5].

The resulting graph is as shown.



Step 3
Determine the zero of the function.
The graph does not cross the x-axis.

Therefore, the solution set is { } or $\varnothing$.

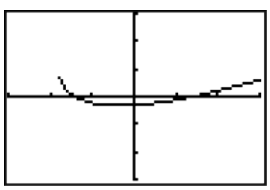
2. **Step 1**
Bring all terms to one side of the equation $-\sqrt{6x+11}=-x-3$.

$$-\sqrt{6x+11}=-x-3$$
$$-\sqrt{6x+11}+x=-3$$
$$-\sqrt{6x+11}+x+3=0$$

Step 2
Graph the related function using a TI-83 or similar graphing calculator.

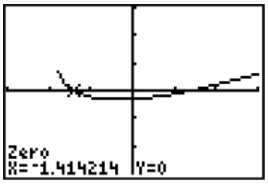
Press Y=, enter the equation as $Y_1=-\sqrt{(6X+11)}+X+3$, and press GRAPH.
An appropriate window setting is x: [–3, 3, 1] and y: [–3, 3, 1].

The resulting graph is as shown.

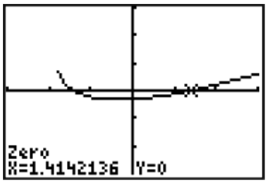


Step 3
Determine the zeros of the function.
Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 2:zero.

For “Left Bound?”, position the cursor just left of one of the zeros, and press $\boxed{\text{ENTER}}$. For “Right Bound?”, position the cursor just right of the same zero, and press $\boxed{\text{ENTER}}$. For “Guess?”, press $\boxed{\text{ENTER}}$. The results are the coordinates of one of the zeros.



Repeat the process to find the second zero.



The zeros of the function are –1.41 and 1.41.

Therefore, the solution set is {–1.41, 1.41}.

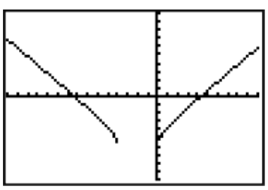
3. **Step 1**
Bring all terms to one side of the equation $\sqrt{x^2+4x}=6$.

$$\sqrt{x^2+4x}=6$$
$$\sqrt{x^2+4x}-6=0$$

Step 2
Graph the related function using a TI-83 or similar graphing calculator.

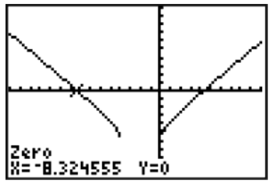
Press $\boxed{\text{Y=}}$, enter the equation as $\text{Y}_1=\sqrt{(\text{X\^{}2}+4\text{X})}-6$, and press $\boxed{\text{GRAPH}}$.
An appropriate window setting is *x*: [–15, 10, 1] and *y*: [–10, 10, 1].

The resulting graph is as shown.

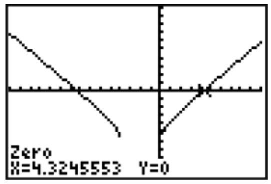


Step 3
Determine the zeros of the function.
Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 2:zero.

For “Left Bound?”, position the cursor just left of the first zero, and press $\boxed{\text{ENTER}}$. For “Right Bound?”, position the cursor just right of the same zero, and press $\boxed{\text{ENTER}}$. For “Guess?”, press $\boxed{\text{ENTER}}$. The results are the coordinates of one of the zeros.



Repeat the process for the second zero.



The zeros of the function are –8.32 and 4.32.

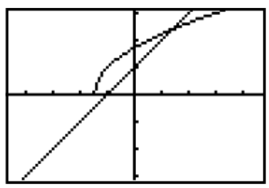
Therefore, the solution set is {–8.32, 4.32}.

4. **Step 1**
Graph each side of the equation $\sqrt{2x+3}=x+1$ using a TI-83 or similar graphing calculator.

Press $\boxed{\text{Y=}}$, enter the equation as $\text{Y}_1=\sqrt{(2\text{X}+3)}$ and $\text{Y}_2=\text{X}+1$, and press $\boxed{\text{GRAPH}}$.

An appropriate window setting is *x*: [–4.7, 4.7, 1] and *y*: [–3.1, 3.1, 1].

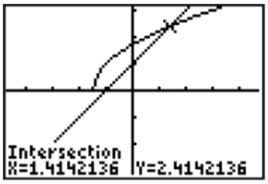
The resulting graph is as shown.



Step 2
Find the point of intersection.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 5:intersect.

For "First curve?", position the cursor just left or right of the intersection point, and press $\boxed{\text{ENTER}}$. For "Second curve?", position the cursor just left or right of the intersection point, and press $\boxed{\text{ENTER}}$. For "Guess?", press $\boxed{\text{ENTER}}$.



The point of intersection is (1.41, 2.41).

Therefore, the solution set is {1.41}.

Practice Test

ANSWERS AND SOLUTIONS

1. **Step 1**
Write the function $\frac{5}{2}y=\sqrt{(8-x)}+20$ in the form

$y=a\sqrt{b(x-h)}+k$.

$$\frac{5}{2}y=\sqrt{(8-x)}+20$$
$$y=\frac{2}{5}\sqrt{(8-x)}+8$$
$$y=\frac{2}{5}\sqrt{(-x+8)}+8$$
$$y=\frac{2}{5}\sqrt{-(x-8)}+8$$

Step 2
Determine how to obtain the graph of
$y=\frac{2}{5}\sqrt{-(x-8)}+8$ from the graph of $y=\sqrt{x}$.

The equation $y=\frac{2}{5}\sqrt{-(x-8)}+8$ is of the form

$y=a\sqrt{b(x-h)}+k$, where $a=\frac{2}{5}$, $b=-1$, $h=8$,

and $k=8$.

Therefore, the graph of $y=\frac{2}{5}\sqrt{-(x-8)}+8$ is obtained

from $y=\sqrt{x}$ by a vertical stretch factor of $\frac{2}{5}$,

a horizontal reflection in the y-axis, and a translation 8 units right and 8 units up.

2. The function $y=\frac{9}{2}\sqrt{-\frac{1}{4}(x-12)}-1$ is of the form

$y=a\sqrt{b(x-h)}+k$, where $a=\frac{9}{2}$, $b=-\frac{1}{4}$, $h=12$,

and $k=-1$.

Step 1

Determine the domain of $y=\frac{9}{2}\sqrt{-\frac{1}{4}(x-12)}-1$.

Parameters b and h have an effect on the domain.
Since $b<0$ and $h=12$, the domain is $x\le 12$.

Step 2

Determine the range of $y=\frac{9}{2}\sqrt{-\frac{1}{4}(x-12)}-1$.

Parameters a and k have an effect on the range.
Since $a>0$ and $k=-1$, the range is $y\ge -1$.

3. **Step 1**
Apply the vertical reflection.
Since the graph of $y=\sqrt{x}$ is reflected in the

x-axis, replace y with $-y$ in the equation $y=\sqrt{x}$.

$$y=\sqrt{x}$$
$$-y=\sqrt{x}$$

Step 2
Apply the vertical stretch.
Since the graph of $y=\sqrt{x}$ is vertically stretched by a

factor of $\frac{2}{3}$, replace y with $\frac{3}{2}y$ in the equation

$-y=\sqrt{x}$.

$$-y=\sqrt{x}$$
$$-\frac{3}{2}y=\sqrt{x}$$

Step 3
Apply the vertical translation.
Since the graph of $y=\sqrt{x}$ is vertically translated 6 units up, replace y with $y-6$ in the equation

$-\frac{3}{2}y=\sqrt{x}$.

$$-\frac{3}{2}y=\sqrt{x}$$
$$-\frac{3}{2}(y-6)=\sqrt{x}$$

Step 4

Solve for y in the equation $-\frac{3}{2}(y-6)=\sqrt{x}$.

$$-\frac{3}{2}(y-6)=\sqrt{x}$$
$$(y-6)=-\frac{2\sqrt{x}}{3}$$
$$y=-\frac{2}{3}\sqrt{x}+6$$

Therefore, the equation of the transformed function is $y=-\frac{2}{3}\sqrt{x}+6$.

4. **Step 1**
Find the x-intercept.
Let $y=0$, and solve for the value of x.

$$y=-\frac{2}{3}\sqrt{x}+6$$
$$0=-\frac{2}{3}\sqrt{x}+6$$
$$-6=-\frac{2}{3}\sqrt{x}$$
$$9=\sqrt{x}$$
$$x=81$$

Therefore, the x-intercept is (81, 0).

Step 2
Find the y-intercept.
Let $x=0$, and solve for the value of y.

$$y=-\frac{2}{3}\sqrt{x}+6$$
$$y=-\frac{2}{3}\sqrt{0}+6$$
$$y=6$$

Therefore, the y-intercept is (0, 6).

5. The domain is $x\geq 0$.

In the function $y=-\frac{2}{3}\sqrt{x}+6$, $a<0$ and $k=6$.

Therefore, the range is $y\leq 6$.

6. **Step 1**
Determine how to obtain the graph of $y=-3\sqrt{(x+20)}+12$ from the graph of $y=\sqrt{x}$.

The equation $y=-3\sqrt{(x+20)}+12$ is of the form $y=a\sqrt{(x-h)}+k$, where $a=-3$, $h=-20$, and $k=12$.

Therefore, the graph of $y=-3\sqrt{(x+20)}+12$ is obtained from $y=\sqrt{x}$ by a vertical reflection in the x-axis, a vertical stretch factor of 3, and a translation –20 units left and 12 units up.

Step 2
Apply the vertical reflection and vertical stretch to the point (169, 13).

The point (169, 13), which is on the graph of $y=\sqrt{x}$, will be transformed to the point $(169,13\times(-3))=(169,-39)$ when the graph of $y=\sqrt{x}$ is reflected in the x-axis and vertically stretched by a factor of 3.

Step 3
Apply the horizontal and vertical translations.
The point (169, –39) will be transformed to the point $(169-20,-39+12)=(149,-27)$.

Thus, the point (169, 13) on the graph of $y=\sqrt{x}$ corresponds to the point (149, –27) on the graph of $y=-3\sqrt{(x+20)}+12$.

7. The function $y-8=-\sqrt{\frac{1}{2}x}$ is equivalent to the function $y=-\sqrt{\frac{1}{2}x}+8$.

Step 1

Determine how to obtain the graph of $y=-\sqrt{\frac{1}{2}x}+8$ from the graph of $y=\sqrt{x}$.

The equation $y=-\sqrt{\frac{1}{2}x}+8$ is of the form $y=a\sqrt{bx}+k$, where $a=-1$, $b=\frac{1}{2}$, and $k=8$.

Therefore, the graph of $y=-\sqrt{\frac{1}{2}x}+8$ is obtained from $y=\sqrt{x}$ by a vertical reflection in the x-axis, a horizontal stretch factor of 2, and a translation 8 units up.

Step 2
Apply the vertical reflection.

Some points on the graph of $y = \sqrt{x}$ are (0, 0), (1, 1), (4, 2), (9, 3), and (16, 4).

Multiply the *y*-coordinates by –1.

$$(0, 0\times(-1)) \to (0,0)$$
$$(1, 1\times(-1)) \to (1,-1)$$
$$(4, 2\times(-1)) \to (4,-2)$$
$$(9, 3\times(-1)) \to (9,-3)$$
$$(16, 4\times(-1)) \to (16,-4)$$

Step 3
Apply the horizontal stretch to the points obtained in step 2.
Multiply the *x*-coordinates by 2.

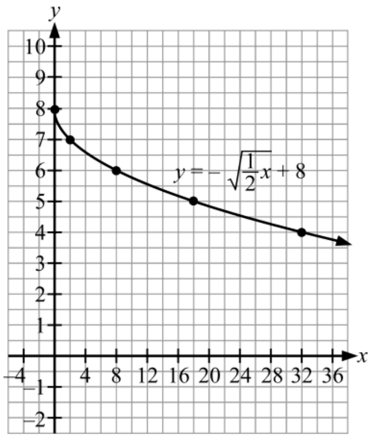
$$(0\times 2, 0) \to (0,0)$$
$$(1\times 2, -1) \to (2,-1)$$
$$(4\times 2, -2) \to (8,-2)$$
$$(9\times 2, -3) \to (18,-3)$$
$$(16\times 2, -4) \to (32,-4)$$

Step 4
Apply the vertical translation to the points obtained in step 3.

Increase the *y*-coordinates by 8 units.

$$(0, 0+8) \to (0,8)$$
$$(2, -1+8) \to (2,7)$$
$$(8, -2+8) \to (8,6)$$
$$(18, -3+8) \to (18,5)$$
$$(32, -4+8) \to (32,4)$$

Step 5
Plot and join the transformed points.



Step 6
State the domain and range.

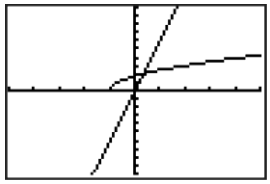
The domain is $x \ge 0$, and the range is $y \le 8$.

8. Step 1
Graph each side of the equation $\sqrt{3x+3} = 6x$ using a TI-83 or similar graphing calculator.

Press $\boxed{\text{Y=}}$, enter the equation as $Y_1 = \sqrt{(3X+3)}$ and $Y_2 = 6X$, and press $\boxed{\text{GRAPH}}$.

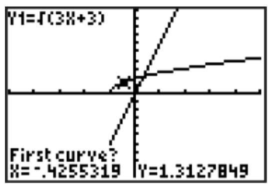
An appropriate window setting is *x*: [–5, 5, 1] and *y*: [–10, 10, 1].

The resulting graph is as shown.

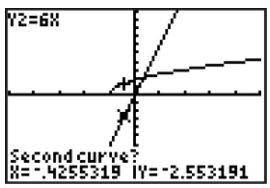


Step 2
Find the point of intersection.
Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 5:intersect.

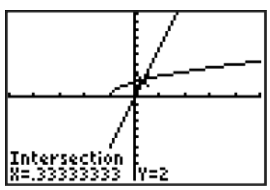
For “First curve?”, position the cursor just left or right of the intersection point, and press $\boxed{\text{ENTER}}$.



For “Second curve?”, position the cursor just left or right of the intersection point, and press $\boxed{\text{ENTER}}$.



For “Guess?”, press $\boxed{\text{ENTER}}$.



The point of intersection is approximately (0.3, 2).

Therefore, the solution set is {0.3}.

RATIONAL FUNCTIONS

Lesson 1—Asymptotes and Holes of a Rational Function

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Step 1
Determine the vertical asymptotes.

Set the denominator equal to zero, and solve for *x*.

$$\begin{aligned}6x+12&=0\\6x&=-12\\x&=-2\end{aligned}$$

The vertical asymptote is $x=-2$.

Step 2
Determine the horizontal asymptotes.
The degree of the numerator $\left(5x^3+1\right)$ is 3, and the degree of the denominator $\left(6x+12\right)$ is 1. Since the degree of the numerator is greater than the degree of the denominator, there are no horizontal asymptotes.

2. Step 1
Determine the vertical asymptotes.

Set the denominator equal to zero, and solve for *x*.

$$\begin{aligned}x^4+8x^2+16&=0\\\left(x^2+4\right)\left(x^2+4\right)&=0\end{aligned}$$

$$\begin{aligned}x^2+4&=0\\x^2&=-4\\x&=\pm\sqrt{-4}\end{aligned}$$

Since the square root of a negative number does not exist, there are no vertical asymptotes.

Step 2
Determine the horizontal asymptotes.

The degree of the numerator $\left(40x^2\right)$ is 2, and the degree of the denominator $\left(x^4+4x^2+16\right)$ is 4. Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote occurs at $y=0$.

3. Step 1
Express the function in factored form, and state the non-permissible values of *x*.

$$f(x)=\frac{x^2+8x+16}{x^3+x^2+12x}$$
$$f(x)=\frac{(x+4)(x+4)}{x(x+3)(x+4)}$$

Set the denominator equal to zero, and solve for *x*.

$$x(x+3)(x+4)=0$$

$$x=0 \qquad \begin{aligned}x+3&=0\\x&=-3\end{aligned} \qquad \begin{aligned}x+4&=0\\x&=-4\end{aligned}$$

The non-permissible values are $x=0$, $x=-3$, and $x=-4$.

Step 2
Determine if the graph of $f(x)$ will have a vertical asymptote or a hole for the non-permissible values of *x*.

The factor $(x+4)$ occurs in both the numerator and denominator.

Express the function in reduced form.

$$f(x)=\frac{(x+4)(x+4)}{x(x+3)(x+4)}$$
$$f(x)=\frac{(x+4)\cancel{(x+4)}}{x(x+3)\cancel{(x+4)}},x\neq 0,-3,-4$$

The factor $(x+4)$ remains in the numerator.

Therefore, $x=-4$ is a hole and $x=0$ and $x=-3$ are vertical asymptotes.

Step 3
Determine the horizontal asymptotes.

The degree of the numerator $\left(x^2+8x+16\right)$ is 2, and the degree of the denominator $\left(x^3+x^2+12x\right)$ is 3. Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote occurs at $y=0$.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. Step 1

Express the function in factored form, and state the non-permissible values of *x*.

$$f(x)=\frac{x^2-2x-8}{x-4}$$
$$f(x)=\frac{(x+2)(x-4)}{x-4}$$

Set the denominator equal to zero, and solve for *x*.

$$x-4=0$$
$$x=4$$

The non-permissible value is $x=4$.

Step 2

Determine if the graph of $f(x)$ will have a vertical asymptote or a hole for the non-permissible value of *x*.

The factor $(x-4)$ occurs in both the numerator and denominator.

Express the function in reduced form.

$$f(x)=\frac{(x+2)\cancel{(x-4)}}{\cancel{x-4}}$$
$$f(x)=(x+2),x\neq 4$$

The factor $(x-4)$ completely cancels out of the numerator and denominator, so $x=4$ is a hole.

Step 3

Determine the horizontal asymptotes.

The degree of the numerator (x^2-2x-8) is 2, and the degree of the denominator $(x-4)$ is 1.

Since the degree of the numerator is more than the degree of the denominator, there are no horizontal asymptotes.

2. Step 1

Express the function in factored form, and state the non-permissible values of *x*.

$$f(x)=\frac{2x^2}{x^2+2}$$
$$f(x)=\frac{2(x)(x)}{x^2+2}$$

Set the denominator equal to zero, and solve for *x*.

$$x^2+2=0$$
$$x^2=-2$$
$$x=\sqrt{-2}$$

Since the square root of a negative number does not exist, there are no non-permissible values. Therefore, there are no vertical asymptotes
or holes.

Step 2

Determine the horizontal asymptotes.

The degree of the numerator $(2x^2)$ is 2, and the degree of the denominator (x^2+2) is 2. Since the degree of the numerator is equal to the degree of the denominator, the horizontal asymptote occurs at $y=\frac{a}{b}$, where *a* and *b* are the leading coefficients of the numerator and denominator, respectively.

$$y=\frac{a}{b}$$
$$y=\frac{2}{1}$$
$$y=2$$

The horizontal asymptote is $y=2$.

3. Step 1

Express the function in factored form, and state the non-permissible values of *x*.

$$f(x)=\frac{2x-16}{3x^3-48x^2+192x}$$
$$f(x)=\frac{2(x-8)}{3x(x^2-16x+64)}$$
$$f(x)=\frac{2(x-8)}{3x(x-8)(x-8)}$$

Set the denominator equal to zero, and solve for *x*.

$$3x(x-8)(x-8)=0$$
$$3x=0 \qquad x-8=0$$
$$x=0 \qquad x=8$$

Therefore, the non-permissible values are $x=0$ and $x=8$.

Step 2

Determine if the graph of $f(x)$ will have a vertical asymptote or a hole for the
non-permissible values of *x*.

The factor $(x-8)$ occurs in both the numerator and denominator.

Express the function in reduced form.

$$f(x)=\frac{2\cancel{(x-8)}}{3x(x-8)\cancel{(x-8)}}$$
$$f(x)=\frac{2}{3x(x-8)},x\neq 0,8$$

The factor $(x-8)$ remains in the denominator. Therefore, $x=0$ and $x=8$ are vertical asymptotes.

Step 3
Determine the horizontal asymptotes.

The degree of the numerator $(2x-16)$ is 1, and the degree of the denominator $(3x^3-48x^2+192x)$ is 3. Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote occurs at $y=0$.

4. **Step 1**
Express the function in factored form, and state the non-permissible values of *x*.

$$f(x)=\frac{x^2-2x}{x^3+4x^2-12x}$$
$$f(x)=\frac{x(x-2)}{x(x+6)(x-2)}$$

Set the denominator equal to zero, and solve for *x*.

$$x(x+6)(x-2)=0$$

$x=0$ $\quad$ $x+6=0$, $x=-6$ $\quad$ $x-2=0$, $x=2$

Therefore, the non-permissible values are $x=0$, $x=-6$, and $x=2$.

Step 2
Determine if the graph of $f(x)$ will have a vertical asymptote or a hole for the non-permissible values of *x*.

The factors (x) and $(x-2)$ occur in both the numerator and denominator.

Express the function in reduced form.

$$f(x)=\frac{\cancel{x}\,\cancel{(x-2)}}{\cancel{x}\,(x+6)\,\cancel{(x-2)}}$$
$$f(x)=\frac{1}{x+6},\ x\neq -6,0,2$$

The factors (x) and $(x-2)$ both cancel out of the numerator and denominator. Therefore, $x=0$ and $x=2$ are holes and $x=-6$ is a vertical asymptote.

Step 3
Determine the horizontal asymptotes.

The degree of the numerator (x^2-2x) is 2, and the degree of the denominator (x^3+4x^2-12x) is 3. Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote occurs at $y=0$.

5. **Step 1**
Express the function in factored form, and state the non-permissible values of *x*.

$$f(x)=\frac{x^4+8x^3-9x^2}{x^2-x}$$
$$f(x)=\frac{x^2(x^2+8x-9)}{x(x-1)}$$
$$f(x)=\frac{(x)(x)(x-1)(x+9)}{x(x-1)}$$

Set the denominator equal to zero, and solve for *x*.

$$x(x-1)=0$$

$x=0$ $\quad$ $x-1=0$, $x=1$

Therefore, the non-permissible values are $x=0$ and $x=1$.

Step 2
Determine if the graph of $f(x)$ will have a vertical asymptote or a hole for the non-permissible values of *x*.

The factors (x) and $(x-1)$ occur in both the numerator and denominator.

Express the function in reduced form.

$$f(x)=\frac{(x)\,\cancel{(x)}\,\cancel{(x-1)}\,(x+9)}{\cancel{x}\,\cancel{(x-1)}}$$
$$f(x)=x(x+9),\ x\neq 0,1$$

The factor (x) remains in the numerator, and $(x-1)$ cancels out.

Therefore, $x=0$ and $x=1$ are holes.

Step 3
Determine the horizontal asymptotes.

The degree of the numerator $(x^4+8x^3-9x^2)$ is 4, and the degree of the denominator (x^2-x) is 2. Since the degree of the numerator is greater than the degree of the denominator, there are no horizontal asymptotes.

Lesson 2—The Behaviour of a Rational Function

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Determine the vertical asymptotes of the function.

The factored form of the function is $y=\dfrac{3x^2-1}{(x+6)(x-1)}$,

where $x\neq -6$ and $x\neq 1$. There are no factors that occur in both the numerator and denominator. Therefore, there is a vertical asymptote at $x=-6$ and $x=1$.

Step 2
Determine the horizontal asymptotes.

The degree of the numerator and denominator is 2.

Since the degree of the numerator is equal to the degree of the denominator, the horizontal asymptote occurs at $y=\dfrac{3}{1}=3$.

Step 3
Determine the behaviour on the left of $x=-6$.

Calculate values of *y* as *x* approaches –6 from the left.

If $x=-6.01$:
$y=\dfrac{3x^2-1}{x^2+5x-6}$ $y=\dfrac{3(-6.01)^2-1}{(-6.01)^2+5(-6.01)-6}$ $y=\dfrac{107.3603}{0.0701}$ $y=1\ 531.530\ 67$
If $x=-6.0001$:
$y=\dfrac{3x^2-1}{x^2+5x-6}$ $y=\dfrac{3(-6.0001)^2-1}{(-6.0001)^2+5(-6.0001)-6}$ $y=\dfrac{107.003\ 603}{0.000\ 700\ 01}$ $y=152\ 860.102$

As *x* approaches –6 from the left, the *y*-values approach positive infinity.

Step 4
Determine the behaviour on the right of $x=-6$.

Calculate values of *y* as *x* approaches –6 from the right.

If $x=-5.99$:
$y=\dfrac{3x^2-1}{x^2+5x-6}$ $y=\dfrac{3(-5.99)^2-1}{(-5.99)^2+5(-5.99)-6}$ $y=\dfrac{106.6403}{-0.0699}$ $y=-1\ 525.612\ 303$
If $x=-5.9999$:
$y=\dfrac{3x^2-1}{x^2+5x-6}$ $y=\dfrac{3(-5.9999)^2-1}{(-5.9999)^2+5(-5.9999)-6}$ $y=\dfrac{106.9964}{-0.000\ 699\ 99}$ $y=-152\ 854.1836$

As *x* approaches –6 from the right, the *y*-values approach negative infinity.

Step 5
Determine the behaviour on the left of $x=1$.

Calculate values of *y* as *x* approaches 1 from the left.

If $x=0.99$:
$y=\dfrac{3x^2-1}{x^2+5x-6}$ $y=\dfrac{3(0.99)^2-1}{(0.99)^2+5(0.99)-6}$ $y=\dfrac{1.9403}{-0.0699}$ $y=-27.7582\ldots$
If $x=0.9999$:
$y=\dfrac{3x^2-1}{x^2+5x-6}$ $y=\dfrac{3(0.9999)^2-1}{(0.9999)^2+5(0.9999)-6}$ $y=\dfrac{1.9994\ldots}{-0.0006\ldots}$ $y=-2856.3265\ldots$

As *x* approaches 1 from the left, the *y*-values approach negative infinity.

Step 6
Determine the behaviour on the right of $x=1$.

Calculate values of y as x approaches 1 from the left.

If $x=1.01$:

$$y=\frac{3x^2-1}{x^2+5x-6}$$
$$y=\frac{3(1.01)^2-1}{(1.01)^2+5(1.01)-6}$$
$$y=\frac{2.0603}{0.0701}$$
$$y=29.3908\ldots$$

If $x = 1.0001$:

$$y=\frac{3x^2-1}{x^2+5x-6}$$
$$y=\frac{3(1.0001)^2-1}{(1.0001)^2+5(1.0001)-6}$$
$$y=\frac{2.0006\ldots}{0.0007\ldots}$$
$$y=2857.9592\ldots$$

As x approaches 1 from the right, the y-values approach positive infinity.

Step 7
Determine the behaviour of the graph as $x\to-\infty$.

If $x = -100$:

$$y=\frac{3x^2-1}{x^2+5x-6}$$
$$y=\frac{3(-100)^2-1}{(-100)^2+5(-100)-6}$$
$$y=\frac{29\ 999}{9\ 494}$$
$$y=3.1597\ldots$$

If $x = -20\ 000$:

$$y=\frac{3x^2-1}{x^2+5x-6}$$
$$y=\frac{3(-20\ 000)^2-1}{(-20\ 000)^2+5(-20\ 000)-6}$$
$$y=\frac{1\ 99\ 999\ 999}{399\ 899\ 994}$$
$$y=3.0007\ldots$$

As $x\to\infty$, the y-values approach 3 from above.

Step 8
Determine the behaviour of the graph as $x\to\infty$.

If $x = 100$:

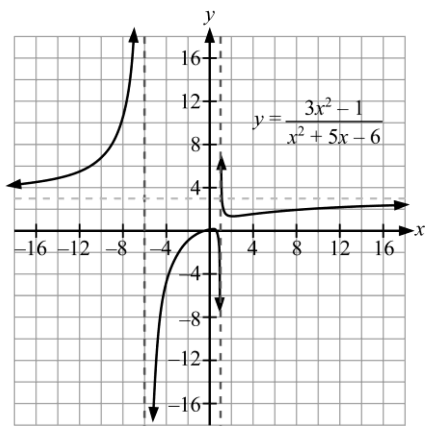
$$y=\frac{3x^2-1}{x^2+5x-6}$$
$$y=\frac{3(100)^2-1}{(100)^2+5(100)-6}$$
$$y=\frac{29999}{10494}$$
$$y=2.8586\ldots$$

If $x=20\ 000$:

$$y=\frac{3x^2-1}{x^2+5x-6}$$
$$y=\frac{3(20000)^2-1}{(20000)^2+5(20000)-6}$$
$$y=\frac{1\ 199\ 999\ 999}{400099994}$$
$$y=2.9992\ldots$$

As $x\to\infty$, the y-values approach 3 from below.

A possible sketch of $y=\dfrac{3x^2-1}{x^2+5x-6}$ is shown.



PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Determine the vertical asymptotes of the function.

The function $y=\dfrac{2x}{x-4}$ is already in factored form, and there are no factors that occur in both the numerator and denominator. Therefore, there is a vertical asymptote at $x=4$.

Step 2
Determine the horizontal asymptotes.

The degree of the numerator and denominator is 1. Since the degree of the numerator is equal to the degree of the denominator, the horizontal asymptote occurs at $y = \frac{2}{1} = 2$.

Step 3
Determine the behaviour on the left of $x = 4$.

Calculate values of *y* as *x* approaches 4 from the left.

If $x = 3.99$:

$$y = \frac{2x}{x-4}$$
$$y = \frac{2(3.99)}{3.99-4}$$
$$y = \frac{7.98}{-0.01}$$
$$y = -798$$

If $x = 3.9999$:

$$y = \frac{2x}{x-4}$$
$$y = \frac{2(3.9999)}{3.9999-4}$$
$$y = \frac{7.9998}{-0.0001}$$
$$y = -79\ 998$$

As *x* approaches 4 from the left, the *y*-values approach negative infinity.

Step 4
Determine the behaviour on the right of $x = 4$.

Calculate values of *y* as *x* approaches 4 from the right.

If $x = 4.01$:

$$y = \frac{2x}{x-4}$$
$$y = \frac{2(4.01)}{4.01-4}$$
$$y = \frac{8.02}{0.01}$$
$$y = 802$$

If $x = 4.0001$:

$$y = \frac{2x}{x-4}$$
$$y = \frac{2(4.0001)}{4.0001-4}$$
$$y = \frac{8.0002}{0.0001}$$
$$y = 80\ 002$$

As *x* approaches 4 from the right, the *y*-values approach positive infinity.

Step 5
Determine the behaviour of the graph as $x \to -\infty$.

If $x = -100$:

$$y = \frac{2x}{x-4}$$
$$y = \frac{2(-100)}{-100-4}$$
$$y = \frac{-200}{-104}$$
$$y = 1.9230\ldots$$

If $x = -20000$:

$$y = \frac{2x}{x-4}$$
$$y = \frac{2(-20000)}{-20000-4}$$
$$y = \frac{-40000}{-20004}$$
$$y = 1.9996\ldots$$

As $x \to -\infty$, the *y*-values approach 2 from below.

Step 6
Determine the behaviour of the graph as $x \to \infty$.

If $x = 100$:

$$y = \frac{2x}{x-4}$$
$$y = \frac{2(100)}{100-4}$$
$$y = \frac{200}{96}$$
$$y = 2.0833\ldots$$

If $x = 20\ 000$:

$$y = \frac{2x}{x-4}$$
$$y = \frac{2(20000)}{20000-4}$$
$$y = \frac{40000}{19996}$$
$$y = 2.0004\ldots$$

As $x \to -\infty$, the *y*-values approach 2 from above.

Lesson 3—Graphing a Rational Function

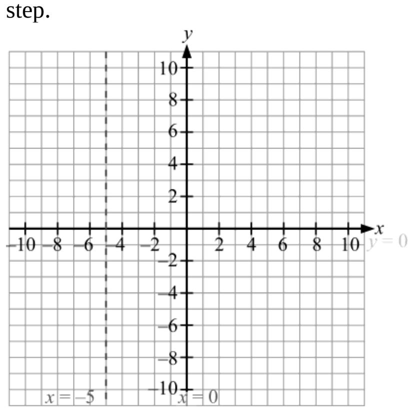
CLASS EXERCISES
ANSWERS AND SOLUTIONS

1. **Step 1**
Determine any asymptotes and holes.

The factored form of the function is
$y=\frac{4(x-2)}{x(x-2)(x+5)}$, where $x\neq -5$, $x\neq 0$, and $x\neq 2$.

The factor $(x-2)$ occurs in both the numerator and denominator. When expressed in reduced form, the function becomes $y=\frac{4}{x(x+5)}$, where $x\neq -5$, $x\neq 0$, and $x\neq 2$. Therefore, there is a hole at $x=2$ and vertical asymptotes at $x = 0$ and $x = -5$.

Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is at $y=0$. Show the vertical asymptote at $x=-5$ using a dotted line. Dotted lines at $x=0$ and $y=0$ are not needed since these lines are simply the y-axis and x-axis, respectively. The hole at $x=2$ will be graphed in a later step.



Step 2
Determine the point at which the graph will intersect or touch the horizontal asymptote.

Let $y = 0$, and solve for x.

$$y=\frac{4x-8}{x^3+3x^2-10x}$$
$$0=4x-8$$
$$8=4x$$
$$x=2$$

Since $x = 2$ was determined to be a non-permissible value in step 1, the graph is undefined at $x = 2$.

Therefore, the graph does not intersect or touch the horizontal asymptote $y = 0$.

Step 3
Find the intercepts.

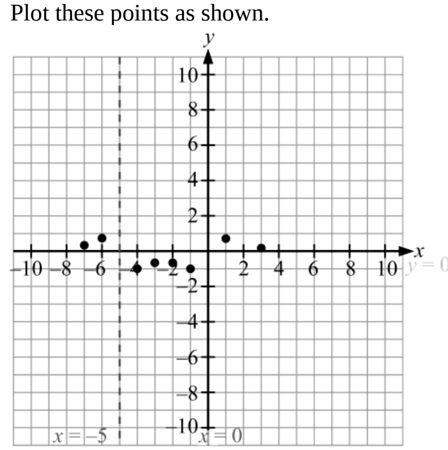
Since the horizontal asymptote is the x-axis and the graph does not cross or touch the horizontal axis, then there are no x-intercepts. One of the vertical asymptotes is the y-axis, which means the graph also has no y-intercept.

Step 4
Plot points on the right and left of each vertical asymptote.

Create a table of values. If necessary, round the y-values to the nearest tenth.

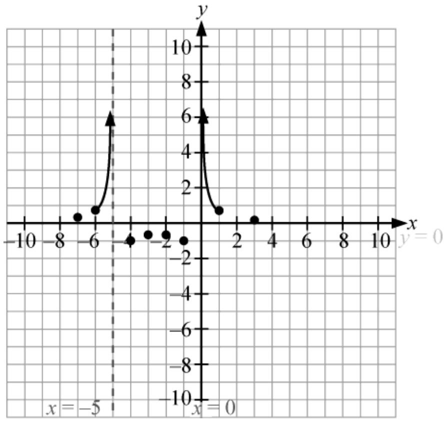
x	*y*
–7	$\frac{4(-7)-8}{(-7)^3+3(-7)^2-10(-7)}=0.3$
–6	$\frac{4(-6)-8}{(-6)^3+3(-6)^2-10(-6)}=0.7$
–4	$\frac{4(-4)-8}{(-4)^3+3(-4)^2-10(-4)}=-1$
–3	$\frac{4(-3)-8}{(-3)^3+3(-3)^2-10(-3)}=-0.7$
–2	$\frac{4(-2)-8}{(-2)^3+3(-2)^2-10(-2)}=-0.7$
–1	$\frac{4(-1)-8}{(-1)^3+3(-1)^2-10(-1)}=-1$
1	$\frac{4(1)-8}{(1)^3+3(1)^2-10(1)}=0.7$
3	$\frac{4(3)-8}{(3)^3+3(3)^2-10(3)}=0.2$

Plot these points as shown.



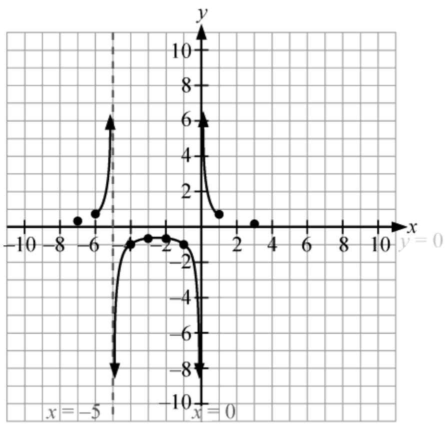
Step 5
From the point (–6, 0.7), show the graph increasing as x approaches the vertical asymptote $x = -5$ from the left.

From the point (1, 0.7), show the graph increasing as x approaches the vertical asymptote $x = 0$ from the right.



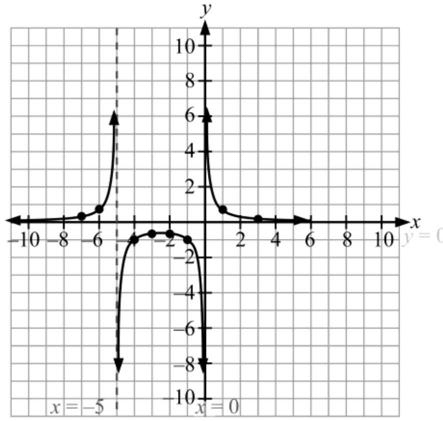
Step 6
From the point (–2, –0.7), show the graph going through (–3, 0.7) and (–4, –1) and decreasing as x approaches the vertical asymptote $x = -5$ from the right.

From the point (–2, –0.7), show the graph going through (–1, –1) and decreasing as x approaches the vertical asymptote $x = 0$ from the left.



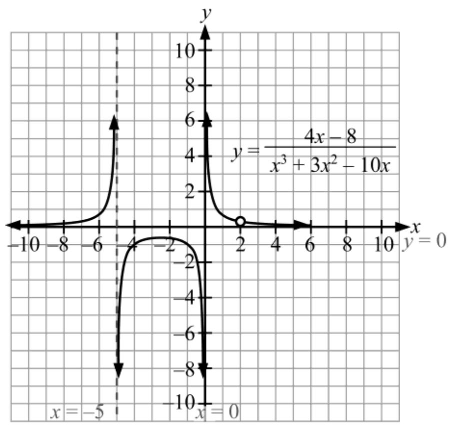
Step 7
From the point (–6, 0.7), show the graph going through (–7, 0.3) and approaching the horizontal asymptote $y = 0$.

From the point (1, 0.7), show the graph going through (3, 0.2) and approaching the horizontal asymptote $y = 0$.



Step 8
Place a hole in the graph at $x = 2$.

The result is the graph of $y = \dfrac{4x-8}{x^3+3x^2-10x}$.



PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Determine any asymptotes and holes.

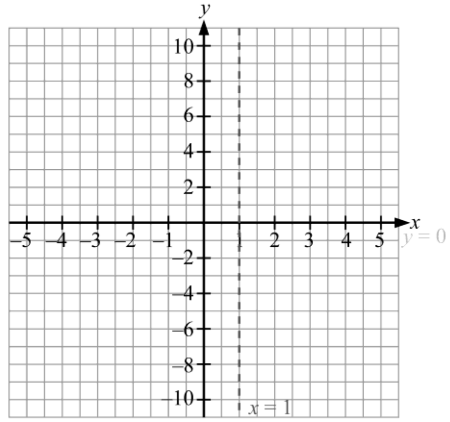
The factored form of the function is $y = \dfrac{x}{x(x-1)(x-1)}$,

where $x \neq 0$ and $x \neq -1$. The factor x occurs in both the numerator and denominator. In reduced form, the function is $y = \dfrac{1}{(x-1)(x-1)}$, where $x \neq 0$ and $x \neq -1$.

Therefore, there is a hole at $x = 0$ and a vertical asymptote at $x = 1$.

Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is at $y = 0$.

Sketch the asymptote at $x = 1$ as a dotted line.
A dotted line at $y = 0$ is not needed since this line is simply the *x*-axis.



Step 2
Determine the point at which the graph will intersect or touch the horizontal asymptote.

Let $y = 0$, and solve for *x*.

$$y = \frac{x}{x^3 - 2x^2 + x}$$
$$0 = \frac{x}{x^3 - 2x^2 + x}$$
$$x = 0$$

Since $x = 0$ was determined to be a non-permissible value in step 1, the graph is undefined at $x = 0$. Therefore, the graph does not intersect or touch the horizontal asymptote $y = 0$.

Step 3
Determine the intercepts.

Since the horizontal asymptote is the *x*-axis and the graph does not cross or touch the horizontal axis, there are no *x*-intercepts.

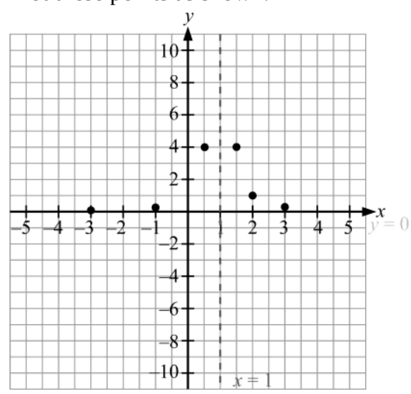
The *y*-intercept occurs when $x = 0$. Since $x = 0$ was determined to be a hole, there is no *y*-intercept; instead, there is a hole on the *y*-axis.

Step 4
Plot points on the right and left of the vertical asymptote.

Create a table of values. If necessary, round the *y*-values to the nearest tenth.

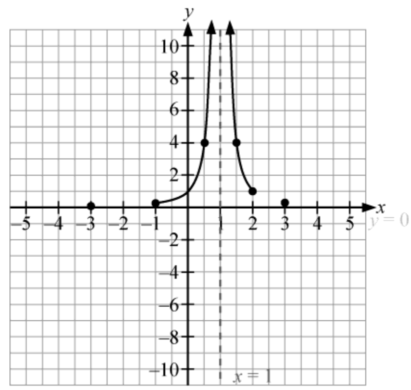
x	*y*
–3	$\frac{-3}{(-3)^3 - 2(-3)^2 + (-3)} = 0.1$
–1	$\frac{-1}{(-1)^3 - 2(-1)^2 + (-1)} = 0.3$
0.5	$\frac{0.5}{(0.5)^3 - 2(0.5)^2 + (0.5)} = 4$
1.5	$\frac{1.5}{(1.5)^3 - 2(1.5)^2 + (0 = 1.5)} = 4$
2	$\frac{2}{(2)^3 - 2(2)^2 + (2)} = 1$
3	$\frac{3}{(3)^3 - 2(3)^2 + (3)} = 0.3$

Plot these points as shown.



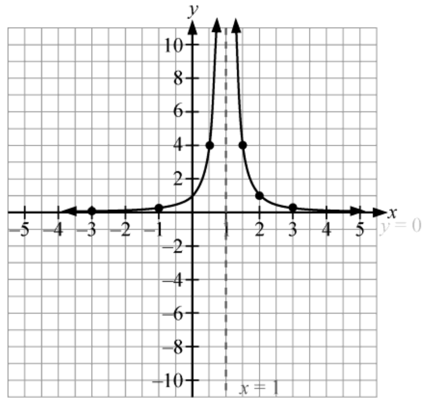
Step 5
From the point (–1, 0.3), show the graph going through (0.5, 4) and increasing as *x* approaches the vertical asymptote $x = 1$ from the left.

From the point (2, 1), show the graph going through (1.5, 4) and increasing as *x* approaches the vertical asymptote $x = -1$ from the right.



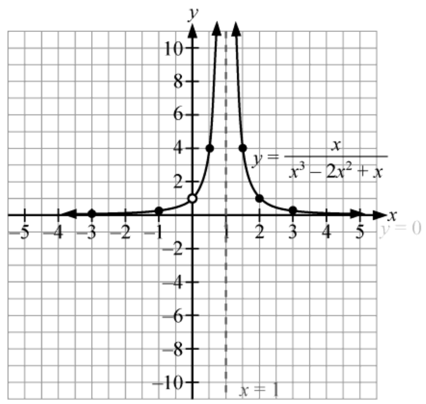
Step 6
From the point (–1, 0.3), show the graph going through (–3, 0.1) and decreasing as x approaches the horizontal asymptote $x = 0$.

From the point (2, 1), show the graph going through (3, 0.3) and decreasing as x approaches the horizontal asymptote $x = 0$.



Step 7
Place a hole in the graph at $x = 0$.

The result is the graph of $y = \dfrac{x}{x^3 - 2x^2 + x}$.



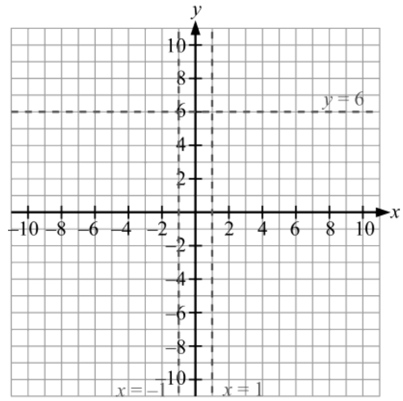
2. **Step 1**
Determine any asymptotes and holes.

The factored form of the function is $y = \dfrac{6x^2 + 1}{(x+1)(x-1)}$,

where $x \neq -1$ and $x \neq 1$.
There are no factors that occur in both the numerator and denominator. Therefore, there are no holes and the vertical asymptotes are $x = –1$
and $x = 1$.

The degree of the numerator is 2, and the degree of the denominator is 2. Since the degree of the numerator is equal to the degree of the denominator, the horizontal asymptote occurs at $y = \dfrac{6}{1} = 6$.

Sketch the asymptotes as dotted lines.



Step 2
Determine the point at which the graph will intersect or touch the horizontal asymptote.

Let $y = 6$, and solve for x.

$$y = \frac{6x^2 + 1}{x^2 - 1}$$
$$6 = \frac{6x^2 + 1}{x^2 - 1}$$
$$6\left(x^2 - 1\right) = 6x^2 + 1$$
$$6x^2 - 6 = 6x^2 + 1$$
$$0 = 7$$

Since no value of x makes $0 = 7$ true, the graph does not intersect or touch the horizontal asymptote.

Step 3
Determine the intercepts.

Find the x-intercepts. Let $y = 0$, and solve for x.

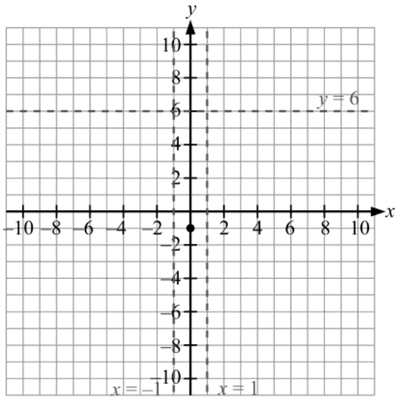
$$y = \frac{6x^2 + 1}{x^2 - 1}$$
$$0 = 6x^2 + 1$$
$$-1 = 6x^2$$
$$-\frac{1}{6} = x^2$$
$$\pm\sqrt{-\frac{1}{6}} = x$$

Since the square root of a negative number does not exist, there are no x-intercepts.

Find the y-intercept. Let $x = 0$, and solve for y.

$$y = \frac{6x^2 + 1}{x^2 - 1}$$
$$y = \frac{6(0)^2 + 1}{(0)^2 - 1}$$
$$y = \frac{1}{-1}$$
$$y = -1$$

The *y*-intercept is located at (0, –1). Plot the point (0, –1) as shown.

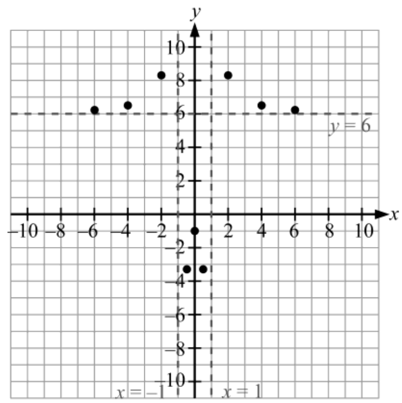


Step 4
Plot points on the right and left of each vertical asymptote.

Create a table of values. If necessary, round the *y*-values to the nearest tenth.

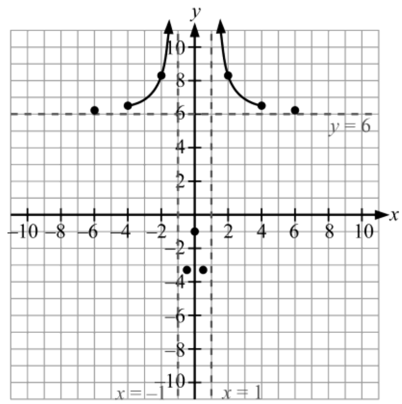
x	***y***
–6	$\frac{6(-6)^2+1}{(-6)^2-1}=6.2$
–4	$\frac{6(-4)^2+1}{(-4)^2-1}=6.5$
–2	$\frac{6(-2)^2+1}{(-2)^2-1}=8.3$
–0.5	$\frac{6(-0.5)^2+1}{(-0.5)^2-1}=-3.3$
0.5	$\frac{6(0.5)^2+1}{(0.5)^2-1}=-3.3$
x	***y***
2	$\frac{6(2)^2+1}{(2)^2-1}=8.3$
4	$\frac{6(4)^2+1}{(4)^2-1}=6.5$
6	$\frac{6(6)^2+1}{(6)^2-1}=6.2$

Plot these points as shown.



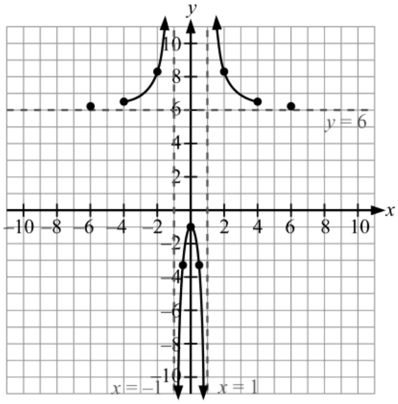
Step 5
From the point (–4, 6.5), show the graph going through (–2, 8.3) and increasing as *x* approaches the vertical asymptote $x=-1$ from the left.

From the point (4, 6.5), show the graph going through (2, 8.3) and increasing as *x* approaches the vertical asymptote $x=1$ from the right.



Step 6
From the point (0, –1), show the graph going through (–0.5, –3.3) and decreasing as *x* approaches the vertical asymptote $x=-1$ from the right.

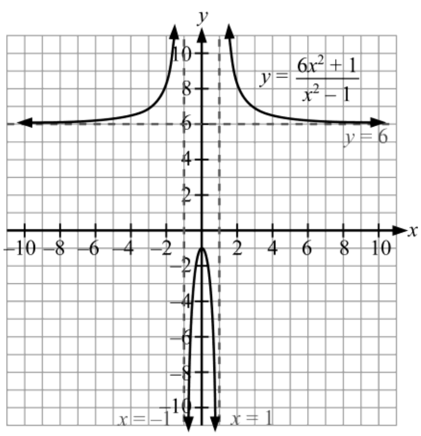
From the point (0, –1), show the graph going through (–0.5, –3.3) and decreasing as *x* approaches the vertical asymptote $x = 1$ from the left.



Step 7
From the point (–4, 6.5), show the graph going through (–6, 6.2) and approaching the horizontal asymptote $y = 6$.

From the point (4, 6.5), show the graph going through (6, 6.2) and approaching the horizontal asymptote $y = 6$.

The result is the graph of $y = \frac{6x^2 + 1}{x^2 - 1}$.



Lesson 4—Solving a Rational Equation Graphically

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Determine the non-permissible values.

$$x^2 + x = 0$$
$$x(x+1) = 0$$
$$x = 0 \qquad x + 1 = 0$$
$$x = -1$$

The non-permissible values are $x = -1$ and $x = 0$.

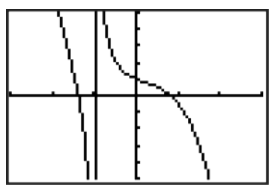
Step 2
Bring all terms to one side of the equation.

$$\frac{x}{x^2 + x} = x^3$$
$$\frac{x}{x^2 + x} - x^3 = 0$$

Step 3
Graph the related function using a TI-83 or similar graphing calculator.

Press $\boxed{\text{Y =}}$, enter the equation as $\text{Y}_1 = \left(\text{X}/\left(\text{X} \wedge 2 + \text{X}\right)\right) - \text{X} \wedge 3$, and press $\boxed{\text{GRAPH}}$.
An appropriate window setting is *x*: [–3, 3, 1] and *y*: [–5, 5, 1].

The resulting graph is as shown.



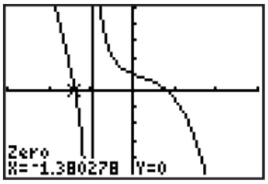
Step 4
Determine the zeros of the function.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 2:zero.

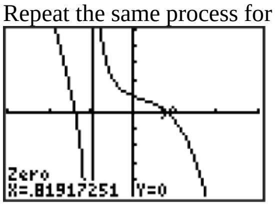
For “Left Bound?”, position the cursor just left of the zero, and press $\boxed{\text{ENTER}}$.

For “Right Bound?”, position the cursor just right of the zero, and press $\boxed{\text{ENTER}}$.

For "Guess?", press $\boxed{\text{ENTER}}$. The results are the coordinates of the zero of the function.



Repeat the same process for the second zero.



The zeros of the function are approximately –1.38 and 0.82.

Therefore, the solution set is {–1.38, 0.82}.

2. **Step 1**
Determine the non-permissible values.

$x-10=0 \qquad x+9=0$
$x=10 \qquad x=-9$

Therefore, the non-permissible values are $x=10$ and $x=-9$.

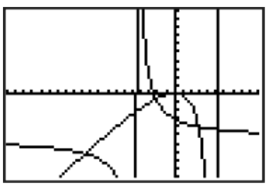
Step 2
Graph each side of the equation using a graphing calculator.

Press $\boxed{\text{Y =}}$, and input each function.

$Y_1 = X \wedge 2/(X-10)$

$Y_2 = -6-((5X+1)/(X+9))$

Press $\boxed{\text{GRAPH}}$. The window setting used to display the two graphs is *x*: [–40, 20, 2] and *y*: [–20, 20, 2].



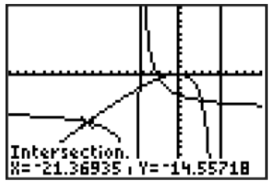
Step 3
Find the points of intersection.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and choose 5:intersect.

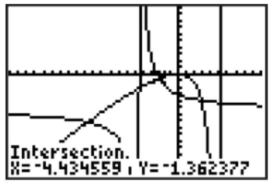
For "First curve?", position the cursor just left or right of the intersection point that is farthest to the left, and press $\boxed{\text{ENTER}}$.

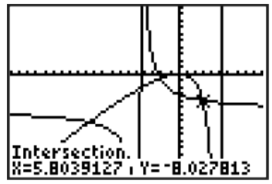
For "Second curve?", position the cursor just left or right of the intersection point that is farthest to the left, and press $\boxed{\text{ENTER}}$.

For "Guess?", press $\boxed{\text{ENTER}}$.



Repeat the process with the second and third points of intersection.





The points of intersection are approximately $(-21.37,-14.56)$, $(-4.43,-1.36)$, and $(5.80,-8.03)$.

Step 4
Determine the solution set of the equation.

The *x*-coordinates of the points of intersection are –21.37, –4.43, and 5.80.

Therefore, the solution set is $\{-21.37,-4.43,5.80\}$.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Determine the non-permissible values.

$x-3=0$
$x=3$

The non-permissible value is 3.

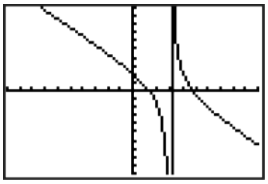
Step 2
Bring all terms to one side of the equation $\frac{3x-6}{x-3}=x$.

$$\frac{3x-6}{x-3}=x$$
$$\frac{3x-6}{x-3}-x=0$$

Step 3
Graph the related function using a TI-83 or similar graphing calculator.

Press $\boxed{\text{Y =}}$, enter the equation as $\text{Y}_1 = \left(\left(3\text{X}-6\right)/\left(\text{X}-3\right)\right)-x$, and press $\boxed{\text{GRAPH}}$.
An appropriate window setting is *x*: [–10, 10, 1] and *y*: [–10, 10, 1].

The resulting graph is as shown.

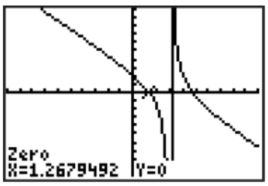


Step 4
Determine the zero of the function.
Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 2:zero.

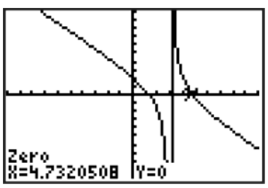
For "Left Bound?", position the cursor just left of the first zero, and press $\boxed{\text{ENTER}}$.

For "Right Bound?", position the cursor just right of the same zero, and press $\boxed{\text{ENTER}}$.

For "Guess?", press $\boxed{\text{ENTER}}$. The results are the coordinates of the first zero of the function.



Repeat the same process to find the second zero.



The zeros of the function are approximately 1.27 and 4.73.

Therefore, the solution set is {1.27, 4.73}.

2. **Step 1**
Determine the non-permissible values.

$$x+4=0$$
$$x=-4$$

The non-permissible value is –4.

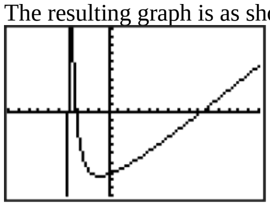
Step 2
Bring all terms to one side of the equation $10-x=\dfrac{10}{x+4}$.

$$10-x=\frac{10}{x+4}$$
$$0=\frac{10}{x+4}-10+x$$

Step 3
Graph the related function using a TI-83 or similar graphing calculator.

Press $\boxed{\text{Y =}}$, enter the equation as $\text{Y}_1 = \left(10/\left(\text{X}+4\right)\right)-10+\text{X}$, and press $\boxed{\text{GRAPH}}$.
An appropriate window setting is *x*: [–10, 15, 1] and *y*: [–10, 10, 1].

The resulting graph is as shown.

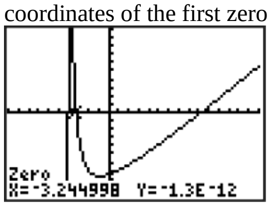


Step 4
Determine the zero of the function.
Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 2:zero.

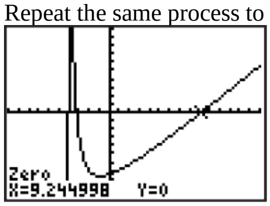
For "Left Bound?", position the cursor just left of the first zero, and press $\boxed{\text{ENTER}}$.

For "Right Bound?", position the cursor just right of the same zero, and press $\boxed{\text{ENTER}}$.

For "Guess?", press $\boxed{\text{ENTER}}$. The results are the coordinates of the first zero of the function.



Repeat the same process to find the second zero.



The zeros of the function are approximately –3.24 and 9.24.

Therefore, the solution set is {–3.24, 9.24}.

3. **Step 1**
Determine the non-permissible values.

$$x + 2 = 0$$
$$x = -2$$

Therefore, the non-permissible value is $x = -2$.

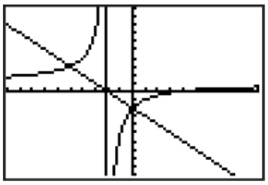
Step 2
Graph each side of the equation using a TI-83.

Press $\boxed{\text{Y =}}$, and input each function.

$Y_1 = (X - 4)/(X + 2)$ and $Y_2 = -X - 2$

Press $\boxed{\text{GRAPH}}$. The window setting used to display the two graphs is *x*: [−10, 10, 1] and

y: [−10, –10, 1].



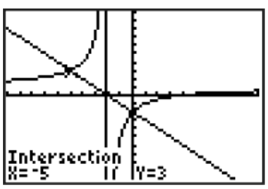
Step 3
Find the points of intersection.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and choose 5:intersect.

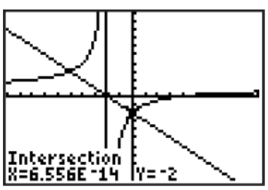
For "First curve?", position the cursor just left or right of the intersection point that is farthest to the left, and press $\boxed{\text{ENTER}}$.

For "Second curve?", position the cursor just left or right of the intersection point that is farthest to the left, and press $\boxed{\text{ENTER}}$.

For "Guess?", press $\boxed{\text{ENTER}}$.



Repeat the process with the second point of intersection.



The points of intersection are approximately (–5, 3) and (0, –2).

Step 4
Determine the solution set of the equation.

The *x*-coordinates of the points of intersection are –5 and 0. Therefore, the solution set is $\{-5, 0\}$.

4. **Step 1**
Determine the non-permissible values.

$x = 0 \qquad x + 2 = 0$
$\qquad\qquad x = -2$

Therefore, the non-permissible values are $x = -2$ and $x = 0$.

Step 2
Graph each side of the equation using a TI-83.

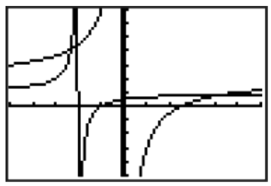
Press $\boxed{\text{Y =}}$, and input each function.

$Y_1 = (2X - 5)/X$

$Y_2 = (X + 1)/(X + 2)$

Press $\boxed{\text{GRAPH}}$. The window setting used to display the two graphs is *x*: [−5, 6, 1] and

y: [−5, 7, 1].



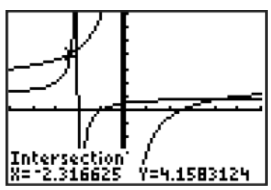
Step 3
Find the points of intersection.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and choose 5:intersect.

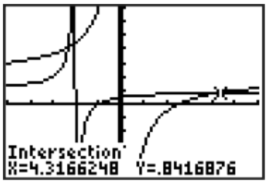
For "First curve?", position the cursor just left or right of the intersection point that is farthest to the left, and press $\boxed{\text{ENTER}}$.

For "Second curve?", position the cursor just left or right of the intersection point that is farthest to the left, and press $\boxed{\text{ENTER}}$.

For "Guess?", press $\boxed{\text{ENTER}}$.



Repeat the process with the second point of intersection.



The points of intersection are approximately $(-2.32, 4.16)$ and $(4.32, 0.84)$.

Step 4
Determine the solution set of the equation.

The x-coordinates of the points of intersection are –2.32 and 4.32.

Therefore, the solution set is $\{-2.32, 4.32\}$.

5. **Step 1**
Determine the non-permissible values.

$x+1=0 \qquad 2x=0$
$x=-1 \qquad x=0$

Therefore, the non-permissible values are $x=-1$ and $x=0$.

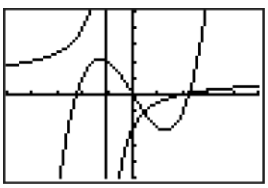
Step 2
Graph each side of the equation using a TI-83.

Press $\boxed{\text{Y =}}$, and input each function.

$\text{Y}_1 = (\text{X}-2)/(\text{X}+1)$

$\text{Y}_2 = (\text{X}^\wedge 2-5)/2\text{X}$

Press $\boxed{\text{GRAPH}}$. The window setting used to display the two graphs is x: [−5, 5, 1] and y: [−5, 5, 1].



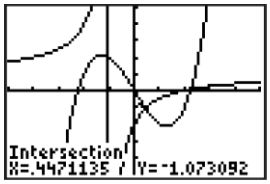
Step 3
Find the points of intersection.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and choose 5:intersect.

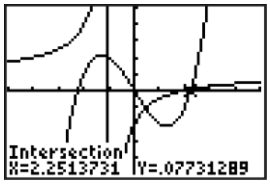
For "First curve?", position the cursor just left or right of the intersection point that is farthest to the left, and press $\boxed{\text{ENTER}}$.

For "Second curve?", position the cursor just left or right of the intersection point that is farthest to the left, and press $\boxed{\text{ENTER}}$.

For "Guess?", press $\boxed{\text{ENTER}}$.



Repeat the process with the second point of intersection.



The points of intersection are approximately $(0.45, -1.07)$ and $(2.25, 0.08)$.

Step 4
Determine the solution set of the equation.

The x-coordinates of the points of intersection are 0.45 and 2.25.

Therefore, the solution set is $\{0.45, 2.25\}$.

Practice Test

ANSWERS AND SOLUTIONS

1. **Step 1**
Express the function in factored form, and state the non-permissible values of x.

$$f(x)=\frac{x^2-7x}{x^3-9x^2+14x}$$
$$f(x)=\frac{x(x-7)}{x(x^2-9x+14)}$$
$$f(x)=\frac{x(x-7)}{x(x-2)(x-7)}$$

Set the denominator equal to zero, and solve for x.

$x=0 \qquad x-2=0 \quad x-7=5$
$\qquad\qquad x=2 \qquad x=7$

The non-permissible values are $x=0$, $x=2$, and $x=7$.

Step 2
Determine if the graph of $f(x)$ will have a vertical asymptote or a hole for the non-permissible values of *x*.

The factors (x) and $(x-7)$ occur in both the numerator and denominator.

Express the function in reduced form.

$$f(x)=\frac{x(x-7)}{x(x-2)(x-7)}$$
$$f(x)=\frac{\cancel{x}\,\cancel{(x-7)}}{\cancel{x}\,(x-2)\,\cancel{(x-7)}}$$

The factors (x) and $(x-7)$ both cancel out from the numerator and denominator. Therefore, $x=0$ and $x=7$ are holes and $x=2$ is a vertical asymptote.

Step 3
Determine the horizontal asymptotes.

The degree of the numerator (x^2-7x) is 2, and the degree of the denominator (x^3-9x^2+14x) is 3.

Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote occurs at $y=0$.

2. **Step 1**
Determine the value of *a*.

The line $y=2$ is a horizontal asymptote.
This means the quotient of the leading coefficient of the numerator and the leading coefficient of the denominator must be equal to 2.

The leading coefficient of $a(x-b)$ or $ax-ab$ is equal to *a*, and the leading coefficient of $x-c$ is 1.

Solve for *a* in the equation $\frac{a}{1}=2$.

$$\frac{a}{1}=2$$
$$a=2$$

Therefore, the value of *a* is 2.

Step 2
Determine the value of *b*.

The *x*-intercept on the graph of $f(x)$ is $(3,0)$.
Therefore, $f(3)=0$.

Let $f(x)=0$, $x=3$, and $a=2$, and solve for *b*.

$$f(x)=\frac{a(x-b)}{x-c}$$
$$0=\frac{2(3-b)}{3-c}$$
$$0=2(3-b)$$
$$0=3-b$$
$$b=3$$

Therefore, the value of *b* is 3.

Step 3
Determine the value of *c*.

The line $x=1$ is a vertical asymptote. This means 1 is a non-permissible value and the denominator of the function must be $x-1$.

Therefore, the value of *c* is 1.

3. **Step 1**
Express the function in factored form, and state the non-permissible values of *x*.

Since $h(-12)=0$, then by the factor theorem, $(x+12)$ is a factor of $h(x)=x^3+10x^2-48x-288$.

Use synthetic division to determine another factor.

$$\begin{array}{r|rrrr} -12 & 1 & 10 & -48 & -288 \\ & \downarrow & -12 & 24 & 288 \\ \hline & 1 & -2 & -24 & 0 \end{array}$$

Another factor of $h(x)$ is the quadratic expression $(x^2-2x-24)$, which factors to $(x+4)(x-6)$.
Therefore, the factored form of the denominator is $(x+12)(x+4)(x-6)$.

The factored form of the rational function can now be expressed as

$$R(x)=\frac{2x^3}{(x+12)(x+4)(x-6)}$$

Set the denominator equal to zero, and solve for *x*.

$$x+12=0 \qquad x+4=0 \qquad x-6=0$$
$$x=-12 \qquad x=-4 \qquad x=6$$

The non-permissible values are $x=-12$, $x=-4$, and $x=6$.

Step 2
Determine if the graph of $f(x)$ will have a vertical asymptote or a hole for the non-permissible values of *x*.

Express the function in reduced form.

$$R(x)=\frac{2x^3}{(x+12)(x+4)(x-6)}$$
$$R(x)=\frac{2(x)(x)(x)}{(x+12)(x+4)(x-6)}$$

There are no factors that occur in both the numerator and denominator.

Therefore, $x=-12$, $x=-4$, and $x=6$ are vertical asymptotes.

Step 3
Determine the horizontal asymptotes.

The degree of the numerator $(2x^3)$ is 2, and the degree of the denominator $(x^3+10x^2-48x-288)$ is 3. The degree of the numerator is equal to the degree of the denominator.

Therefore, the horizontal asymptote is $y=\frac{2}{1}=2$.

4. **Step 1**
Determine the vertical asymptotes of the function.

The factored form of the function is
$y=\frac{2x(x-2)}{(x-2)(x+8)}$, where $x\neq 2$ and $x\neq -8$.
The factor $(x-2)$ occurs in both the numerator and denominator. When expressed in reduced form, the function becomes $y=\frac{2x}{(x+8)}$, where $x\neq 2$ and $x\neq -8$.

Therefore, there is a vertical asymptote at $x=-8$ and a hole at $x=2$.

Step 2
Determine the behaviour on the left of $x=-8$.

Calculate values of *y* as *x* approaches –8 from the left.

If $x=-8.01$:
$y=\frac{2(-8.01)^2-4(-8.01)}{(-8.01)^2+6(-8.01)-16}$ $y=\frac{160.3602}{0.1001}$ $y=1602$
If $x=-8.0001$:
$y=\frac{2(-8.0001)^2-4(-8.0001)}{(-8.0001)^2+6(-8.0001)-16}$ $y=\frac{160.0036}{0.00100001}$ $y=160002$

As *x* approaches –8 from the left, the *y*-values approach positive infinity.

Step 3
Determine the behaviour on the right of $x=-8$.

Calculate values of *y* as *x* approaches –8 from the right.

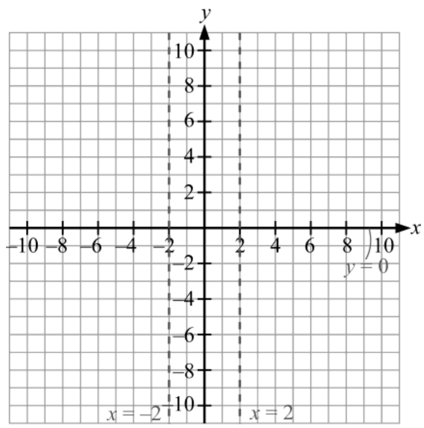
If $x=-7.99$:
$y=\frac{2(-7.99)^2-4(-7.99)}{(-7.99)^2+6(-7.99)-16}$ $y=\frac{159.6402}{-0.0999}$ $y=-1598$
If $x=-7.9999$:
$y=\frac{2(-7.9999)^2-4(-7.9999)}{(-7.9999)^2+6(-7.9999)-16}$ $y=\frac{159.9964}{-0.00099999}$ $y=-159998$

As *x* approaches –8 from the right, the *y*-values approach negative infinity.

5. **Step 1**
Determine any asymptotes and holes.

The factored form of the function is

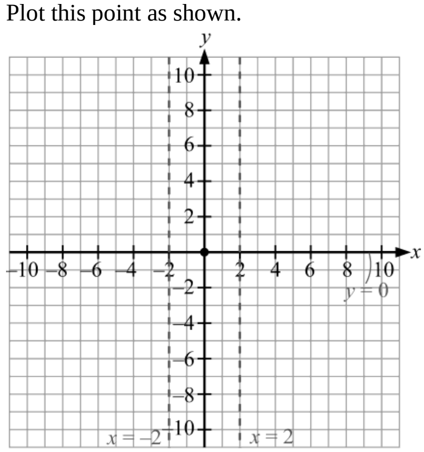
$y=\dfrac{2x}{(x+2)(x-2)}$, where $x\neq -2$ and $x\neq 2$. There are no factors that occur in both the numerator and denominator. Therefore, there are no holes and the vertical asymptotes are $x=-2$ and $x=2$. The degree of the numerator is 1, and the degree of the denominator is 2. Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote occurs at $y=0$. Sketch the asymptotes as dotted lines.



Step 2
Determine the point at which the graph will intersect or touch the horizontal asymptote.
Let $y=0$, and solve for x.

$$y=\frac{2x}{x^2-4}$$
$$0=\frac{2x}{x^2-4}$$
$$0=2x$$
$$0=x$$

Therefore, the graph of the rational function will intersect or touch the horizontal asymptote at (0, 0). Plot this point as shown.



Step 3
Determine the intercepts.

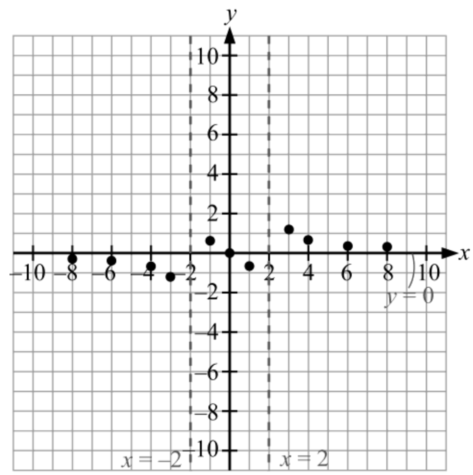
Since the graph intersects the horizontal asymptote at (0, 0), the x-intercept is located at the point (0, 0). Since the y-intercept occurs when $x=0$, then the y-intercept is also located at (0, 0).

Step 4
Plot points on the right and left of each vertical asymptote.

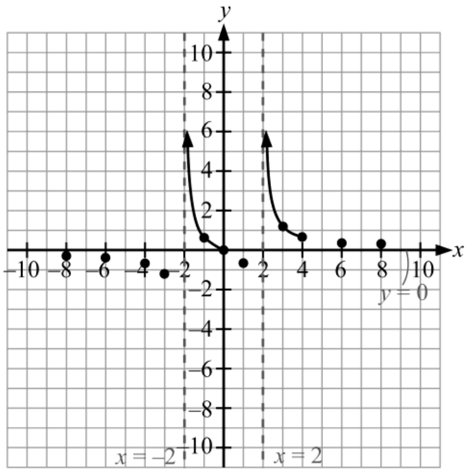
Create a table of values. If necessary, round the y-values to the nearest tenth.

x	***y***	***x***	***y***
–8	$\frac{2(-8)}{(-8)^2-4}=-0.3$	1	$\frac{2(1)}{(1)^2-4}=-0.7$
–6	$\frac{2(-6)}{(-6)^2-4}=-0.4$	3	$\frac{2(3)}{(3)^2-4}=1.2$
–4	$\frac{2(-4)}{(-4)^2-4}=-0.7$	4	$\frac{2(4)}{(4)^2-4}=0.7$
–3	$\frac{2(-3)}{(-3)^2-4}=-1.2$	6	$\frac{2(6)}{(6)^2-4}=0.4$
–1	$\frac{2(-1)}{(-1)^2-4}=0.7$	8	$\frac{2(8)}{(8)^2-4}=0.3$

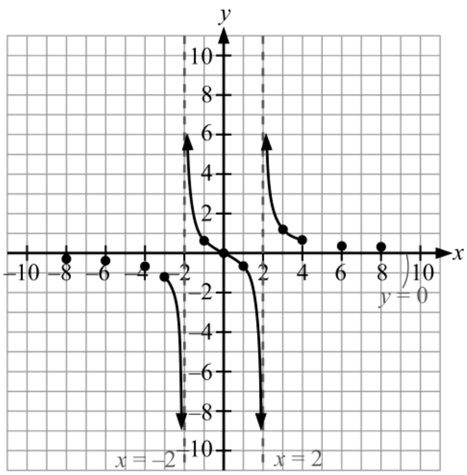
Plot these points as shown.



Step 5
From the point (0, 0), show the graph going through (–1, 0.7) and increasing as *x* approaches the vertical asymptote $x = -2$ from the right.
From the point (4, 0.7), show the graph going through (3, 1.2) and increasing as *x* approaches the vertical asymptote $x = 2$ from the right.



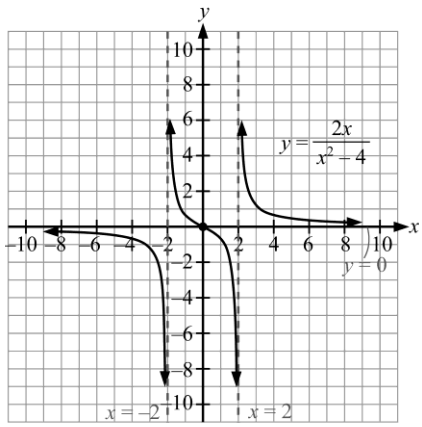
Step 6
From the point (–3, –1.2), show the graph decreasing as *x* approaches the vertical asymptote $x = -2$ from the left.
From the point (0, 0), show the graph going through (1, –0.7) and decreasing as *x* approaches the vertical asymptote $x = 2$ from the left.



Step 7
From the point (–3, –1.2), show the graph going through (–4, –0.7), (–6, –0.4), and (–8, –0.3) and approaching the *x*-axis.

From the point (4, 0.7), show the graph going through (6, 0.4) and (8, 0.3) and approaching the *x*-axis.

The result is the graph of $y = \dfrac{2x}{x^2 - 4}$.



Step 8
State the domain and range.

The domain is $x \neq -2$ or $x \neq 2$, $x \in R$. The range is $y \neq 0$, $y \in R$.

6. **Step 1**
Determine the non-permissible values.

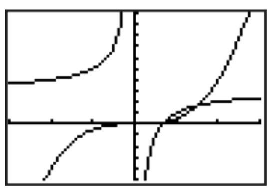
The non-permissible value is $x = 0$.

Step 2
Graph each side of the equation using a graphing calculator.
Press $\boxed{\text{Y =}}$, and input each function.

$$\text{Y}_1 = \left(\left(\text{X} - 2\right)/\text{X}\right) + 2$$

$$\text{Y}_2 = \left(\text{X}^\wedge 3\right)/2$$

Press $\boxed{\text{GRAPH}}$. The window setting used to display the two graphs is *x*: [−3, 3, 1] and *y*: [−5, 10, 1].



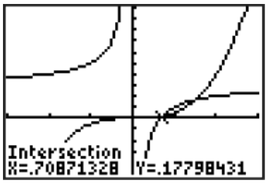
Step 3
Find the points of intersection.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and choose 5:intersect.

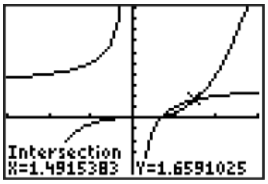
For "First curve?", position the cursor just left or right of the intersection point that is farthest to the left, and press $\boxed{\text{ENTER}}$.

For "Second curve?", position the cursor just left or right of the intersection point that is farthest to the left, and press $\boxed{\text{ENTER}}$.

For "Guess?", press $\boxed{\text{ENTER}}$.



Repeat the process with the intersection point that is farthest to the right.



The points of intersection are approximately (0.71, 0.18) and (1.49, 1.66).

Step 4
Determine the solution set of the equation.
The x-coordinates of the points of intersection are 0.71 and 1.49.
Therefore, the solution set is $\{0.71, 1.49\}$.

EXPONENTIAL FUNCTIONS

Lesson 1—The Graph of an Exponential Function

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Apply the vertical stretch.

Since the function is vertically stretched by a factor of 5, replace y in $y=\left(\frac{1}{4}\right)^x$ with $\frac{1}{5}y$.

$$y=\left(\frac{1}{4}\right)^x$$
$$\frac{1}{5}y=\left(\frac{1}{4}\right)^x$$

Step 2
Apply the horizontal stretch.

Since the function is horizontally stretched by a factor of $\frac{1}{2}$, replace x in $\frac{1}{5}y=\left(\frac{1}{4}\right)^x$ with $2x$.

$$\frac{1}{5}y=\left(\frac{1}{4}\right)^x$$
$$\frac{1}{5}y=\left(\frac{1}{4}\right)^{2x}$$

Step 3
Apply the vertical translation.

Since the function is translated 10 units down, replace y in $\frac{1}{5}y=\left(\frac{1}{4}\right)^{2x}$ with $y-(-10)$.

$$\frac{1}{5}y=\left(\frac{1}{4}\right)^{2x}$$
$$\frac{1}{5}\left(y-(-10)\right)=\left(\frac{1}{4}\right)^{2x}$$

Step 4
Isolate *y*.

$$\frac{1}{5}\left(y-(-10)\right)=\left(\frac{1}{4}\right)^{2x}$$
$$\frac{1}{5}\left(y+10\right)=\left(\frac{1}{4}\right)^{2x}$$
$$y+10=5\left(\frac{1}{4}\right)^{2x}$$
$$y=5\left(\frac{1}{4}\right)^{2x}-10$$

The equation of the transformed graph is

$y=5\left(\frac{1}{4}\right)^{2x}-10$.

Step 5
Determine the *y*-intercept of the transformed graph.

Let $x=0$, and solve for *y*.

$$y=5\left(\frac{1}{4}\right)^{2(0)}-10$$
$$y=5\left(\frac{1}{4}\right)^{0}-10$$
$$y=5(1)-10$$
$$y=-5$$

Therefore, the *y*-intercept is (0, –5).

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. The equation $y-9=6^{x+5}$ becomes $y=6^{x+5}+9$.

The domain is $x\in R$, and the range is $y>9$.
The horizontal asymptote is $y=9$.

Let $x=0$, and solve for *y*.

$$y=6^{x+5}+9$$
$$y=6^{0+5}+9$$
$$y=6^{5}+9$$
$$y=7785$$

Therefore, the *y*-intercept is (0, 7 785).

2. The equation $2\left(y+3\right)=\left(\frac{1}{2}\right)\left(6\right)^{x}$ becomes

$y=\left(\frac{1}{4}\right)\left(6\right)^{x}-3$.

The domain is $x\in R$, and the range is $y>-3$. The horizontal asymptote is $y=-3$.

Let $x=0$, and solve for *y*.

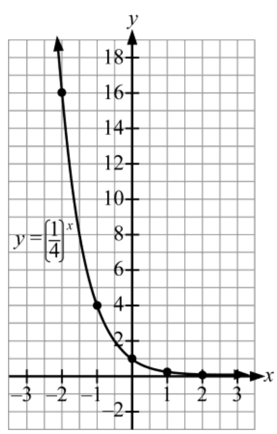
$$y=\left(\frac{1}{4}\right)\left(6\right)^{0}-3$$
$$y=\frac{1}{4}-3$$
$$y=\frac{1}{4}-3$$
$$y=-\frac{11}{4}$$

Therefore, the *y*-intercept is $\left(0,-\frac{11}{4}\right)$.

3. **Step 1**

Sketch the graph of $y=\left(\frac{1}{4}\right)^{x}$ using a table of values.

x	*y*
–2	16
–1	4
0	1
1	0.25
2	0.0625



Step 2
Apply the vertical stretch.

Since the graph of $y=\left(\frac{1}{4}\right)^x$ is vertically stretched by a factor of 6, multiply the *y*-coordinates by 6.

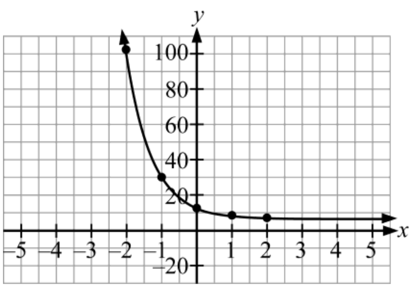
$$(-2,16\times 6)\to(-2,96)$$
$$(-1,4\times 6)\to(-1,24)$$
$$(0,1\times 6)\to(0,6)$$
$$(1,0.25\times 6)\to(0,1.5)$$
$$(2,0.0625\times 6)\to(0,0.375)$$

Step 3
Apply the vertical translation.

Since the graph of $y=\left(\frac{1}{4}\right)^x$ is vertically translated 7 units up, increase the *y*-coordinates obtained in step 2 by 7.

$$(-2,96+7)\to(-2,103)$$
$$(-1,24+7)\to(-1,31)$$
$$(0,6+7)\to(0,13)$$
$$(1,1.5+7)\to(1,8.5)$$
$$(2,0.375+7)\to(2,7.375)$$

Step 4
Plot and join the transformed points.



Step 5
State the domain and range.

The domain is $x\in R$.

Since the graph of $y=\left(\frac{1}{4}\right)^x$ is translated 7 units up, the range is $y>7$.

4. Apply the transformations in the order shown.

i) $y=4(5)^x$

ii) $y=4(5)^{\frac{3}{2}x}$

iii) $y=4(5)^{\left[\frac{3}{2}(x+11)\right]}$

The equation of the transformed function is $y=4(5)^{\left[\frac{3}{2}(x+11)\right]}$.

5. **Step 1**

Determine the horizontal asymptote of $y=4(5)^{\left[\frac{3}{2}(x+11)\right]}$.

The horizontal asymptote of $y=5^x$ is $y=0$. Since there is no vertical translation, the horizontal asymptote of the transformed function is $y=0$.

Step 2
State the domain and range.

The domain is $x\in R$, and the range is $y>0$.

6. **Step 1**

In the transformed function $y=4(5)^{\left[\frac{3}{2}(x+11)\right]}$, substitute –9 for *x* and *k* for *y*.

$$y=4(5)^{\left[\frac{3}{2}(x+11)\right]}$$
$$k=4(5)^{\left[\frac{3}{2}(-9+11)\right]}$$

Step 2
Determine the value of *k*.

$$k=4(5)^{\left[\frac{3}{2}(-9+11)\right]}$$
$$k=4(5)^{\left[\frac{3}{2}(2)\right]}$$
$$k=4(5)^3$$
$$k=500$$

The value of *k* is 500.

7. The function $g(x)=\left(\frac{5}{2}\right)^{4x+1}-3$ is equivalent to

$$g(x)=\left(\frac{5}{2}\right)^{4\left(x+\frac{1}{4}\right)}-3.$$

Step 1

Describe the transformations applied to $f(x)=\left(\frac{5}{2}\right)^x$ to get the graph of $g(x)=\left(\frac{5}{2}\right)^{4\left(x+\frac{1}{4}\right)}-3$.

The function $g(x)=\left(\frac{5}{2}\right)^{4\left(x+\frac{1}{4}\right)}-3$ is of the form $y=\left(\frac{5}{2}\right)^{b(x-h)}+k$, where $b=4$, $h=-\frac{1}{4}$, and $k=-3$. Therefore, the graph $g(x)=\left(\frac{5}{2}\right)^{4\left(x+\frac{1}{4}\right)}-3$ is obtained from $f(x)=\left(\frac{5}{2}\right)^{x}$ by a horizontal stretch factor of $\frac{1}{4}$, a horizontal translation of $\frac{1}{4}$ or 0.25 left, and a vertical translation 3 units down.

Step 2
Apply the transformations to the point (1, 2.5).

Apply the horizontal stretch so the point becomes $\left(1\times\frac{1}{4}, 2.5\right)=(0.25, 2.5)$. Then, apply the translations so the point becomes
$(0.25-0.25, 2.5-3)=(0,-0.5)$.

Therefore, the point (1, 2.5) on the graph of *f* transforms to the point (0, –0.5) on the graph of *g*.

Lesson 2—Solving Exponential Equations

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Since the bases on each side of the equation are equal, then the exponents on each side of the equation must also be equal.

Equate the exponents, and solve for *x*.

$$4x^2-9=3x^2$$
$$x^2-9=0$$
$$(x+3)(x-3)=0$$
$$x+3=0 \qquad x-3=0$$
$$x=-3 \qquad x=3$$

The values of *x* are –3 and 3.

2. **a)** **Step 1**
Write the equation so that the powers have the same base.

Since both sides can be written as powers with base 5, replace 125 with 5^3 and $\frac{1}{25}$ with 5^{-2}.

$$125^{4x}=\left(\frac{1}{25}\right)^{x+9}$$
$$\left(5^3\right)^{4x}=\left(5^{-2}\right)^{x+9}$$

Step 2
Simplify the equation using the laws of exponents.

Apply the power law of exponents to both sides of the equation.

$$\left(5^3\right)^{4x}=\left(5^{-2}\right)^{x+9}$$
$$5^{12x}=5^{-2x-18}$$

Step 3
Apply the property that if $b^x=b^y$, where $b\neq 0$ and $b\neq 1$, then $x=y$.

Both sides of the equation have the same base, so equate the exponents.

$$12x=-2x-18$$

Step 4
Solve for *x*.

$$12x=-2x-18$$
$$14x=-18$$
$$x=-\frac{18}{14}$$
$$x=-\frac{9}{7}$$

Therefore, the value of *x* is $-\frac{9}{7}$.

b) **Step 1**
Write the equation so that the powers have the same base.

Since both sides can be written as powers with base 3, replace 9 with 3^2 and 81 with 3^4.

$$\frac{3^{m-6}}{9^{5m-1}}=81^m$$
$$\frac{3^{m-6}}{\left(3^2\right)^{5m-1}}=\left(3^4\right)^m$$

Step 2
Apply the power rule to both sides of the equation.

$$\frac{3^{m-6}}{\left(3^2\right)^{5m-1}}=\left(3^4\right)^m$$
$$\frac{3^{m-6}}{3^{10m-2}}=3^{4m}$$

Step 3
Apply the quotient law to the left side of the equation.

$$\frac{3^{m-6}}{3^{10m-2}} = 3^{4m}$$
$$3^{m-6-(10m-2)} = 3^{4m}$$
$$3^{-9m-4} = 3^{4m}$$

Step 4
Apply the property that if $b^x = b^y$, where $b \neq 0$ and $b \neq 1$, then $x = y$.

Both sides of the equation have the same base, so equate the exponents.

$$-9m - 4 = 4m$$
$$-4 = 13m$$
$$m = -\frac{4}{13}$$

Therefore, the value of m is $-\frac{4}{13}$.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. The equation $6^{12x^2-4x} - 2 = 4$ can be rewritten as $6^{12x^2-4x} = 6$.

Since the bases on each side of the equation are equal, the exponents on each side of the equation must be equal. Equate the exponents, and solve for x.

$$12x^2 - 4x = 1$$
$$12x^2 - 4x - 1 = 0$$
$$(6x+1)(2x-1) = 0$$
$$6x + 1 = 0 \qquad 2x - 1 = 0$$
$$6x = -1 \qquad 2x = 1$$
$$x = -\frac{1}{6} \qquad x = \frac{1}{2}$$

Therefore, the values of x are $-\frac{1}{6}$ and $\frac{1}{2}$.

2. **Step 1**

Write the equation $12^{x+3} = \left(\frac{1}{12}\right)^{8x}$ so that the powers have the same base.

Replace $\frac{1}{12}$ with 12^{-1}.

$$12^{x+3} = \left(\frac{1}{12}\right)^{8x}$$
$$12^{x+3} = \left(12^{-1}\right)^{8x}$$

Step 2
Apply the power law to the right side of the equation.

$$12^{x+3} = \left(12^{-1}\right)^{8x}$$
$$12^{x+3} = 12^{-8x}$$

Step 3
Solve for x.
Both sides of the equation have the same base, so equate the exponents and solve for x.

$$x + 3 = -8x$$
$$9x + 3 = 0$$
$$9x = -3$$
$$x = -\frac{3}{9}$$
$$x = -\frac{1}{3}$$

Therefore, the value of x is $-\frac{1}{3}$.

3. **Step 1**

Divide both sides of the equation $7\left(2^{x^2-14}\right) = 28$ by 7.

$$7\left(2^{x^2-14}\right) = 28$$
$$2^{x^2-14} = 4$$

Step 2

Write the equation $2^{x^2-14} = 4$ so that the powers have the same base.

Replace 4 with 2^2.

$$2^{x^2-14} = 4$$
$$2^{x^2-14} = 2^2$$

Step 3
Solve for x.

Both sides of the equation have the same base, so equate the exponents and solve for x.

$$x^2 - 14 = 2$$
$$x^2 = 16$$
$$x = \pm\sqrt{16}$$
$$x = \pm 4$$

Therefore, the values of x are –4 and 4.

4. **Step 1**
Write the equation $16\left(4^{2x-3}\right) = 64^{x-5}$ so that the powers have the same base.

Replace 16 with 4^2 and 64 with 4^3.

$$16\left(4^{2x-3}\right) = 64^{x-5}$$
$$4^2\left(4^{2x-3}\right) = \left(4^3\right)^{x-5}$$

Step 2
Apply the product law to the left side of the equation.

$$\begin{aligned} 4^2\left(4^{2x-3}\right) &= \left(4^3\right)^{x-5} \\ 4^{2+2x-3} &= \left(4^3\right)^{x-5} \\ 4^{2x-1} &= \left(4^3\right)^{x-5} \end{aligned}$$

Step 3
Apply the power law to the right side of the equation.

$$\begin{aligned} 4^{2x-1} &= \left(4^3\right)^{x-5} \\ 4^{2x-1} &= 4^{3x-15} \end{aligned}$$

Step 4
Solve for x.

Both sides of the equation have the same base,
so equate the exponents and solve for x.

$$\begin{aligned} 2x-1 &= 3x-15 \\ -x-1 &= -15 \\ -x &= -14 \\ x &= 14 \end{aligned}$$

Therefore, the value of x is 14.

5. Step 1
Rewrite $\sqrt{5}$ as a power.

$$\begin{aligned} 25^{4x+7} &= \left(\sqrt{5}\right)^{3x+11} \\ 25^{4x+7} &= \left(5^{\frac{1}{2}}\right)^{3x+11} \end{aligned}$$

Step 2
Write the equation so that the powers have the same base.

Replace 25 with 5^2.

$$\begin{aligned} 25^{4x+7} &= \left(5^{\frac{1}{2}}\right)^{3x+11} \\ \left(5^2\right)^{4x+7} &= \left(5^{\frac{1}{2}}\right)^{3x+11} \end{aligned}$$

Step 3
Apply the power law to both sides of the equation.

$$\begin{aligned} \left(5^2\right)^{4x+7} &= \left(5^{\frac{1}{2}}\right)^{3x+11} \\ 5^{8x+14} &= 5^{\frac{3x+11}{2}} \end{aligned}$$

Step 4
Solve for x.

Both sides of the equation have the same base,
so equate the exponents and solve for x.

$$\begin{aligned} 8x+14 &= \frac{3x+11}{2} \\ 2(8x+14) &= 3x+11 \\ 16x+28 &= 3x+11 \\ 13x &= -17 \\ x &= -\frac{17}{13} \end{aligned}$$

Therefore, the value of x is $-\frac{17}{13}$.

6. Step 1
Rewrite $\sqrt[3]{256}$ as a power.

$$\begin{aligned} \sqrt[3]{256}^{\,x+11} &= 4\left[\frac{1}{1024}^{\,3}\right]^{x+2} \\ \left(256^{\frac{1}{3}}\right)^{x+11} &= 4\left[\frac{1}{1024}^{\,3}\right]^{x+2} \end{aligned}$$

Step 2
Write the equation so that the powers have the same base.

$$\begin{aligned} \left(256^{\frac{1}{3}}\right)^{x+11} &= 4\left[\frac{1}{1024}^{\,3}\right]^{x+2} \\ \left(\left(2^8\right)^{\frac{1}{3}}\right)^{x+11} &= \left(2^2\right)\left(\left(2^{-10}\right)^3\right)^{x+2} \end{aligned}$$

Step 3
Apply the power law to both sides of the equation.

$$\begin{aligned} \left(\left(2^8\right)^{\frac{1}{3}}\right)^{x+11} &= \left(2^2\right)\left(\left(2^{-10}\right)^3\right)^{x+2} \\ \left(2^{\frac{8}{3}}\right)^{x+11} &= \left(2^2\right)\left(2^{-30}\right)^{x+2} \\ 2^{\frac{8x+88}{3}} &= \left(2^2\right)\left(2^{-30x-60}\right) \end{aligned}$$

Step 4
Apply the product law to the right side of the equation.

$$\begin{aligned} 2^{\frac{8x+88}{3}} &= \left(2^2\right)\left(2^{-30x-60}\right) \\ 2^{\frac{8x+88}{3}} &= 2^{-30x-58} \end{aligned}$$

Step 5
Solve for *x*.

Both sides of the equation have the same base, so equate the exponents and solve for *x*.

$$\frac{8x+88}{3} = -30x - 58$$
$$8x + 88 = -90x - 174$$
$$98x = -262$$
$$x = -\frac{262}{98}$$
$$x = -\frac{131}{49}$$

Therefore, the value of *x* is $-\frac{131}{49}$.

Lesson 3—Applications of Exponential Functions

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Identify the variables whose values are given in the problem with respect to the function $N_t = N_0 \times R^{\frac{t}{p}}$.

When $t = 0$, there are 32 spiders $(N_0 = 32)$.
Since the doubling period is required, the growth rate is $R = 2$. After 5 weeks, the spider population is 32 768.

Therefore, in 5 weeks, the spider population will double $\frac{5}{p}$ times.

Step 2
Substitute the values $N_0 = 32$, $R = 2$, and $t = 5$ into the function $N_t = N_0 \times R^{\frac{t}{p}}$.

$$N_t = N_0 \times R^{\frac{t}{p}}$$
$$32768 = 32(2)^{\frac{5}{p}}$$

Step 3
Solve for *p*.

Write the equation $32768 = 32(2)^{\frac{5}{p}}$ so the powers have the same base.

$$32768 = 32(2)^{\frac{5}{p}}$$
$$1024 = 2^{\frac{5}{p}}$$
$$2^{10} = 2^{\frac{5}{p}}$$

Equate the exponents, and solve for *p*.

$$10 = \frac{5}{p}$$
$$p = \frac{5}{10}$$
$$p = \frac{1}{2}$$

Therefore, it takes half a week for the species of spiders to double.

2. **Step 1**

Determine the function $N_t = N_0 \times R^{\frac{t}{p}}$ that represents the value of the car after *t* years.
The car was originally worth \$45 000, so $N_0 = 45\ 000$.

The value of the car depreciates at a rate of 8.5%, so the decay rate is $R = (1 - 0.085) = 0.915$. Since the value of the car depreciates every year, $p = 1$.

Substitute the values $N_0 = 45\ 000$, $R = 0.915$, and $p = 1$ into the function $N_t = N_0 \times R^{\frac{t}{p}}$.

$$N_t = N_0 \times R^{\frac{t}{p}}$$
$$N_t = 45\ 000(0.915)^{\frac{t}{1}}$$
$$N_t = 45\ 000(0.915)^{t}$$

The function $N_t = 45\ 000(0.915)^t$ represents the value of the car after *t* years.

Step 2
Determine the value of the car after 18 years.

The elapsed time is 18 years, so solve for N_{18}.

$$N_t = 45\ 000(0.915)^t$$
$$N_{18} = 45\ 000(0.915)^{18}$$
$$N_{18} = 9\ 094.780\ 186$$

After 18 years, the value of the car is \$9 095.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Identify the variables whose values are given in the problem with respect to the function $N_t = N_0 \times R^{\frac{t}{p}}$.

When $t = 0$, the population is 1800 $(N_0 = 1800)$. If the population of mice increases 15% each year, then the growth rate is $R = 1.15$. It takes 1 year for the population to increase 15%, so the period is $p = 1$.

Step 2
Substitute the values $N_0 = 1800$, $R = 1.15$,

and $p = 1$ into the function $N_t = N_0 \times R^{\frac{t}{p}}$.

$$N_t = N_0 \times R^{\frac{t}{p}}$$
$$N_t = 1800 \times (1.15)^{\frac{t}{1}}$$
$$N_t = 1800 \times (1.15)^t$$

The function that models the population after *t* years is $N_t = 1800 \times (1.15)^t$.

2. Since the elapsed time is 4 years, substitute 4 for *t* in the function $N_t = 1800 \times (1.15)^t$, and solve for N_4.

$$N_t = 1800 \times (1.15)^t$$
$$N_4 = 1800 \times (1.15)^4$$
$$N_4 = 3148.21125$$

Therefore, the population of mice after 4 years is approximately 3 148.

3. **Step 1**
In the function $N_t = 1\ 800 \times (1.15)^t$, let $N_t = 1\ 800 \times 2 = 3\ 600$ and write the equation so that the powers have the same base.

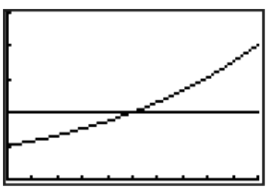
$$N_t = 1\ 800 \times (1.15)^t$$
$$3\ 600 = 1800 \times (1.15)^t$$
$$2 = (1.15)^t$$

The powers cannot be written with the same base.

Step 2
Solve the equation $2 = (1.15)^t$ using a TI-83 or similar graphing calculator.

Enter each side of the equation as $Y_1 = 2$ and $Y_2 = (1.15)\wedge X$. Press GRAPH.

An appropriate setting is *x*: [0, 10, 1] and *y*: [0, 5, 1].

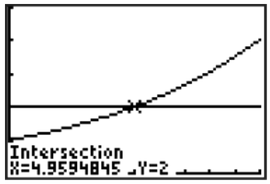


Press 2nd TRACE, and choose 5:intersect.

For "First curve?", position the cursor just left or right of the intersection point, and press ENTER.

For "Second curve?", position the cursor just left or right of the intersection point, and press ENTER.

For "Guess?", press ENTER.



The point of intersection is $(4.959\ 484\ 5, 2)$.

$$4.959\ 484\ 5\text{ years} \times \frac{12\text{ months}}{1\text{ year}} \approx 60\text{ months}$$

Therefore, it will take approximately 60 months for the population of mice to double.

4. **Step 1**
Identify the variables whose values are given in the problem with respect to the function $V(t) = V_0 \times R^{\frac{t}{p}}$.

The initial amount of the investment is \$25 000, so $V_0 = 25\ 000$. If the annual average rate of return is 6.5%, then the growth rate is $R = 1 + 0.065 = 1.065$. It takes 1 year for the investment to increase 6.5%, so the period is $p = 1$.

Step 2
Substitute the values $V_0 = 25000$, $R = 1.065$, and $p = 1$ into the function $V(t) = V_0 \times R^{\frac{t}{p}}$.

$$V(t) = V_0 \times R^{\frac{t}{p}}$$
$$V(t) = 25\ 000(1.065)^{\frac{t}{1}}$$
$$V(t) = 25\ 000(1.065)^t$$

The function that models the stock after *t* years is $V(t) = 25\ 000(1.065)^t$.

5. **Step 1**
Determine the value of *t*.

The variable *t* is the number of years, so the value of *t* is $2002 - 1989 = 13\text{ years}$.

Step 2
Determine the value of the stock on January 1, 2002.

Evaluate $V(13)$.

$$V(t) = 25\ 000(1.065)^t$$
$$V(13) = 25\ 000(1.065)^{13}$$
$$V(13) = 25\ 000(2.267\ 487\ 497)$$
$$V(13) = 56\ 687.187\ 43$$

Therefore, the value of the stock on January 1, 2002 is $56 687.19.

6. When the investment is doubled, it becomes $\$25000 \times 2 = \$50\ 000$.

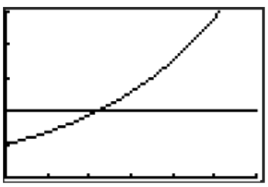
Determine the value of t when $V(t) = 50\ 000$.

$$V(t) = 25000(1.065)^t$$
$$50000 = 25000(1.065)^t$$
$$2 = (1.065)^t$$

Solve the equation $2 = (1.065)^t$ using a TI-83 or similar graphing calculator.

Enter each side of the equation as $Y_1 = 2$ and $Y_2 = 1.065 \wedge X$. Press $\boxed{\text{GRAPH}}$.

An appropriate setting is x: [0, 30, 5] and y: [0, 5, 1].

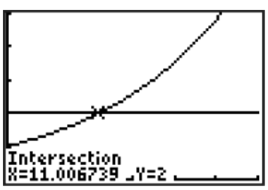


Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and choose 5:intersect.

For "First curve?", position the cursor just left or right of the intersection point, and press $\boxed{\text{ENTER}}$.

For "Second curve?", position the cursor just left or right of the intersection point, and press $\boxed{\text{ENTER}}$.

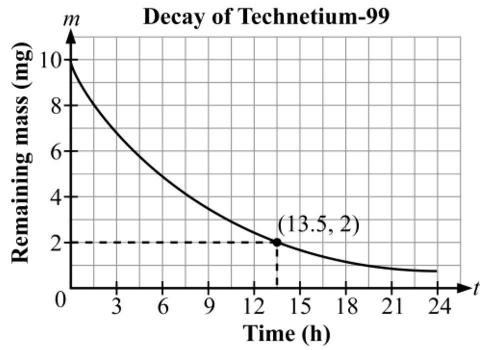
For "Guess?", press $\boxed{\text{ENTER}}$.



The point of intersection is (11.006 739, 1.6). It will take approximately 11 years in order for the stock to be worth $50 000.

Thus, the year in which the stock is worth double the initial investment is 1989 + 11 = 2000.

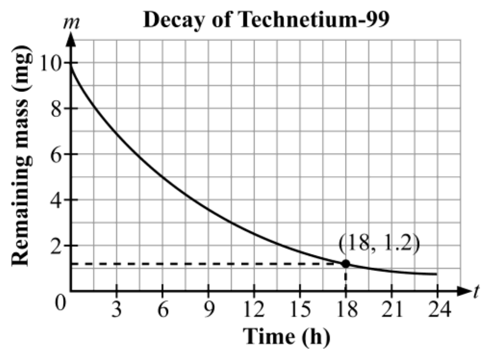
7. Using the given graph of the exponential function, determine the value of t when m = 2.



The graph of the exponential function shows that when $m = 2$, $t = 13.5$.

Therefore, it takes 13.5 h until only 2 mg of the isotope remains in the patient.

8. Using the given graph of the exponential function, determine the value of m when t = 18.



After 18 h, there is approximately 1.2 mg of isotope left in the patient's body.

Therefore, the amount of isotope that has metabolized after 18 h is $10 - 1.2 = 8.8\,\text{mg}$.

9. If the original mass of the isotope is 10 mg at t = 0, then the half-life is the time taken for this mass to decay to half its original amount.

$$\frac{10}{2} = 5\text{ mg}$$

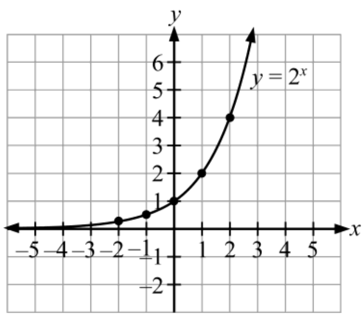
According to the graph, the mass of 5 mg occurs at 6.0 h. Therefore, the half-life of technetium-99 is 6.0 h.

Practice Test

ANSWERS AND SOLUTIONS

1. Step 1
Sketch the graph of $y = 2^x$ using a table of values.

x	*y*
–2	0.25
–1	0.5
0	1
1	2
2	4



Step 2
Describe the transformations applied to $y = 2^x$ to get the graph of $y = 2^{x-5} + 4$.

The function $y = 2^{x-5} + 4$ is of the form $y = 2^{x-h} + k$, where $h = 5$ and $k = 4$.

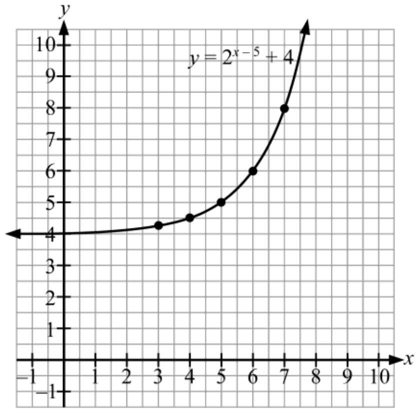
Therefore, the graph $y = 2^{x-5} + 4$ is obtained from $y = 2^x$ by a horizontal translation 5 units right and a vertical translation 4 units up.

Step 3
Apply the transformations to the points on the graph of $y = 2^x$.

Increase the *x*-coordinates by 5 and the *y*-coordinates by 4.

$$(-2+5, 0.25+4) \to (3, 4.25)$$
$$(-1+5, 0.5+4) \to (4, 4.5)$$
$$(0+5, 1+4) \to (5, 5)$$
$$(1+5, 2+4) \to (6, 6)$$
$$(2+5, 4+4) \to (7, 8)$$

Step 4
Plot and join the transformed points.



2. The domain is $x \in R$, and the range is $y > -12$.

The horizontal asymptote is $y = -12$.

Let $x = 0$, and solve for *y*.

$$y = \left(\frac{8}{3}\right)^{x+6} - 12$$
$$y = \left(\frac{8}{3}\right)^{0+6} - 12$$
$$y = \left(\frac{8}{3}\right)^{6} - 12$$
$$y = 347.593\,964\,3$$

Therefore, the *y*-intercept is approximately (0, 347.6).

3. Step 1
Write the equation so that the powers have the same base.

$$6^{x^2-2x} = 216^5$$
$$6^{x^2-2x} = \left(6^3\right)^5$$

Step 2
Apply the power law to the right side of the equation.

$$6^{x^2-2x} = \left(6^3\right)^5$$
$$6^{x^2-2x} = 6^{15}$$

Step 3
Equate the exponents, and solve for *x*.

$$x^2 - 2x = 15$$
$$x^2 - 2x - 15 = 0$$
$$(x-3)(x+5) = 0$$
$$x - 3 = 0 \qquad x + 5 = 0$$
$$x = 3 \qquad x = -5$$

Therefore, the values of *x* are 3 and –5.

4. **Step 1**
Write $\sqrt[5]{256}$ as a power.

$$\left(4^{4x+1}\right)^3 = \left(\sqrt[5]{256}^{5x-15}\right)^4$$
$$\left(4^{4x+1}\right)^3 = \left(\left(256^{\frac{1}{5}}\right)^{5x-15}\right)^4$$

Step 2
Write the equation so that the powers have the same base.

$$\left(4^{4x+1}\right)^3 = \left(\left(256^{\frac{1}{5}}\right)^{5x-15}\right)^4$$
$$\left(4^{4x+1}\right)^3 = \left(\left(\left(4^4\right)^{\frac{1}{5}}\right)^{5x-15}\right)^4$$

Step 3
Apply the power law to both sides of the equation.

$$\left(4^{4x+1}\right)^3 = \left(\left(\left(4^4\right)^{\frac{1}{5}}\right)^{5x-15}\right)^4$$
$$\left(4^{4x+1}\right)^3 = \left(\left(4^{\frac{4}{5}}\right)^{5x-15}\right)^4$$
$$4^{12x+3} = \left(4^{4x-12}\right)^4$$
$$4^{12x+3} = 4^{16x-48}$$

Step 4
Equate the exponents, and solve for x.

$$12x+3 = 16x-48$$
$$-4x = -51$$
$$x = \frac{51}{4}$$

Therefore, the value of x is $\frac{51}{4}$.

5. **Step 1**
Write the equation so that the powers have the same base.

$$7^{x+4} \times \left(\frac{1}{343}\right)^{x+1} = 49^{2x-3}$$
$$7^{x+4} \times \left(\frac{1}{7^3}\right)^{x+1} = \left(7^2\right)^{2x-3}$$
$$7^{x+4} \times \left(7^{-3}\right)^{x+1} = \left(7^2\right)^{2x-3}$$

Step 2
Apply the power rule to both sides of the equation.

$$7^{x+4} \times \left(7^{-3}\right)^{x+1} = \left(7^2\right)^{2x-3}$$
$$7^{x+4} \times 7^{-3x-3} = 7^{4x-6}$$

Step 3
Apply the product law to the left side of the equation.

$$7^{x+4} \times 7^{-3x-3} = 7^{4x-6}$$
$$7^{x+4-3x-3} = 7^{4x-6}$$
$$7^{-2x+1} = 7^{4x-6}$$

Step 4
Equate the exponents, and solve for x.

$$-2x+1 = 4x-6$$
$$-6x = -7$$
$$x = \frac{7}{6}$$

Therefore, the value of x is $\frac{7}{6}$.

6. **Step 1**
Write the equation so that the powers have the same base.

$$\left(\frac{125}{8}\right)^{2x} = \left(\frac{4}{25}\right)^{x+5}$$
$$\left(\left(\frac{5}{2}\right)^3\right)^{2x} = \left(\left(\frac{2}{5}\right)^2\right)^{x+5}$$
$$\left(\left(\frac{5}{2}\right)^3\right)^{2x} = \left(\left(\frac{5}{2}\right)^{-2}\right)^{x+5}$$

Step 2
Apply the power law to both sides of the equation.

$$\left(\left(\frac{5}{2}\right)^3\right)^{2x} = \left(\left(\frac{5}{2}\right)^{-2}\right)^{x+5}$$
$$\left(\frac{5}{2}\right)^{6x} = \left(\frac{5}{2}\right)^{-2x-10}$$

Step 3
Equate the exponents, and solve for x.

$$6x = -2x-10$$
$$8x = -10$$
$$x = -\frac{10}{8}$$
$$x = -\frac{5}{4}$$

Therefore, the value of x is $-\frac{5}{4}$.

7. **Step 1**
Identify the variables whose values are given in the problem with respect to the function $N_t = N_0 \times R^{\frac{t}{p}}$.

The initial number of customers was 30 000, so $N_0 = 30\ 000$. The number of customers increases by 25% every 6 months, so the rate of growth is $R = 1 + 0.25 = 1.25$ and $p = 6$.

Step 2
Substitute the values $N_0 = 30\ 000$, $R = 1.25$, and $p = 6$ into the function $N_t = N_0 \times R^{\frac{t}{p}}$.

$$N_t = N_0 \times R^{\frac{t}{p}}$$
$$N_t = 30\ 000(1.25)^{\frac{t}{6}}$$

The function that represents the number of cellphones after *t* months is $N_t = 30\ 000(1.25)^{\frac{t}{6}}$.

8. **Step 1**
Determine the value of *t*.

There are 3 years between January 1, 1996 and January 1, 1999. Since *t* is in months, the value of *t* is $\left(3\,\text{years} \times \frac{12\,\text{months}}{1\,\text{year}}\right) = 36\,\text{months}$.

Step 2
Evaluate N_{36}.

$$N_t = 30\ 000(1.25)^{\frac{t}{6}}$$
$$N_{36} = 30\ 000(1.25)^{\frac{36}{6}}$$
$$N_{36} = 114\ 440.918$$

Therefore, there were approximately 114 441 customers on January 1, 1999.

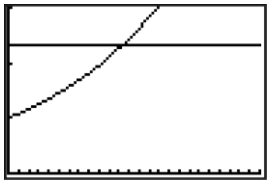
9. **Step 1**
Let $N_t = 70\ 000$, and simplify.

$$N_t = 30\ 000(1.25)^{\frac{t}{6}}$$
$$70\ 000 = 30\ 000(1.25)^{\frac{t}{6}}$$
$$\frac{7}{3} = 1.25^{\frac{t}{6}}$$

Step 2
Solve the equation $\frac{7}{3} = 1.25^{\frac{t}{6}}$ using a TI-83 or similar graphing calculator.

Enter each side of the equation as $\mathrm{Y}_1 = 7/3$ and $\mathrm{Y}_2 = 1.25 \wedge (\mathrm{X}/6)$. Press $\boxed{\text{GRAPH}}$.

An appropriate setting is *x*: [0, 50, 2] and *y*: [0, 3, 1].

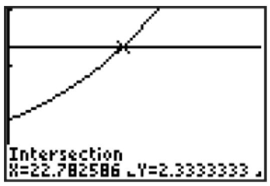


Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and choose 5:intersect.

For “First curve?”, position the cursor just left or right of the intersection point, and press $\boxed{\text{ENTER}}$.

For “Second curve?”, position the cursor just left or right of the intersection point, and press $\boxed{\text{ENTER}}$.

For “Guess?”, press $\boxed{\text{ENTER}}$.



There will be 70 000 cellphone customers about 23 months after January 1, 1996, or at the beginning of December 1997.

10. **Step 1**
Identify the variables whose values are given in the problem with respect to the function $P(t) = P_0 \times R^{\frac{t}{p}}$.

The initial population of salmon is 1 000 000, so $P_0 = 1\ 000\ 000$. The number of salmon decreases by 6.5% every 4 years, so the rate of decay is $R = 1 - 0.065 = 0.935$ and $p = 4$.

Step 2
Substitute the values $P_0 = 1\ 000\ 000$, $R = 1.065$, and $p = 4$ into the function $P(t) = P_0 \times R^{\frac{t}{p}}$.

$$P(t) = P_0 \times R^{\frac{t}{p}}$$
$$P(t) = 1000\ 000(0.935)^{\frac{t}{4}}$$

The function that represents the population of salmon after *t* years is $P(t) = 1\ 000\ 000(0.935)^{\frac{t}{4}}$.

11. Evaluate $P(10)$.

$$P(t) = 1\ 000\ 000(0.935)^{\frac{t}{4}}$$
$$P(10) = 1\ 000000(0.935)^{\frac{10}{4}}$$
$$P(10) = 845\ 335.3434$$

Therefore, the population of salmon after 10 years is about 845 335.

12. One-third of 1 000 000 can be expressed as $\frac{1}{3}(1\ 000\ 000)$.

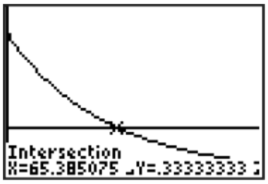
Step 1
Let $P(t) = \frac{1}{3}(1\ 000\ 000)$, and simplify.

$$P(t) = 1\ 000\ 000(0.935)^{\frac{t}{4}}$$
$$\frac{1}{3}(1\ 000\ 000) = 1\ 000\ 000(0.935)^{\frac{t}{4}}$$
$$\frac{1}{3} = 0.935^{\frac{t}{4}}$$

Step 2
Solve the equation $\frac{1}{3} = 0.935^{\frac{t}{4}}$ using a TI-83 or similar graphing calculator.

Enter each side of the equation as $Y_1 = 1/3$ and $Y_2 = 0.935 \wedge (X/4)$. Press GRAPH.

An appropriate setting is *x*: [0, 50, 2] and *y*: [0, 3, 1].

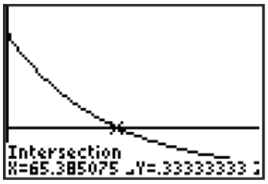


Press 2nd TRACE, and choose 5:intersect.

For “First curve?”, position the cursor just left or right of the intersection point, and press ENTER.

For “Second curve?”, position the cursor just left or right of the intersection point, and press ENTER.

For “Guess?”, press ENTER.



The point of intersection is (65.385 075, 0.333 333 33).

Therefore, it will take approximately 65.4 years for the population of salmon to decrease to one-third its original amount.

13. **Step 1**
Determine the function that represents the population growth of wolves.

Let *W*(*t*) represent the population of wolves after time *t* in months.

The initial population of wolves is 65, so $W_0 = 65$. The population of wolves increased by 25% every 2 months, so $R = 1 + 0.25 = 1.25$ and $p = 2$.

Therefore, the function that represents the population of wolves after *t* months is $W(t) = 65(1.25)^{\frac{t}{2}}$.

Step 2
Determine the function that represents the population growth of rabbits.

Let *R*(*t*) represent the population of rabbits after time *t* in months.

The initial population of rabbits is 50, so $R_0 = 50$. The population of rabbits increased by 30% every month, so $R = 1 + 0.30 = 1.30$ and $p = 1$.

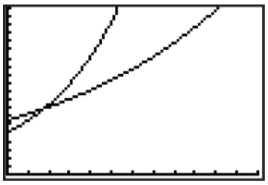
Therefore, the function that represents the population of rabbits after *t* months is $R(t) = 50(1.30)^t$.

Step 3

Using a TI-83, graph the functions $W(t)=65(1.25)^{\frac{t}{2}}$ and $R(t)=50(1.30)^{t}$.

Enter each function as $Y_1=65(1.25)^\wedge(X/2)$ and $Y_2=50(1.30)^\wedge X$. Press $\boxed{\text{GRAPH}}$.

An appropriate setting is *x*: [0, 12, 1] and *y*: [0, 200, 10].



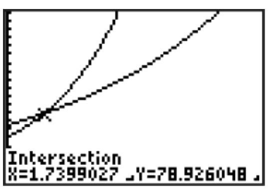
Step 4
Determine the point of intersection.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and choose 5:intersect.

For "First curve?", position the cursor just left or right of the intersection point, and press $\boxed{\text{ENTER}}$.

For "Second curve?", position the cursor just left or right of the intersection point, and press $\boxed{\text{ENTER}}$.

For "Guess?", press $\boxed{\text{ENTER}}$.



The point of intersection is (1.739 902 7, 78.926 048).

Therefore, it took 1.7 months for the populations of wolves and rabbits to become equal.

14. Step 1
Determine the wolf population at the end of September 2000.

Since September is the ninth month of the year, evaluate *W*(9).

$$W(t)=65(1.25)^{\frac{t}{2}}$$
$$W(9)=65(1.25)^{\frac{9}{2}}$$
$$W(9)=65(2.729575168)$$
$$W(9)=177.4223859$$

The wolf population is approximately 177 in September 2000.

Step 2
Determine the rabbit population at the end of September 2000.

Evaluate $R(9)$.

$$R(t)=50(1.30)^{t}$$
$$R(9)=50(1.30)^{9}$$
$$R(9)=50(10.60449937)$$
$$R(9)=530.2249687$$

The rabbit population is approximately 530 at the end of September 2000.

Step 3
Determine the difference between the wolf and rabbit populations at the end of September 2000.

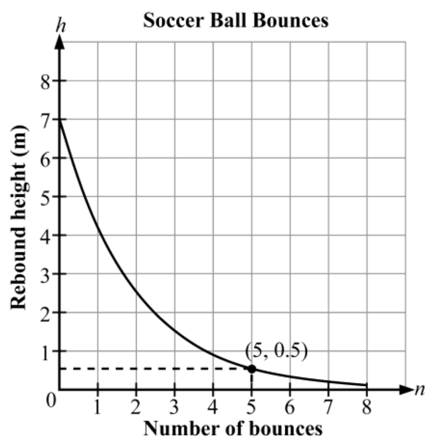
$$R(9)-W(9)$$
$$=530-177$$
$$=353$$

Therefore, the approximate difference between the wolf and rabbit populations at the end of September 2000 is 353.

15. The height of the ledge corresponds to when the number of bounces is 0. On the graph, the height is 7 m when *n* = 0.

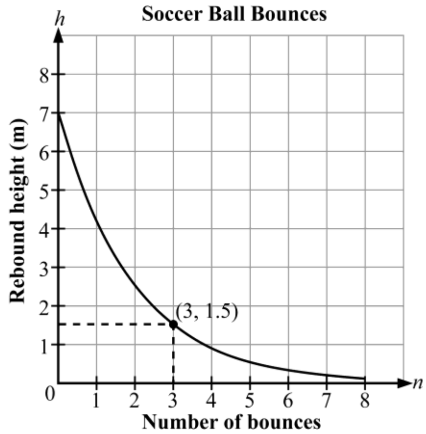
Therefore, the ledge is 7 m tall.

16. Use the given graph to determine the value of *h* when *n* = 5



After 5 bounces, the rebound height of the soccer ball is approximately 0.5 m.

17. Use the given graph to determine the value of n when $h = 1.5$.



The rebound height of the soccer ball is approximately 1.5 m after 3 bounces.

LOGARITHMIC FUNCTIONS

Lesson 1—Defining Logarithms

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. a) Step 1

Divide both sides by 150.

$$30000 = 150(2)^{\frac{m}{40}}$$
$$200 = 2^{\frac{m}{40}}$$

Step 2

Apply the property that if $a = b^c$, then $c = \log_b a$.

The logarithmic form of $30000 = 150(2)^{\frac{m}{40}}$ is $\frac{m}{40} = \log_2 200$.

b) Apply the property that if $c = \log_b a$, then $a = b^c$.

The exponential form of $4a = \log_2(b-10)$ is $b - 10 = 2^{4a}$.

2. a) Step 1

Let $\log_2\left(\frac{1}{64}\right) = x$. Rewrite $\log_2\left(\frac{1}{64}\right) = x$ in exponential form.

The exponential form of $\log_2\left(\frac{1}{64}\right) = x$ is $2^x = \frac{1}{64}$.

Step 2

Solve the exponential equation $2^x = \frac{1}{64}$.

$$2^x = \frac{1}{64}$$
$$2^x = 64^{-1}$$
$$2^x = \left(2^6\right)^{-1}$$
$$2^x = 2^{-6}$$

The value of x is –6. Therefore, $\log_2\left(\frac{1}{64}\right) = -6$.

b) If $\log_3 81 = x$ and $\log_3\left(\frac{1}{27}\right) = y$, determine $x + y$.

Step 1
Determine the value of *x*.

The exponential form of $\log_3 81 = x$ is $3^x = 81$.

Since $3^4 = 81$, the value of *x* is 4.

Step 2
Determine the value of *y*.

The exponential form of $\log_3\left(\frac{1}{27}\right) = y$ is $3^y = \frac{1}{27}$.

Since $3^{-3} = \frac{1}{27}$, the value of *y* is –3.

Step 3
Determine the value of $\log_3 81 + \log_3\left(\frac{1}{27}\right)$.

Find the value of $x + y$.
$x + y = 4 + (-3)$
$x + y = 1$

Therefore, $\log_3 81 + \log_3\left(\frac{1}{27}\right) = 1$.

3. **Step 1**
Determine two integers that the value of $\log_2 580$ lies between.

Let $\log_2 580 = \frac{x}{2}$, which is equivalent to $2^{\frac{x}{2}} = 580$.

Since $2^9 = 512$ and $2^{10} = 1024$, the value of *x* must lie between 9 and 10.

Step 2
Use systematic trial.

Try an exponent of 9.1.
$2^{9.1} = 548.748\ldots$
This value is too low.

Try an exponent of 9.2.
$2^{9.2} = 588.133\ldots$
This value is too high.

Try an exponent of 9.15.
$2^{9.15} = 568.099\ldots$
This value is too low.

Try an exponent of 9.179.
$2^{9.179} = 579.634\ldots$
This value is slightly too low.

Try an exponent of 9.180.
$2^{9.18} = 580.036\ldots$

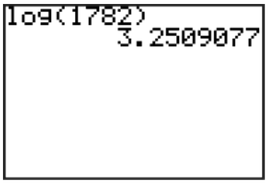
The value of $\log_2 580$ is approximately 9.180.

4. Let $\log\left(\frac{1}{100\,000}\right) = x$. The exponential form of $\log\left(\frac{1}{100\,000}\right) = x$ is $10^x = \frac{1}{100\,000}$.
Since $10^{-5} = \frac{1}{100\,000}$, the value of *x* is –5.

Therefore, $\log\left(\frac{1}{100\,000}\right) = -5$.

5. Use a TI-83 or similar calculator to evaluate $\log 1782$.

Press $\boxed{\text{LOG}}$ $\boxed{1}$ $\boxed{7}$ $\boxed{8}$ $\boxed{2}$ $\boxed{)}$ $\boxed{\text{ENTER}}$.



Rounded to the nearest thousandth, the value of $\log 1\,782$ is 3.251.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. The logarithmic form of $6^2 = 36$ is $2 = \log_6 36$.

2. The logarithmic form of $9^0 = 1$ is $0 = \log_9 1$.

3. The logarithmic form of $5^x = y - 11$ is $x = \log_5(y - 11)$.

4. The logarithmic form of $\left(\frac{1}{3}\right)^{x+1} = 9$ is $x + 1 = \log_{\frac{1}{3}} 9$.

5. The logarithmic form of $10^{\log(0.001)} = x$ is $\log(0.001) = \log_{10} x$ or $\log(0.001) = \log x$.

6. The exponential form of $\log 0.0001 = -4$ is $10^{-4} = 0.0001$.

7. The exponential form of $-5 = \log_{\left(\frac{1}{4}\right)} 1024$ is

$\left(\frac{1}{4}\right)^{-5} = 1024$.

8. Divide both sides by 8.

$$4x = 8\log_2 256$$
$$\frac{4x}{8} = \frac{8\log_2 256}{8}$$
$$\frac{1}{2}x = \log_2 256$$

The exponential form of $4x = 8\log_2 256$ is

$2^{\frac{1}{2}x} = 256$.

9. The exponential form of $\log(x+9) = 108$ is $10^{108} = x+9$.

10. Multiply both sides by 2.

$$\frac{\log_3 \sqrt{27}}{2} = -a$$
$$\log_3 \sqrt{27} = -2a$$

The exponential form of $\frac{\log_3 \sqrt{27}}{2} = -a$ is

$3^{-2a} = \sqrt{27}$.

11. Let $\log_5 25 = x$. The exponential form of $\log_5 25 = x$ is $5^x = 25$.

Since $5^2 = 25$, the value of x is 2.

Therefore, $\log_5 25 = 2$.

12. Let $\log_2 1 = x$. The exponential form of $\log_2 1 = x$ is $2^x = 1$.

Since $2^0 = 1$, the value of x is 0.

Therefore, $\log_2 1 = 0$.

13. Let $4\log_6\left(\frac{1}{6}\right) = x$.

Step 1
Divide both sides by 4.

$$4\log_6\left(\frac{1}{6}\right) = x$$
$$\log_6\left(\frac{1}{6}\right) = \frac{x}{4}$$

Step 2

Rewrite the equation $\log_6\left(\frac{1}{6}\right) = \frac{x}{4}$ in exponential form.

$$\log_6\left(\frac{1}{6}\right) = \frac{x}{4}$$
$$6^{\frac{x}{4}} = \frac{1}{6}$$

Step 3
Solve for x.

Rewrite $\frac{1}{6}$ as 6^{-1}.

$$6^{\frac{x}{4}} = 6^{-1}$$

Equate the exponents, and solve for x.

$$\frac{x}{4} = -1$$
$$x = -4$$

Therefore, $4\log_6\left(\frac{1}{6}\right) = -4$.

14. If $\log 10\ 000 = x$ and $\log 0.000\ 01 = y$, determine $x - y$.

Step 1
Determine the value of x.

The exponential form of $\log 10\ 000 = x$ is $10^x = 10\,000$.

Since $10^4 = 10\ 000$, the value of x is 4.

Step 2
Determine the value of y.

The exponential form of $\log 0.000\ 01$ is $10^y = 0.000\ 01$.

Since $10^{-5} = 0.000\ 01$, the value of y is –5.

Step 3
Determine the value of log 10 000 – log 0.000 01.

Find the value of $x-y$.

$x-y=4-(-5)$
$x-y=4+5$
$x-y=9$

Therefore, $\log 10\ 000-\log 0.000\ 01=9$.

15. If $\log_5 1=x$ and $\log_5\left(\frac{1}{125}\right)=y$, determine $x+y$.

Step 1
Determine the value of *x*.

The exponential form of $\log_5 1=x$ is $5^x=1$.

Since $5^0=1$, the value of *x* is 0.

Step 2
Determine the value of *y*.

The exponential form of $\log_5\left(\frac{1}{125}\right)=y$ is $5^y=\frac{1}{125}$.

Since $5^{-3}=\frac{1}{125}$, the value of *y* is –3.

Step 3

Determine the value of $\log_5 1+\log_5\left(\frac{1}{125}\right)$.

Find the value of $x+y$.

$x+y=0+(-3)$
$x+y=-3$

Therefore, $\log_5 1+\log_5\left(\frac{1}{125}\right)=-3$.

16. Step 1
Determine two integers that the value of $\log_2 33$ lies between.

Let $\log_2 33=x$. The exponential form of $\log_2 33=x$ is $2^x=33$.

Since $2^5=32$ and $2^6=64$, the value of *x* must lie between 5 and 6.

Step 2
Use systematic trial.

Try an exponent of 5.1.
$2^{5.1}=34.296\ldots$
This value is slightly too high.

Try an exponent of 5.04.
$2^{5.04}=32.899\ldots$
This value is slightly too low.

Try an exponent of 5.045.
$2^{5.045}=33.013$
This value is slightly too high.

Try an exponent of 5.044.
$2^{5.044}=32.990\ldots$

The value of $\log_2 33=x$ is approximately 5.044.

17. Step 1
Determine two integers that the value of $\log_4 325$ lies between.

Let $\log_4 325=x$. The exponential form of $\log_4 325=x$ is $4^x=325$.

Since $4^4=256$ and $4^5=1024$, the value of *x* must lie between 4 and 5.

Step 2
Use systematic trial.

Try an exponent of 4.1.
$4^{4.1}=294.066\ldots$
This value is too low.

Try an exponent of 4.2.
$4^{4.2}=337.794\ldots$
This value is too high.

Try an exponent of 4.15.
$4^{4.15}=315.173\ldots$
This value is too low.

Try an exponent of 4.17.
$4^{4.17}=324.033\ldots$
This value is slightly too low.

Try an exponent of 4.171.
$4^{4.171}=324.483\ldots$
This value is slightly too low.

Try an exponent of 4.172.
$4^{4.172}=324.933\ldots$

The value of $\log_4 325$ is approximately 4.172.

18. Step 1
Determine two integers that the value of $\log_3 1457$ lies between.
Let $\log_3 1457 = x$. The exponential form of $\log_3 1457 = x$ is $3^x = 1457$.

Since $3^6 = 729$ and $3^7 = 2\ 187$, the value of x must lie between 6 and 7.

Step 2
Use systematic trial.

Try an exponent of 6.1.
$3^{6.1} = 813.653$
This value is too low.

Try an exponent of 6.6.
$3^{6.6} = 1\ 409.289\ldots$
This value is too low.

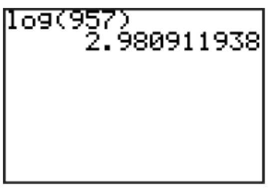
Try an exponent of 6.63.
$3^{6.63} = 1\ 456.511\ldots$
This value is too low.

Try an exponent of 6.631.
$3^{6.631} = 1\ 458.112\ldots$
This value is slightly too high.

Try an exponent of 6.630.
$3^{6.630} = 1\ 456.511\ldots$ $3^{6.630} = 1456.511\ldots$

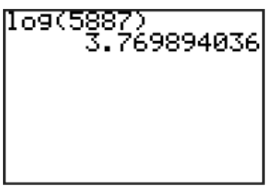
The value of $\log_3 1\ 457$ is approximately 6.630.

19. Use a TI-83 or similar calculator to evaluate log 957.
Press $\boxed{\text{LOG}}$ $\boxed{9}$ $\boxed{5}$ $\boxed{7}$ $\boxed{)}$ $\boxed{\text{ENTER}}$



Rounded to the nearest thousandth, the value of log 957 is 2.981.

20. Use a TI-83 or similar calculator to evaluate log 5 887.
Press $\boxed{\text{LOG}}$ $\boxed{5}$ $\boxed{8}$ $\boxed{8}$ $\boxed{7}$ $\boxed{)}$ $\boxed{\text{ENTER}}$



Rounded to the nearest thousandth, the value of log 5 887 is 3.770.

Lesson 2—Sketching a Logarithmic Function

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Step 1
Replace $f(x)$ with y.

$$f(x) = \log_4(5x)$$
$$y = \log_4(5x)$$

Step 2
Interchange variables x and y.

$$y = \log_4(5x)$$
$$x = \log_4(5y)$$

Step 3
Solve for y.

Write the equation $x = \log_4(5y)$ in exponential form.

$$x = \log_4(5y) \to 4^x = 5y$$

Divide both sides by 5.

$$4^x = 5y$$
$$\frac{4^x}{5} = y$$
$$y = \frac{1}{5}\left(4^x\right)$$

Therefore, the equation of the inverse function is $y = \frac{1}{5}\left(4^x\right)$ or $f^{-1}(x) = \frac{1}{5}\left(4^x\right)$.

2. Step 1
Determine the relationship between $y = 3^x$ and $y = \log_3^x$.

The function $y = \log_3^x$ is the inverse of the function $y = 3^x$. This can be verified as follows:

$$y = 3^x$$
$$x = 3^y$$

The logarithmic form of $x = 3^y$ is $y = \log_3 x$.

Step 2
Determine how to obtain the graph of $y = \log_3^x$ from the graph of $y = 3^x$. Since $y = 3^x$ and $y = \log_3^x$ are inverses of each other, the graph of $y = \log_3 x$ is obtained from the graph of $y = 3^x$ by a reflection in the line $y = x$.

Step 3
Sketch the graph of $y = \log_3{}^x$.

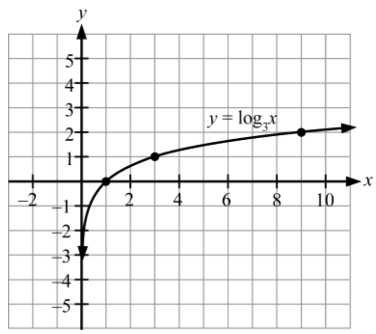
A table of values for the function $y = 3^x$ is as shown.

$y = 3^x$	
x	***y***
0	1
1	3
2	9

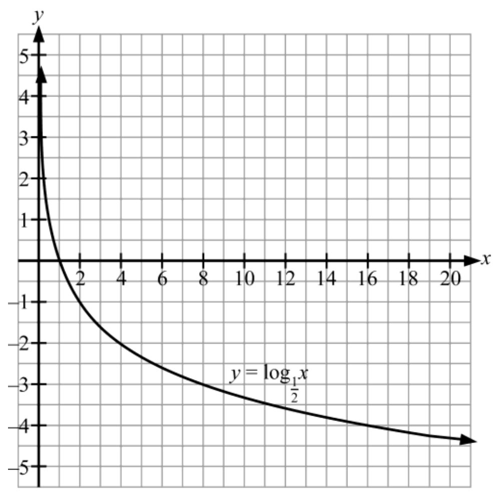
Interchange the *x*- and *y*-coordinates of $y = 3^x$ to obtain points on the graph of $y = \log_3 x$.

$y = \log_3 x$	
x	***y***
1	0
3	1
9	2

Plot and join the points to obtain the graph of $y = \log_3 x$.



3. Step 1
Sketch the graph of $y = \log_{\frac{1}{2}} x$.



Step 2
Describe the transformations applied to $y = \log_{\frac{1}{2}} x$ to get the graph of $y = 2\log_{\frac{1}{2}}(4x)$.

The function $y = 2\log_{\frac{1}{2}}(4x)$ is of the form $y = a\log_{\frac{1}{2}}(bx)$, where $a = 2$ and $b = 4$. Therefore, the graph of $y = 2\log_{\frac{1}{2}}(4x)$ is obtained from $y = \log_{\frac{1}{2}} x$ by a vertical stretch factor of 2 and a horizontal stretch factor of $\frac{1}{4}$.

Step 3
Apply the vertical stretch.

Multiply the *y*-coordinates by 2.

Some points on the graph of $y = \log_{\frac{1}{2}} x$ are (1, 0), (4, –2), and (16, –4).

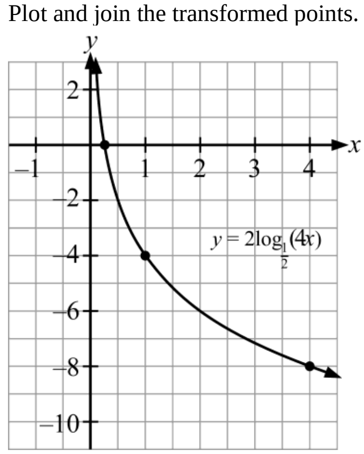
$$(1, 0 \times 2) \to (1, 0)$$
$$(4, -2 \times 2) \to (4, -4)$$
$$(16, -4 \times 2) \to (16, -8)$$

Step 4
Apply the horizontal stretch.

Multiply the *x*-coordinates from step 1 by $\frac{1}{4}$ or 0.25.

$$(1 \times 0.25, 0) \to (0.25, 0)$$
$$(4 \times 0.25, -4) \to (1, -4)$$
$$(16 \times 0.25, -8) \to (4, -8)$$

Step 5
Plot and join the transformed points.



Step 6
Determine the vertical asymptote, and state the domain and range.

There is no horizontal translation, so the vertical asymptote remains as $x=0$.

The domain is $x>0$, and the range is $y \in R$.

ANSWERS AND SOLUTIONS PRACTICE EXERCISES

1. **Step 1**
Replace $f(x)$ with *y*.

$$f(x)=\left(\frac{2}{3}\right)^{x-3}+5$$
$$y=\left(\frac{2}{3}\right)^{x-3}+5$$

Step 2
Interchange *x* and *y*.

$$y=\left(\frac{2}{3}\right)^{x-3}+5$$
$$x=\left(\frac{2}{3}\right)^{y-3}+5$$

Step 3
Write the equation in logarithmic form.
Subtract 5 from both sides.

$$x=\left(\frac{2}{3}\right)^{y-3}+5$$
$$x-5=\left(\frac{2}{3}\right)^{y-3}$$
$$\log_{\frac{2}{3}}(x-5)=y-3$$

Step 4
Solve for *y*.

$$\log_{\frac{2}{3}}(x-5)=y-3$$
$$\log_{\frac{2}{3}}(x-5)+3=y$$

Therefore, the equation of the inverse function is $y=\log_{\frac{2}{3}}(x-5)+3$ and can also be written as $f^{-1}(x)=\log_{\frac{2}{3}}(x-5)+3$.

2. **Step 1**
Replace $f(x)$ with *y*.

$$f(x)=8\log(x)-19$$
$$y=8\log(x)-19$$

Step 2
Interchange *x* and *y*.

$$y=8\log(x)-19$$
$$x=8\log(y)-19$$

Step 3
Write the equation in exponential form.

Isolate the logarithm.

$$x=8\log(y)-19$$
$$x+19=8\log y$$
$$\frac{x+19}{8}=\log y$$
$$\frac{x+19}{8}=\log y \rightarrow y=10^{\frac{x+19}{8}}$$

Therefore, the equation of the inverse function is $y=10^{\frac{x+19}{8}}$. The function can also be written as $f^{-1}(x)=10^{\frac{x+19}{8}}$.

3. **Step 1**
Determine the function $f^{-1}(x)$.

Let $y=10^{x-15}-7$. Interchange *x* and *y*, and write the equation in logarithmic form.

$$y=10^{x-15}-7$$
$$x=10^{y-15}-7$$
$$x+7=10^{y-15}$$
$$\log(x+7)=y-15$$
$$y=\log(x+7)+15$$

Therefore, the inverse function is $y=\log(x+7)+15$, which can also be written as $f^{-1}(x)=\log(x+7)+15$.

Step 2
Determine the value of $f^{-1}(4)$.

$$f^{-1}(x)=\log(x+7)+15$$
$$f^{-1}(4)=\log(4+7)+15$$
$$f^{-1}(4)=\log(11)+15$$
$$f^{-1}(4)=16.041\ldots$$

Therefore, the value of $f^{-1}(4)$ to the nearest hundredth is 16.04.

4. C

Step 1
Determine the range of the graph.

The range of the graph of $y = \log_6 x$ is $y \in R$, not $y > 0$, since the graph extends infinitely upward and downward.

Step 2
Determine the *y*-intercept of the graph.

The *y*-intercept of the graph of $y = \log_6 x$ is not (0, 1). The graph does not have a *y*-intercept. Instead, the *x*-intercept of the graph is (1, 0).

Step 3
Determine the equation of the horizontal asymptote of the graph.

The graph of $y = \log_6 x$ does not have a horizontal asymptote. Instead, the equation of its vertical asymptote is $x = 0$.

Step 4
Determine whether the graph passes through the point $\left(\frac{1}{6}, -1\right)$.

Substitute $\frac{1}{6}$ for *x* and –1 for *y* in the equation $y = \log_6 x$, and convert to exponential form.

$$y = \log_6 x$$
$$-1 = \log_6\left(\frac{1}{6}\right)$$
$$6^{-1} = \frac{1}{6}$$

Since $6^{-1} = \frac{1}{6}$, the graph of $y = \log_6 x$ does pass through $\left(\frac{1}{6}, -1\right)$.

5. Step 1
Apply the horizontal stretch.

Replace *x* in the equation $y = \log_{\frac{3}{8}} x$ with $3x$.

$$y = \log_{\frac{3}{8}} x$$
$$y = \log_{\frac{3}{8}}(3x)$$

Step 2
Apply the horizontal translation.

Replace *x* in the equation $y = \log_{\frac{3}{8}}(3x)$ with *x* – (–5).

$$y = \log_{\frac{3}{8}}(3x)$$
$$y = \log_{\frac{3}{8}}\left(3\left(x - (-5)\right)\right)$$
$$y = \log_{\frac{3}{8}}\left(3(x+5)\right)$$
$$y = \log_{\frac{3}{8}}(3x+15)$$

Step 3
Apply the vertical translation.

Replace *y* in the equation $y = \log_{\frac{3}{8}}(3x+15)$ with $y + 8$.

$$y = \log_{\frac{3}{8}}(3x+15)$$
$$y + 8 = \log_{\frac{3}{8}}(3x+15)$$

Step 4
Solve for *y*.

$$y + 8 = \log_{\frac{3}{8}}(3x+15)$$
$$y = \log_{\frac{3}{8}}(3x+15) - 8$$

Therefore, the equation of the transformed function is $y = \log_{\frac{3}{8}}(3x+15) - 8$.

6. Step 1
Sketch the graph of $y = \log_{\frac{1}{3}} x$.

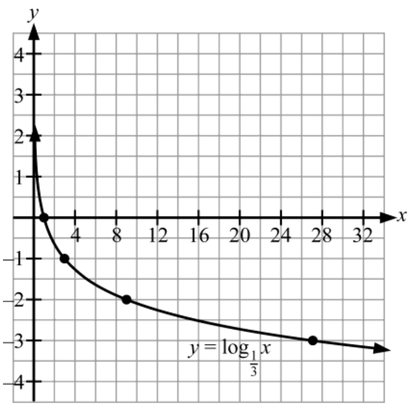
The inverse function of $y = \log_{\frac{1}{3}} x$ is $y = \left(\frac{1}{3}\right)^x$. Build a table of values for the function $y = \left(\frac{1}{3}\right)^x$.

$y = \left(\frac{1}{3}\right)^x$	
x	***y***
0	1
–1	3
–2	9
–3	27

The table of values for $y=\log_{\frac{1}{3}} x$ is obtained by interchanging the *x*- and *y*-values of each ordered pair in the table of values for $y=\log_{\frac{1}{3}} x$.

$y=\log_{\frac{1}{3}} x$	
x	***y***
1	0
3	–1
9	–2
27	–3

Plot and join the points.



Step 2
Determine how to obtain the graph of $y=\frac{3}{2}\log_{\frac{1}{3}}(x)-6$ from the graph of $y=\log_{\frac{1}{3}} x$.

The function $y=\frac{3}{2}\log_{\frac{1}{3}}(x)-6$ is of the form $y=a\log_{\frac{1}{3}} x+k$, where $a=\frac{3}{2}$ and $k=-6$.

Therefore, the graph of $y=\frac{3}{2}\log_{\frac{1}{3}}(x)-6$ is obtained from $y=\log_{\frac{1}{3}} x$ by a vertical stretch factor of $\frac{3}{2}$ and a vertical translation 6 units down.

Step 3
Apply the vertical stretch.
Some points on the graph of $y=\log_{\frac{1}{3}} x$ are (1, 0), (3, –1), (9, –2), and (27, –3).

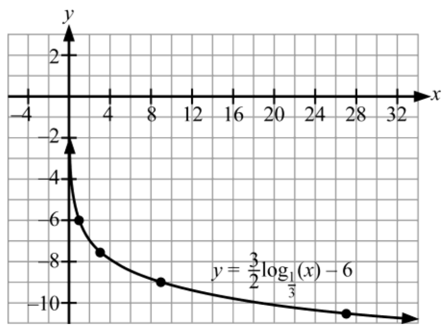
Multiply the *y*-coordinates by $\frac{3}{2}$ or 1.5.

$$(1,0\times1.5)\rightarrow(1,0)$$
$$(3,-1\times1.5)\rightarrow(3,-1.5)$$
$$(9,-2\times1.5)\rightarrow(9,-3)$$
$$(27,-3\times1.5)\rightarrow(27,-4.5)$$

Step 4
Apply the vertical translation.
Decrease the *y*-coordinates in step 3 by 6.

$$(1,0-6)\rightarrow(1,-6)$$
$$(3,-1.5-6)\rightarrow(3,-7.5)$$
$$(9,-3-6)\rightarrow(9,-9)$$
$$(27,-4.5-6)\rightarrow(27,-10.5)$$

Step 5
Plot and join the transformed points.



Step 6
Determine the vertical asymptote, and state the domain and range.

The vertical asymptote is $x=0$. The domain is $x>0$, and the range is $y\in R$.

7. **C**
The function $f(x)=\log_b x$ is increasing when $b>1$ and decreasing when $0<b<1$, so the change from decreasing to increasing will occur when *b* changes from $\frac{1}{2}$ to 3.

8. **Step 1**
Determine the equation of the vertical asymptote.

Since the graph of $y=\log_5 x$ is translated 7 units right, the vertical asymptote becomes $x=0+7=7$.

Step 2
State the domain.

The horizontal translation 7 units right results in a domain of $x > (0+7)$ or $x > 7$.

Step 3
State the range.

The horizontal stretch and translation has no effect on the range.

Therefore, the range remains as $y \in R$.

Step 4
Determine the intercepts.

Since the domain is $x > 7$, there is no *y*-intercept.
The *x*-intercept on the graph of $y = \log_5 x$ is (1, 0).

After the horizontal stretch by a factor of $\frac{5}{2}$,

the *x*-intercept becomes $\left(1 \times \frac{5}{2}, 0\right) = \left(\frac{5}{2}, 0\right)$. Finally, after the horizontal translation 7 units right, the *x*-intercept becomes $\left(\frac{5}{2} + 7, 0\right) = \left(\frac{19}{2}, 0\right)$.

9. **Step 1**
Apply the horizontal stretch.

Replace *x* with $\frac{5}{2}x$ in the equation $y = \log_5 x$.

$$y = \log_5 x$$
$$y = \log_5\left(\frac{5}{2}x\right)$$

Step 2
Apply the horizontal translation.

Replace *x* with $(x-7)$ in the equation $y = \log_5\left(\frac{5}{2}x\right)$.

$$y = \log_5\left(\frac{5}{2}x\right)$$
$$y = \log_5\left(\frac{5}{2}(x-7)\right)$$

Therefore, the equation of the transformed function is $y = \log_5\left(\frac{5}{2}(x-7)\right)$.

Lesson 3—The Laws of Logarithms

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Write the logarithm of the product $(25 \times a)$ as the logarithm of 25 plus the logarithm of *a*.
$\log_5(25a) = \log_5 25 + \log_5 a$

Step 2
Evaluate each logarithm, and add the results.
$\log_5 25 + \log_5 a$
$= 2 + \log_5 a$

No further simplification is possible, so $\log_5 25a = 2 + \log_5 a$.

2. Write the sum of the logarithm of $(x+1)$ and the logarithm of $(x-3)$ as the logarithm of the product $(x+1)(x-3)$.
$\log(x+1) + \log(x-3) = \log[(x+1)(x-3)]$

3. **Step 1**
Apply the product law of logarithms.
$\log_3(27 \times 9 \times m) = \log_3 27 + \log_3 9 + \log_3 m$

Step 2
Evaluate, and simplify.
$\log_3 27 + \log_3 9 + \log_3 m$
$= 3 + 2 + \log_3 m$
$= 5 + \log_3 m$

Since $\log_3 m = x$, the expression equivalent to $\log_3(27 \times 9 \times m)$ is $5 + x$.

4. **Step 1**
Write the logarithm of the quotient $\left(\frac{36}{m}\right)$ as the logarithm of 36 minus the logarithm of *m*.
$\log_6\left(\frac{36}{m}\right) = \log_6 36 - \log_6 m$

Step 2
Evaluate $\log_6 36$.
$\log_6 36 - \log_6 m$
$= 2 - \log_6 m$

Since $\log_6 m = x$, the expression equivalent to $\log_6\left(\frac{36}{m}\right)$ is $2 - x$.

5. **Step 1**
Write the difference between the logarithm of $72x^2$ and the logarithm of $9x$ as the logarithm of the quotient $\left(\frac{72x^2}{9x}\right)$.

$\log_2\left(72x^2\right)-\log_2\left(9x\right)=\log_2\left(\frac{72x^2}{9x}\right)$

Step 2
Simplify.

$\log_2\left(\frac{72x^2}{9x}\right)$
$=\log_2\left(8x\right)$

Therefore, $\log_2\left(72x^2\right)-\log_2\left(9x\right)=\log_2\left(8x\right)$.

6. **Step 1**
Apply the power law of logarithms.

$\log_3\left(243\right)^{5-k}$
$=\left(5-k\right)\log_3 243$

Step 2
Evaluate $\log_3 243$.

$\left(5-k\right)\log_3 243$
$=\left(5-k\right)\log_3\left(3^5\right)$
$=\left(5-k\right)5$

Step 3
Simplify.

$\left(5-k\right)5$
$=25-5k$

Therefore, the expression $25-5k$ is equivalent to $\log_3\left(243^{5-k}\right)$.

7. **Step 1**
Apply the power law of logarithms to the middle term in the expression, and simplify.

$\log 100+3\log 10-\log 10000$
$=\log 100+\log\left(10^3\right)-\log 10000$
$=\log 100+\log 1000-\log 10000$

Step 2
Apply the product law to the first two terms in the expression, and simplify.

$\log 100+\log 1000-\log 10000$
$=\log 100+\log 1000-\log 10000$
$=\log\left(100\times 1000\right)-\log 10000$
$=\log 100000-\log 10000$

Step 3
Apply the quotient law of logarithms, and simplify.

$1\log 100000-\log 10000$
$=1\log\left(\frac{100000}{10000}\right)$
$=1\log 10$

Step 4
Evaluate log 10.

$\log 10=1$

Therefore, $\log 100+3\log 10-\log 10000=1$.

8. **Step 1**
Write $\sqrt[4]{x}$ as a power.

$\log_8\left(\sqrt[4]{x}\right)+3\log_8 y-\log_8\left(y^4\right)-\log_8 x$
$=\log_8\left(x^{\frac{1}{4}}\right)+3\log_8 y-\log_8\left(y^4\right)-\log_8 x$

Step 2
Apply the power law of logarithms to the second term of the expression.

$\log_8\left(x^{\frac{1}{4}}\right)+3\log_8 y-\log_8\left(y^4\right)-\log_8 x$
$=\log_8\left(x^{\frac{1}{4}}\right)+\log_8\left(y^3\right)-\log_8\left(y^4\right)-\log_8 x$

Step 3
Factor out –1 from the last two terms.

$\log_8\left(x^{\frac{1}{4}}\right)+\log_8\left(y^3\right)-\log_8\left(y^4\right)-\log_8 x$
$=\log_8\left(x^{\frac{1}{4}}\right)+\log_8\left(y^3\right)-\left[\log_8\left(y^4\right)+\log_8 x\right]$

Step 4
Apply the product law to the first two terms and the last two terms.

$\log_8\left(x^{\frac{1}{4}}\right)+\log_8\left(y^3\right)-\left[\log_8\left(y^4\right)+\log_8 x\right]$
$=\log_8\left(x^{\frac{1}{4}}y^3\right)-\log_8\left(y^4x\right)$

Step 5
Apply the quotient law of logarithms, and simplify.

$\log_8\left(x^{\frac{1}{4}}y^3\right)-\log_8\left(y^4x\right)$
$=\log_8\left(\frac{x^{\frac{1}{4}}y^3}{y^4x}\right)$
$=\log_8\left(x^{-\frac{3}{4}}y^{-1}\right)$
$=\log_8\left(\frac{1}{x^{\frac{3}{4}}y}\right)$

Therefore,

$\log_8\left(\sqrt[4]{x}\right)+3\log_8 y-\log_8\left(y^4\right)-\log_8 x=\log_8\left(\frac{1}{x^{\frac{3}{4}}y}\right)$

PRACTICE EXERCISES
ANSWERS AND SOLUTIONS

1. **Step 1**
Apply the product law of logarithms.
$\log_3(9\times 27)$
$=\log_3 9+\log_3 27$

Step 2
Evaluate each logarithm, and add the results.
$\log_3 9+\log_3 27$
$=2+3$
$=5$

2. **Step 1**
Apply the quotient law of logarithms.
$\log_2 1536-\log_2 3$
$=\log_2\left(\frac{1536}{3}\right)$
$=\log_2 512$

Step 2
Evaluate $\log_2 512$.
$\log_2 512$
$=\log_2\left(2^9\right)$
$=9$

Therefore, $\log_2 1\,536 - \log_2 3 = 9$.

3. **Step 1**
Apply the quotient law of logarithms.
$\log_9\left(\frac{\sqrt[3]{729}}{81}\right)$
$=\log_9\left(\sqrt[3]{729}\right)-\log_9 81$

Step 2
Evaluate each logarithm, and subtract the results.
$\log_9\left(\sqrt[3]{729}\right)-\log_9 81$
$=\log_9 9-\log_9 81$
$=1-2$
$=-1$

Therefore, $\log_9\left(\frac{\sqrt[3]{729}}{81}\right)=-1$.

4. **Step 1**
Apply the product law of logarithms.
$\log_5 2+\log_5\left(\frac{25}{2}\right)$
$=\log_5\left(2\times\frac{25}{2}\right)$
$=\log_5 25$

Step 2
Evaluate $\log_5 25$.
$\log_5 25=2$

Therefore, $\log_5 2+\log_5\left(\frac{25}{2}\right)=2$.

5. **Step 1**
Apply the quotient law of logarithms.
$\log_3 6-\log_3 54$
$=\log_3\left(\frac{6}{54}\right)$
$=\log_3\left(\frac{1}{9}\right)$

Step 2
Evaluate $\log_2 512$.
$\log_3\left(\frac{1}{9}\right)$
$=\log_3\left(\frac{1}{3^2}\right)$
$=\log_3\left(3^{-2}\right)$
$=-2$

Therefore, $\log_3 6-\log_3 54=-2$.

6. **Step 1**
Apply the product law of logarithms.
$\log_7 14+\log_7\left(\frac{1}{2}\right)+\log_7\left(\frac{1}{7}\right)$
$=\log_7\left(14\times\frac{1}{2}\times\frac{1}{7}\right)$
$=\log_7\left(7\times\frac{1}{7}\right)$
$=\log_7 7$

Step 2
Evaluate $\log_7 7$.
$\log_7 7=1$

Therefore, $\log_7 14+\log_7\left(\frac{1}{2}\right)+\log_7\left(\frac{1}{7}\right)=1$.

7. **Step 1**
Apply the power law of logarithms.
$\log_4\left(64^{10}\right)$
$=10\log_4 64$

Step 2
Evaluate $10\log_4 64$.

$$\begin{aligned}&10\log_4 64\\&=10(3)\\&=30\end{aligned}$$

Therefore, $\log_4\left(64^{10}\right)$.

8. Step 1
Apply the power law of logarithms.

$$\begin{aligned}&\log_{\frac{1}{2}}\left(64^{17}\right)\\&=17\log_{\frac{1}{2}} 64\end{aligned}$$

Step 2
Evaluate $17\log_{\frac{1}{2}} 64$.

$$\begin{aligned}&17\log_{\frac{1}{2}} 64\\&=17\log_{\frac{1}{2}}\left(2^6\right)\\&=17\log_{\frac{1}{2}}\left[\left(\frac{1}{2}\right)^{-6}\right]\\&=17(-6)\\&=-102\end{aligned}$$

Therefore, $\log_{\frac{1}{2}}\left(64^{17}\right)=-102$.

9. Step 1
Apply the product law to the first two terms of the expression.

$$\begin{aligned}&\log 5+\log 4-\log 200\\&=\log(5\times 4)-\log 200\\&=\log 20-\log 200\end{aligned}$$

Step 2
Apply the quotient law.

$$\begin{aligned}&\log 20-\log 200\\&=\log\left(\frac{20}{200}\right)\end{aligned}$$

Step 3
Evaluate $\log\left(\frac{20}{200}\right)$.

$$\begin{aligned}&\log\left(\frac{20}{200}\right)\\&=\log\left(\frac{1}{10}\right)\\&=\log\left(10^{-1}\right)\\&=-1\end{aligned}$$

Therefore, $\log 5+\log 4-\log 200=-1$.

10. Step 1
Apply the quotient law to the first two terms of the expression.

$$\begin{aligned}&\log_3 162-\log_3 4+\log_3 6\\&=\log_3\left(\frac{162}{4}\right)+\log_3 6\\&=\log_3\left(\frac{81}{2}\right)+\log_3 6\end{aligned}$$

Step 2
Apply the product law.

$$\begin{aligned}&\log_3\left(\frac{81}{2}\right)+\log_3 6\\&=\log_3\left(\frac{81}{2}\times 6\right)\\&=\log_3 243\end{aligned}$$

Step 3
Evaluate $\log_3 243$.

$$\begin{aligned}&\log_3 243\\&=\log_3\left(3^5\right)\\&=5\end{aligned}$$

Therefore, $\log_3 162-\log_3 4+\log_3 6=5$.

11. Step 1
Apply the power rule of logarithms to all terms of the expression.

$$\begin{aligned}&2\log_{\sqrt{3}} 6-2\log_{\sqrt{3}} 2-5\log_{\sqrt{3}} 1\\&=\log_{\sqrt{3}}\left(6^2\right)-\log_{\sqrt{3}}\left(2^2\right)-\log_{\sqrt{3}}\left(1^5\right)\\&=\log_{\sqrt{3}} 36-\log_{\sqrt{3}} 4-\log_{\sqrt{3}} 1\\&=\log_{\sqrt{3}} 36-\log_{\sqrt{3}} 4-0\\&=\log_{\sqrt{3}} 36-\log_{\sqrt{3}} 4\end{aligned}$$

Step 2
Apply the quotient law of logarithms.

$$\begin{aligned}&\log_{\sqrt{3}} 36-\log_{\sqrt{3}} 4\\&=\log_{\sqrt{3}}\left(\frac{36}{4}\right)\\&=\log_{\sqrt{3}} 9\end{aligned}$$

Step 3
Evaluate $\log_{\sqrt{3}} 9$.

$$\begin{aligned}&\log_{\sqrt{3}} 9\\&=\log_{\sqrt{3}}\left(\sqrt{3}^4\right)\\&=4\end{aligned}$$

Therefore, $2\log_{\sqrt{3}} 6-2\log_{\sqrt{3}} 2-5\log_{\sqrt{3}} 1=4$.

12. Step 1
Apply the power law of logarithms to the first two terms of the expression.

$$3\log_{\frac{1}{6}} 12 + 2\log_{\frac{1}{6}}\left(\frac{1}{2}\right) - \log_{\frac{1}{6}} 2$$
$$= \log_{\frac{1}{6}}\left(12^3\right) + \log_{\frac{1}{6}}\left[\left(\frac{1}{2}\right)^2\right] - \log_{\frac{1}{6}} 2$$
$$= \log_{\frac{1}{6}} 1728 + \log_{\frac{1}{6}}\left(\frac{1}{4}\right) - \log_{\frac{1}{6}} 2$$

Step 2
Apply the product law of logarithms to the first two terms of the expression.

$$\log_{\frac{1}{6}} 1728 + \log_{\frac{1}{6}}\left(\frac{1}{4}\right) - \log_{\frac{1}{6}} 2$$
$$= \log_{\frac{1}{6}}\left(1728 \times \frac{1}{4}\right) - \log_{\frac{1}{6}} 2$$
$$= \log_{\frac{1}{6}} 432 - \log_{\frac{1}{6}} 2$$

Step 3
Apply the quotient law of logarithms.

$$\log_{\frac{1}{6}} 432 - \log_{\frac{1}{6}} 2$$
$$= \log_{\frac{1}{6}}\left(\frac{432}{2}\right)$$
$$= \log_{\frac{1}{6}} 216$$

Step 4
Evaluate $\log_{\frac{1}{6}} 216$.

$$\log_{\frac{1}{6}} 216$$
$$= \log_{\frac{1}{6}}\left(6^3\right)$$
$$= \log_{\frac{1}{6}}\left[\left(\frac{1}{6}\right)^{-3}\right]$$
$$= -3$$

Therefore, $3\log_{\frac{1}{6}} 12 + 2\log_{\frac{1}{6}}\left(\frac{1}{2}\right) - \log_{\frac{1}{6}} 2 = -3$.

13. Step 1
Apply the power law of logarithms to the first term.

$$3\log_2 x + 2\log_2(5x)$$
$$= \log_2\left(x^3\right) + \log_2\left((5x)^2\right)$$

Step 2
Apply the product law of logarithms.

$$\log_2\left(x^3\right) + \log_2\left[(5x)^2\right]$$
$$= \log_2\left[x^3(5x)^2\right]$$
$$= \log_2\left(25x^5\right)$$

Therefore, $3\log_2 x + 2\log_2(5x) = \log_2\left(25x^5\right)$.

14. Step 1
Apply the power law of logarithms to the first term.

$$2\log_b(3m) - \log_b(18m)$$
$$= \log_b\left[(3m)^2\right] - \log_b(18m)$$
$$= \log_b\left(9m^2\right) - \log_b(18m)$$

Step 2
Apply the quotient law of logarithms.

$$\log_b\left(9m^2\right) - \log_b(18m)$$
$$= \log_b\left(\frac{9m^2}{18m}\right)$$
$$= \log_b\left(\frac{1}{2}m\right)$$

Therefore, $2\log_b(3m) - \log_b(18m) = \log_b\left(\frac{1}{2}m\right)$.

15. Step 1
Apply the power law of logarithms to the last two terms of the expression.

$$\log_x a + 3\log_x b - 2\log_x a$$
$$= \log_x a + \log_x\left(b^3\right) - \log_x\left(a^2\right)$$

Step 2
Apply the product law of logarithms to the first two terms of the expression.

$$\log_x a + \log_x\left(b^3\right) - \log_x\left(a^2\right)$$
$$= \log_x\left(ab^3\right) - \log_x\left(a^2\right)$$

Step 3
Apply the quotient law of logarithms.

$$\log_x\left(ab^3\right) - \log_x\left(a^2\right)$$
$$= \log_x\left(\frac{ab^3}{a^2}\right)$$
$$= \log_x\left(\frac{b^3}{a}\right)$$

Therefore, $\log_x a + 3\log_x b - 2\log_x a = \log_x\left(\frac{b^3}{a}\right)$.

16. Step 1
Apply the power law of logarithms to the first and last terms of the expression.

$$\frac{1}{4}\log_3 x+\log_3 y-\log_3(xy)+4\log_3 y$$
$$=\log_3\left(x^{\frac{1}{4}}\right)+\log_3 y-\log_3(xy)+\log_3\left(y^4\right)$$

Step 2
Factor out –1 from the last two terms.

$$\log_3\left(x^{\frac{1}{4}}\right)+\log_3 y-\log_3(xy)+\log_3\left(y^4\right)$$
$$=\log_3\left(x^{\frac{1}{4}}\right)+\log_3 y-\left[\log_3(xy)-\log_3\left(y^4\right)\right]$$

Step 3
Apply the product law of logarithms to the first two terms of the expression.

$$\log_3\left(x^{\frac{1}{4}}\right)+\log_3 y-\left[\log_3(xy)-\log_3\left(y^4\right)\right]$$
$$=\log_3\left(x^{\frac{1}{4}}y\right)-\left[\log_3(xy)-\log_3\left(y^4\right)\right]$$

Step 4
Apply the quotient law of logarithms to the last two terms of the expression.

$$\log_3\left(x^{\frac{1}{4}}y\right)-\left[\log_3(xy)-\log_3\left(y^4\right)\right]$$
$$=\log_3\left(x^{\frac{1}{4}}y\right)-\log_3\left(\frac{xy}{y^4}\right)$$
$$=\log_3\left(x^{\frac{1}{4}}y\right)-\log_3\left(\frac{x}{y^3}\right)$$

Step 5
Apply the quotient law of logarithms.

$$\log_3\left(x^{\frac{1}{4}}y\right)-\log_3\left(\frac{x}{y^3}\right)$$
$$=\log_3\left(\frac{x^{\frac{1}{4}}y^4}{x}\right)$$
$$=\log_3\left(x^{-\frac{3}{4}}y^4\right)$$

Therefore,

$$\frac{1}{4}\log_3 x+\log_3 y-\log_3(xy)+4\log_3 y$$
$$=\log_3\left(x^{-\frac{3}{4}}y^4\right)$$

17. Step 1

It is given that $\log_b 9=m$, so rewrite $\log_b\left(\frac{1}{81}\right)$ so that it includes $\log_b 9$.

$$\log_b\left(\frac{1}{81}\right)$$
$$=\log_b\left(\frac{1}{9^2}\right)$$
$$=\log_b\left(9^{-2}\right)$$

Step 2
Apply the power law of logarithms.

$$\log_b\left(9^{-2}\right)$$
$$=-2\log_b 9$$

Step 3
Substitute m for $\log_b 9$.

$$-2\log_b 9=-2m$$

Therefore, the expression $\log_b\left(\frac{1}{81}\right)$ in terms of m is $-2m$.

Lesson 4—Solving a Logarithmic Equation

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Step 1
Express the logarithmic equation in exponential form.

The equation in exponential form is $3^3=\frac{1}{2}x+7$ or $27=\frac{1}{2}x+7$.

Step 2
Solve the resulting equation.

$$27=\frac{1}{2}x+7$$
$$20=\frac{1}{2}x$$
$$x=40$$

Step 3
Verify the solution by substituting it into the original equation.

The solution is verified if the LHS = RHS.

$x=40$	
LHS	RHS
3	$\log_3\left(\frac{1}{2}(40)+7\right)$ $=\log_3(20+7)$ $=\log_3 27$ $=3$

Since the LHS = RHS, the solution $x=40$ is verified.

2. **Step 1**
Apply the product law of logarithms.

$$\log_3(x+20)+\log_3(88-x)=7$$
$$\log_3[(x+20)(88-x)]=7$$
$$\log_3(88x-x^2+1760-20x)=7$$
$$\log_3(-x^2+68+1760)=7$$

Step 2
Express the logarithmic equation in exponential form.

The equation in exponential form is $3^7=-x^2+68+1760$ or $-x^2+68+1760=2187$.

Step 3
Solve the exponential equation.

$$-x^2+68x+1760=2187$$
$$-x^2+68x-427=0$$
$$-(x^2-68x+427)=0$$
$$-(x-7)(x-61)=0$$
$$x-7=0 \qquad x-61=0$$
$$x=7 \qquad x=61$$

Step 4
Verify the solutions by substituting them into the original equation.

The solution is verified if the LHS = RHS.

$x=7$	
LHS	RHS
$\log_3(7+20)+\log_3(88-7)$ $=\log_3 27+\log_3 81$ $=3+4$ $=7$	7

Since the LHS $=$ RHS, the solution $x=7$ is verified.

$x=61$	
LHS	RHS
$\log_3(61+20)+\log_3(88-61)$ $=\log_3 81+\log_3 27$ $=4+3$ $=7$	7

Since the LHS = RHS, the solution $x=61$ is verified.

In the equation $\log_3(x+20)+\log_3(88-x)=7$, the solutions are $x=7$ and $x=61$.

3. **Step 1**
Write the logarithmic expressions on the same side of the equation.

$$\log_6(x+6)=-2+\log_6(45x-54)$$
$$\log_6(x+6)-\log_6(45x-54)=-2$$

Step 2
Apply the quotient law of logarithms.

$$\log_6(x+6)-\log_6(45x-54)=-2$$
$$\log_6\left(\frac{x+6}{45x-54}\right)=-2$$

Step 3
Express the logarithmic equation in exponential form.

The equation in exponential form is $6^{-2}=\frac{x+6}{45x-54}$ or $\frac{x+6}{45x-54}=\frac{1}{36}$.

Step 4
Solve the exponential equation.

$$\frac{x+6}{45x-54}=\frac{1}{36}$$
$$\frac{45x-54}{x+6}=36$$
$$\frac{5x-6}{x+6}=4$$
$$5x-6=4x+24$$
$$x=30$$

Step 5
Verify the solution by substituting it into the original equation.

The solution is verified if the LHS = RHS.

$x=30$	
LHS	RHS
$\log_6(30+6)$ $=\log_6 36$ $=2$	$-2+\log_6[45(30)-54]$ $=-2+\log_6(1350-54)$ $=-2+\log_6 1296$ $=-2+4$ $=2$

Since the LHS = RHS, the solution $x=30$ is verified.

In the equation $\log_6(x+6)=-2+\log_6(45x-54)$, the solution is $x=30$.

4. **Step 1**
Apply the change of base formula using base 10.

$$\log_4 x+\log_8 x=5$$
$$\frac{\log x}{\log 4}+\frac{\log x}{\log 8}=5$$

Step 2
Remove the common factor $\log x$.

$$\frac{\log x}{\log 4}+\frac{\log x}{\log 8}=5$$
$$\log x\left[\frac{1}{\log 4}+\frac{1}{\log 8}\right]=5$$

Step 3
Isolate $\log x$.

Divide both sides of the equation by $\left[\frac{1}{\log 4}+\frac{1}{\log 8}\right]$.

$$\log x\left[\frac{1}{\log 4}+\frac{1}{\log 8}\right]=5$$
$$\log x=\frac{5}{\left[\frac{1}{\log 4}+\frac{1}{\log 8}\right]}$$

Step 4

Solve for x. The exponential form is $10^{\frac{5}{\left[\frac{1}{\log 4}+\frac{1}{\log 8}\right]}}=x$.

$$10^{\frac{5}{\left[\frac{1}{\log 4}+\frac{1}{\log 8}\right]}}=x$$
$$10^{1.806179\ldots}=x$$
$$64=x$$

Step 5
Verify the solution by substituting it into the original equation.

$x=64$	
LHS	RHS
$\log_4 64+\log_8 64$ $=\frac{\log 64}{\log 4}+\frac{\log 64}{\log 8}$ $=3+2$ $=5$	5

Since the LHS = RHS, the solution $x = 64$ is verified.

In the equation $\log_4 x+\log_8 x=5$, the solution is $x = 64$.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Express the logarithmic equation in exponential form.

The equation in exponential form is $5^3=\frac{20x+50}{x-41}$ or $125=\frac{20x+50}{x-41}$.

Step 2
Solve the resulting equation.

$$125=\frac{20x+50}{x-41}$$
$$125(x-41)=20x+50$$
$$125(x-41)=5(4x+10)$$
$$25(x-41)=4x+10$$
$$25x-1025=4x+10$$
$$21x=1035$$
$$x=\frac{345}{7}$$

Step 3
Verify the solution by substituting it into the original equation.

The solution is verified if the LHS = RHS.

$x=\frac{345}{7}$	
LHS	RHS
3	$\log_5\left[\frac{20\left(\frac{345}{7}\right)+50}{\frac{345}{7}-41}\right]$ $=\log_5 125$ $=3$

Since the LHS = RHS, the solution $x = \frac{345}{7}$ is verified.

2. **Step 1**
Express the logarithmic equation in exponential form.

The equation in exponential form is $4^{-1} = \log_{\frac{1}{16}} x$ or $\frac{1}{4} = \log_{\frac{1}{16}} x$. The exponential form $\frac{1}{4} = \log_{\frac{1}{16}} x$ is a logarithmic equation.

Step 2
Express the logarithmic equation $\frac{1}{4} = \log_{\frac{1}{16}} x$ in exponential form. The equation in exponential form is $x = \left(\frac{1}{16}\right)^{\frac{1}{4}}$.

Step 3
Solve $x = \left(\frac{1}{16}\right)^{\frac{1}{4}}$.

$$x = \left(\frac{1}{16}\right)^{\frac{1}{4}}$$
$$x = \sqrt[4]{\left(\frac{1}{16}\right)}$$
$$x = \frac{1}{2}$$

Step 4
Verify the solution by substituting it into the original equation. The solution is verified if the LHS = RHS.

$x = \frac{1}{2}$	
LHS	RHS
$\log_4\left[\log_{\frac{1}{16}} x\right]$ $= \log_4\left[\log_{\frac{1}{16}}\left(\frac{1}{2}\right)\right]$ $= \log_4\left(\frac{1}{4}\right)$ $= -1$	-1

Since the LHS = RHS, the solution $x = \frac{1}{2}$ is verified.

3. **Step 1**
Write the logarithmic expressions on the same side of the equation.

$$\log(3x+1) = 1 - \log x$$
$$\log(3x+1) + \log x = 1$$

Step 2
Apply the product law of logarithms.

$$\log(3x+1) + \log x = 1$$
$$\log\left[x(3x+1)\right] = 1$$
$$\log\left(3x^2 + x\right) = 1$$

Step 3
Express the logarithmic equation in exponential form.

The equation in exponential form is $10^1 = 3x^2 + x$ or $10 = 3x^2 + x$.

Step 4
Solve the resulting equation.

$$10 = 3x^2 + x$$
$$0 = 3x^2 + x - 10$$
$$0 = (3x-5)(x+2)$$
$$3x - 5 = 0 \qquad x + 2 = 0$$
$$3x = 5 \qquad x = -2$$
$$x = \frac{5}{3}$$

Step 5
Verify the solutions by substituting them into the original equation.

The solution is verified if the LHS = RHS.

$x = \frac{5}{3}$	
LHS	RHS
$\log\left[3\left(\frac{5}{3}\right)+1\right]$ $= \log(5+1)$ $= \log 6$ $= 0.778151\ldots$	$1 - \log\left(\frac{5}{3}\right)$ $= 0.778151\ldots$

Since the LHS = RHS, the solution $x = \frac{5}{3}$ is verified.

$x = -2$	
LHS	RHS
$\log\left[3(-2)+1\right]$ $= \log(-6+1)$ $= \log(-5)$	$1 - \log(-2)$

The expressions $\log_2(-5)$ and $\log_2(-2)$ cannot be evaluated because the logarithm of a negative number is undefined. Therefore, $x=-2$ is an extraneous root and not part of the solution.

Thus, the solution is $x=\frac{5}{3}$.

4. **Step 1**
Write the logarithmic expressions on the same side of the equation.

$$2\log_2(x+1)=\log x+2$$
$$2\log_2(x+1)-\log_2 x=2$$

Step 2
Apply the power law of logarithms.

$$2\log_2(x+1)-\log_2 x=2$$
$$\log_2\left[(x+1)^2\right]-\log_2 x=2$$

Step 3
Apply the quotient law of logarithms.

$$\log_2\left[(x+1)^2\right]-\log_2 x=2$$
$$\log_2\left[\frac{(x+1)^2}{x}\right]=2$$

Step 4
Express the logarithmic equation in exponential form.

The equation in exponential form is $2^2=\frac{(x+1)^2}{x}$ or $4=\frac{(x+1)^2}{x}$.

Step 5
Solve the resulting equation.

$$4=\frac{(x+1)^2}{x}$$
$$4x=(x+1)^2$$
$$4x=x^2+2x+1$$
$$0=x^2-2x+1$$
$$0=(x-1)^2$$
$$0=x-1$$
$$1=x$$

Step 6
Verify the solution by substituting it into the original equation.

The solution is verified if the LHS = RHS.

$x=1$	
LHS	RHS
$2\log_2(1+1)$ $=2\log_2 2$ $=2(1)$ $=2$	$\log_2 1+2$ $=\log_2 1+2$ $=0+2$ $=2$

Since the LHS = RHS, the solution $x=1$ is verified.

5. **Step 1**
Write the logarithmic expressions on the same side of the equation.

$$4-\log_{\sqrt{2}}(x+3)=\log_{\sqrt{2}}(2-x)$$
$$4=\log_{\sqrt{2}}(2-x)+\log_{\sqrt{2}}(x+3)$$

Step 2
Apply the product law of logarithms.

$$4=\log_{\sqrt{2}}(2-x)+\log_{\sqrt{2}}(x+3)$$
$$4=\log_{\sqrt{2}}\left[(2-x)(x+3)\right]$$
$$4=\log_{\sqrt{2}}\left(-x^2-x+6\right)$$

Step 3
Express the logarithmic equation in exponential form.

The equation in exponential form is $\left(\sqrt{2}\right)^4=-x^2-x+6$ or $4=-x^2-x+6$.

Step 4
Solve the resulting equation.

$$4=-x^2-x+6$$
$$0=-x^2-x+2$$
$$0=-\left(x^2+x-2\right)$$
$$0=x^2+x-2$$
$$0=(x-1)(x+2)$$
$$x-1=0 \qquad x+2=0$$
$$x=1 \qquad x=-2$$

Step 5
Verify the solutions by substituting them into the original equation.

The solution is verified if the LHS = RHS.

$x=1$	
LHS	RHS
$4-\log_{\sqrt{2}}(1+3)$ $=4-4$ $=0$	$\log_{\sqrt{2}}(2-1)$ $=\log_{\sqrt{2}}1$ $=0$

Since the LHS = RHS, the solution $x=1$ is verified.

$x=-2$	
LHS	RHS
$4-\log_{\sqrt{2}}(-2+3)$ $=4-0$ $=4$	$\log_{\sqrt{2}}[2-(-2)]$ $=\log_{\sqrt{2}}4$ $=4$

Since the LHS = RHS, the solution $x=-2$ is verified.

6. Step 1
Write the logarithmic expressions on the same side of the equation.

$$\log_8(x+3)+1=\log_8(9x-37)$$
$$\log_8(x+3)-\log_8(9x-37)=-1$$

Step 2
Apply the quotient law of logarithms.

$$\log_8(x+3)-\log_8(9x-37)=-1$$
$$\log_8\left(\frac{x+3}{9x-37}\right)=-1$$

Step 3
Express the logarithmic equation in exponential form.

The equation in exponential form is $8^{-1}=\frac{x+3}{9x-37}$ or $\frac{1}{8}=\frac{x+3}{9x-37}$.

Step 4
Solve the resulting equation.

$$\frac{1}{8}=\frac{x+3}{9x-37}$$
$$9x-37=8(x+3)$$
$$9x-37=8x+24$$
$$x=61$$

Step 5
Verify the solution by substituting it into the original equation.

The solution is verified if the LHS = RHS.

$x=61$	
LHS	RHS
$\log_8(61+3)+1$ $=\log_8 64+1$ $=2+1$ $=3$	$\log_8[9(61)-37]$ $=\log_8(549-37)$ $=\log_8 512$ $=3$

Since the LHS = RHS, the solution $x=61$ is verified.

7. B
Michael's first error occurred in step 2. When the equation $\log_5\left(\frac{12x+1}{x-1}\right)=2$ is written in exponential form, the result is $\frac{12x+1}{x-1}=5^2$

Michael's solution should have appeared as follows:

$$\log_5\left(\frac{12x+1}{x-1}\right)=2$$
$$\frac{12x+1}{x-1}=5^2$$
$$\frac{12x+1}{x-1}=25$$
$$12x+1=25(x-1)$$
$$12x+1=25x-25$$
$$-13x=-26$$
$$x=2$$

Michael also made an error in step 4. If his work in the first three steps had been correct, then $x=-\frac{3}{10}$ in step 4 rather than $x=-\frac{10}{3}$.

8. Step 1
Write the logarithmic expressions on the same side of the equation.

$$\log x=3-\log_8 x$$
$$\log x+\log_8 x=3$$

Step 2
Apply the change of base formula using base 10.

$$\log x+\log_8 x=3$$
$$\log x+\frac{\log x}{\log 8}=3$$

Step 3
Remove the common factor $\log x$.

$$\log x+\frac{\log x}{\log 8}=3$$
$$\log x\left[1+\frac{1}{\log 8}\right]=3$$

Step 4
Isolate $\log x$.

Divide both sides of the equation by $\left[1+\frac{1}{\log 8}\right]$.

$$\log x\left[1+\frac{1}{\log 8}\right]=3$$
$$\log x=\frac{3}{\left[1+\frac{1}{\log 8}\right]}$$

Step 5
Solve for *x*.

The exponential form is $10^{\frac{3}{\left[1+\frac{1}{\log 8}\right]}}=x$.

$$10^{\frac{3}{\left[1+\frac{1}{\log 8}\right]}}=x$$
$$10^{1.423261\ldots}=x$$
$$26.5226\ldots=x$$

Step 6
Verify the solution by substituting it into the original equation.

$x=26.5226\ldots$	
LHS	RHS
$\log(26.5226\ldots)$ $=1.42361\ldots$	$3-\log_8(26.5226\ldots)$ $=3-\frac{\log(26.5226\ldots)}{\log 8}$ $=1.42361\ldots$

Since the LHS = RHS, the solution $x=26.523$ is verified.

In the equation $\log x=3-\log_8 x$, the solution is $x=26.523$.

9. **Step 1**
Write the logarithmic expressions on the same side of the equation.

$$\log_5(2x)-1=\log_6(2x)$$
$$\log_5(2x)-\log_6(2x)=1$$

Step 2
Apply the change of base formula using base 10.

$$\log_5(2x)-\log_6(2x)=1$$
$$\frac{\log(2x)}{\log 5}-\frac{\log(2x)}{\log 6}=1$$

Step 3
Remove the common factor $\log(2x)$.

$$\frac{\log(2x)}{\log 5}-\frac{\log(2x)}{\log 6}=1$$
$$\log(2x)\left[\frac{1}{\log 5}-\frac{1}{\log 6}\right]=1$$

tep 4
Isolate $\log(2x)$.

Divide both sides of the equation by $\left[\frac{1}{\log 5}-\frac{1}{\log 6}\right]$.

$$\log(2x)\left[\frac{1}{\log 5}-\frac{1}{\log 6}\right]=1$$
$$\log(2x)=\frac{1}{\left[\frac{1}{\log 5}-\frac{1}{\log 6}\right]}$$

Step 5
Solve for *x*.

The exponential form is $10^{\frac{1}{\left[\frac{1}{\log 5}-\frac{1}{\log 6}\right]}}=2x$.
Solve the exponential equation.

$$10^{\frac{1}{\left[\frac{1}{\log 5}-\frac{1}{\log 6}\right]}}=2x$$
$$\frac{10^{\frac{1}{\left[\frac{1}{\log 5}-\frac{1}{\log 6}\right]}}}{2}=x$$
$$\frac{10^{6.86910\ldots}}{2}=x$$
$$3\ 698\ 930.212=x$$

Step 6
Verify the solution by substituting it into the original equation.

$x=3\ 698\ 930.212$	
LHS	RHS
$\log_5[2(3698930.212)]-1$ $=\frac{\log[2(3698930.212)]}{\log 5}-1$ $=8.827\ 46\ldots$	$\log_6[2(3698930.212)]$ $=\frac{\log[2(3698930.212)]}{\log 6}$ $=8.82746\ldots$

Since the LHS = RHS, the solution $x = 3698930.212$ is verified.

In the equation $\log_5(2x) - 1 = \log_6(2x)$, the solution is $x = 3\ 698\ 930.212$.

10. **Step 1**
Write the logarithmic expressions on the same side of the equation.

$$\log_7(x-3) - 5 = -\log_4(x-3)$$
$$\log_7(x-3) + \log_4(x-3) = 5$$

Step 2
Apply the change of base formula using base 10.

$$\log_7(x-3) + \log_4(x-3) = 5$$
$$\frac{\log(x-3)}{\log 7} + \frac{\log(x-3)}{\log 4} = 5$$

Step 3
Remove the common factor $\log(x-3)$.

$$\frac{\log(x-3)}{\log 7} + \frac{\log(x-3)}{\log 4} = 5$$
$$\log(x-3)\left[\frac{1}{\log 7} + \frac{1}{\log 4}\right] = 5$$

Step 4
Isolate $\log(x-3)$.

Divide both sides of the equation by $\left[\frac{1}{\log 7} + \frac{1}{\log 4}\right]$.

$$\log(x-3)\left[\frac{1}{\log 7} + \frac{1}{\log 4}\right] = 5$$
$$\log(x-3) = \frac{5}{\left[\frac{1}{\log 7} + \frac{1}{\log 4}\right]}$$

Step 5
Solve for *x*.

The exponential form is $10^{\frac{5}{\left[\frac{1}{\log 7} + \frac{1}{\log 4}\right]}} = x - 3$.
Solve the exponential equation.

$$10^{\frac{5}{\left[\frac{1}{\log 7} + \frac{1}{\log 4}\right]}} = x - 3$$
$$10^{\frac{5}{\left[\frac{1}{\log 7} + \frac{1}{\log 4}\right]}} + 3 = x$$
$$10^{1.75792...} + 3 = x$$
$$60.2700... = x$$

Step 6
Verify the solution by substituting it into the original equation.

$x \approx 60.2700\ldots$	
LHS	RHS
$\log_7(60.2700\ldots - 3) - 5$ $= \frac{\log(60.2700\ldots - 3)}{\log 7} - 5$ $= -2.91985\ldots$	$-\log_4(60.270 - 3)$ $= -\frac{\log(60.270 - 3)}{\log 4}$ $= -2.919\ldots$

Since the LHS = RHS, the solution $x = 60.270$ is verified. In the equation $\log_7(x-3) - 5 = -\log_4(x-3)$, the solution is $x = 60.270$.

Lesson 5—Solving an Exponential Equation Using Logarithms

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Isolate the power.

$$80^{2x} - 4 = 5$$
$$80^{2x} = 9$$

Step 2
Write the equation in logarithmic form.

$$80^{2x} = 9$$
$$\log_{80} 9 = 2x$$

Step 3
Apply the change of base formula to the expression $\log_{80} 9$.

$$\log_{80} 9 = 2x$$
$$\frac{\log 9}{\log 80} = 2x$$

Step 4
Isolate *x*.

Divide both sides by 2.

$$\frac{\log 9}{\log 80} = 2x$$
$$\frac{\log 9}{2\log 80} = x$$

Step 5
Find an approximate solution using a calculator.

$$x = \frac{\log 9}{2\log 80}$$
$$x \approx 0.251$$

Therefore, the value of x is approximately 0.251.

2. **Step 1**
Take the common logarithm of each side of the equation.

$$7^{x-4} = 6^{x-2}$$
$$\log\left(7^{x-4}\right) = \log\left(6^{x-2}\right)$$

Step 2
Apply the power law of logarithms.

$$\log\left(7^{x-4}\right) = \log\left(6^{x-2}\right)$$
$$(x-4)\log 7 = (x-2)\log 6$$

Step 3
Apply the distributive property.

$$(x-4)\log 7 = (x-2)\log 6$$
$$x\log 7 - 4\log 7 = x\log 6 - 2\log 6$$

Step 4
Isolate x.

$$x\log 7 - 4\log 7 = x\log 6 - 2\log 6$$
$$x\log 7 - x\log 6 = -2\log 6 + 4\log 7$$
$$x(\log 7 - \log 6) = -2\log 6 + 4\log 7$$
$$x = \frac{-2\log 6 + 4\log 7}{\log 7 - \log 6}$$

Step 5
Find an approximate solution using a calculator.

Therefore, the value of x is approximately 27.247.

3. **Step 1**
Take the common logarithm of each side of the equation.

$$15\left(5^{x-2}\right) = 8^{2x+1}$$
$$\log\left[15\left(5^{x-2}\right)\right] = \log\left(8^{2x+1}\right)$$

Step 2
Apply the product law of logarithms.

$$\log\left[15\left(5^{x-2}\right)\right] = \log\left(8^{2x+1}\right)$$
$$\log 15 + \log\left(5^{x-2}\right) = \log\left(8^{2x+1}\right)$$

Step 3
Apply the power law of logarithms.

$$\log 15 + \log\left(5^{x-2}\right) = \log\left(8^{2x+1}\right)$$
$$\log 15 + (x-2)\log 5 = (2x+1)\log 8$$

Step 4
Apply the distributive property.

$$\log 15 + (x-2)\log 5 = (2x+1)\log 8$$
$$\log 15 + x\log 5 - 2\log 5 = 2x\log 8 + \log 8$$

Step 5
Isolate x.

$$\log 15 + x\log 5 - 2\log 5 = 2x\log 8 + \log 8$$
$$\log 15 - 2\log 5 - \log 8 = 2x\log 8 - x\log 5$$
$$\log 15 - 2\log 5 - \log 8 = x[2\log 8 - \log 5]$$
$$x = \frac{\log 15 - 2\log 5 - \log 8}{2\log 8 - \log 5}$$

Step 6
Find an approximate solution using a calculator.

$$x = \frac{\log 15 - 2\log 5 - \log 8}{2\log 8 - \log 5}$$
$$x = \frac{-1.124\ldots}{1.107\ldots}$$
$$x \approx -1.016$$

Therefore, the value of x is approximately –1.016.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Isolate the power.

$$4(5)^{x-2} = 232$$
$$5^{x-2} = 58$$

Step 2
Write the equation in logarithmic form.

$$5^{x-2} = 58$$
$$\log_5 58 = x - 2$$

Step 3
Apply the change of base formula to the expression $\log_5 58$.

$$\log_5 58 = x - 2$$
$$\frac{\log 58}{\log 5} = x - 2$$

Step 4
Isolate x.

Add 2 to both sides.

$$\frac{\log 58}{\log 5} = x - 2$$
$$\frac{\log 58}{\log 5} + 2 = x$$

Step 5
Find an approximate solution using a calculator.

$$x = \frac{\log 58}{\log 5} + 2$$
$$x \approx 4.52$$

Therefore, the value of x is approximately 4.52.

2. **Step 1**
Isolate the power.

$$13 = 125^{4x-1} - 7$$
$$20 = 125^{4x-1}$$

Step 2
Take the common logarithm of each side of the equation.

$$20 = 125^{4x-1}$$
$$\log 20 = \log\left(125^{4x-1}\right)$$

Step 3
Apply the power law of logarithms.

$$\log 20 = \log\left(125^{4x-1}\right)$$
$$\log 20 = (4x-1)\log 125$$

Step 4
Divide both sides by $\log 125$.

$$\log 20 = (4x-1)\log 125$$
$$\frac{\log 20}{\log 125} = 4x-1$$

Step 5
Add 1 to both sides.

$$\frac{\log 20}{\log 125} = 4x-1$$
$$\frac{\log 20}{\log 125} + 1 = 4x$$

Step 6
Divide both sides by 4.

$$\frac{\log 20}{\log 125} + 1 = 4x$$
$$\frac{\frac{\log 20}{\log 125} + 1}{4} = x$$
$$\frac{1}{4}\left(\frac{\log 20}{\log 125} + 1\right) = x$$

Step 7
Find an approximate solution using a calculator.

$$x = \frac{1}{4}\left(\frac{\log 20}{\log 125} + 1\right)$$
$$x \approx 0.41$$

Therefore, the value of x is approximately 0.41.

3. **Step 1**
Take the common logarithm of each side of the equation.

$$8^{6x-5} = 9^{11x}$$
$$\log\left(8^{6x-5}\right) = \log\left(9^{11x}\right)$$

Step 2
Apply the power law of logarithms.

$$\log\left(8^{6x-5}\right) = \log\left(9^{11x}\right)$$
$$(6x-5)\log 8 = 11x\log 9$$

Step 3
Apply the distributive property.

$$(6x-5)\log 8 = 11x\log 9$$
$$6x\log 8 - 5\log 8 = 11x\log 9$$

Step 4
Subtract $6x\log 8$ from both sides of the equation.

$$6x\log 8 - 5\log 8 = 11x\log 9$$
$$-5\log 8 = 11x\log 9 - 6x\log 8$$

Step 5
Factor out x on the right side of the equation.

$$-5\log 8 = 11x\log 9 - 6x\log 8$$
$$-5\log 8 = x(11\log 9 - 6\log 8)$$

Step 6
Divide both sides by $(11\log 9 - 6\log 8)$.

$$-5\log 8 = x(11\log 9 - 6\log 8)$$
$$\frac{-5\log 8}{11\log 9 - 6\log 8} = x$$

Step 7
Find an approximate solution using a calculator.

$$x = \frac{-5\log 8}{11\log 9 - 6\log 8}$$
$$x \approx -0.89$$

Therefore, the value of x is approximately –0.89.

4. **Step 1**
Take the common logarithm of each side of the equation.

$$3^{4x-5} = 2^{x-1}$$
$$\log\left(3^{4x-5}\right) = \log\left(2^{x-1}\right)$$

Step 2
Apply the power law of logarithms.

$$\log\left(3^{4x-5}\right) = \log\left(2^{x-1}\right)$$
$$(4x-5)\log 3 = (x-1)\log 2$$

Step 3
Apply the distributive property.

$$\begin{aligned}(4x-5)\log 3 &= (x-1)\log 2\\ 4x\log 3-5\log 3 &= x\log 2-\log 2\end{aligned}$$

Step 4
Isolate x.

$$\begin{aligned}4x\log 3-5\log 3 &= x\log 2-\log 2\\ 4x\log 3-x\log 2 &= -\log 2+5\log 3\\ x(4\log 3-\log 2) &= -\log 2+5\log 3\\ x &= \frac{-\log 2+5\log 3}{4\log 3-\log 2}\end{aligned}$$

Step 5
Find an approximate solution using a calculator.

$$x=\frac{-\log 2+5\log 3}{4\log 3-\log 2}$$
$$x\approx 1.30$$

Therefore, the value of x is approximately 1.30.

5. **Step 1**
Write the left side of the equation as a single power using exponent laws.

$$\begin{aligned}8^{x+4}\times 16^{x-5} &= 7^{2-x}\\ \left(2^3\right)^{x+4}\times\left(2^4\right)^{x-5} &= 7^{2-x}\\ 2^{3x+12}\times 2^{4x-20} &= 7^{2-x}\\ 2^{7x-8} &= 7^{2-x}\end{aligned}$$

Step 2
Take the common logarithm of each side of the equation.

$$2^{7x-8}=7^{2-x}$$
$$\log\left(2^{7x-8}\right)=\log\left(7^{2-x}\right)$$

Step 3
Apply the power law of logarithms.

$$\begin{aligned}\log\left(2^{7x-8}\right) &= \log\left(7^{2-x}\right)\\ (7x-8)\log 2 &= (2-x)\log 7\end{aligned}$$

Step 4
Apply the distributive property.

$$\begin{aligned}(7x-8)\log 2 &= (2-x)\log 7\\ 7x\log 2-8\log 2 &= 2\log 7-x\log 7\end{aligned}$$

Step 5
Subtract $(-8\log 2)$ and $(-x\log 7)$ from both sides.

$$\begin{aligned}7x\log 2-8\log 2 &= 2\log 7-x\log 7\\ 7x\log 2+x\log 7 &= 2\log 7+8\log 2\end{aligned}$$

Step 6
Factor out x on the left side of the equation.

$$\begin{aligned}7x\log 2+x\log 7 &= 2\log 7+8\log 2\\ x(7\log 2+\log 7) &= 2\log 7+8\log 2\end{aligned}$$

Step 7
Divide both sides by $(7\log 2+\log 7)$.

$$\begin{aligned}x(7\log 2+\log 7) &= 2\log 7+8\log 2\\ x &= \frac{2\log 7+8\log 2}{7\log 2+\log 7}\end{aligned}$$

Step 8
Find an approximate solution using a calculator.

$$x=\frac{2\log 7+8\log 2}{7\log 2+\log 7}$$
$$x\approx 1.39$$

Therefore, the value of x is approximately 1.39.

6. **Step 1**
Write the right side of the equation as a single power using exponent laws.

$$\begin{aligned}12^{6x-1} &= 3^{5x}\times 27^{x-4}\\ 12^{6x-1} &= 3^{5x}\times\left(3^3\right)^{x-4}\\ 12^{6x-1} &= 3^{5x}\times 3^{3x-12}\\ 12^{6x-1} &= 3^{8x-12}\end{aligned}$$

Step 2
Take the common logarithm of each side of the equation.

$$\begin{aligned}12^{6x-1} &= 3^{8x-12}\\ \log\left(12^{6x-1}\right) &= \log\left(3^{8x-12}\right)\end{aligned}$$

Step 3
Apply the power law of logarithms

$$\begin{aligned}\log\left(12^{6x-1}\right) &= \log\left(3^{8x-12}\right)\\ (6x-1)\log 12 &= (8x-12)\log 3\end{aligned}$$

Step 4
Apply the distributive property.

$$\begin{aligned}(6x-1)\log 12 &= (8x-12)\log 3\\ 6x\log 12-\log 12 &= 8x\log 3-12\log 3\end{aligned}$$

Step 5
Subtract $(8x\log 3)$ and $(-\log 12)$ from both sides.

$$\begin{aligned}6x\log 12-\log 12 &= 8x\log 3-12\log 3\\ 6x\log 12-8x\log 3 &= -12\log 3+\log 12\end{aligned}$$

Step 6
Factor out x on the left side of the equation.

$$\begin{aligned}6x\log 12-8x\log 3 &= -12\log 3+\log 12\\ x(6\log 12-8\log 3) &= -12\log 3+\log 12\end{aligned}$$

Step 7

Divide both sides by $(6\log 12 - 8\log 3)$.

$$x(6\log 12 - 8\log 3) = -12\log 3 + \log 12$$
$$x = \frac{-12\log 3 + \log 12}{6\log 12 - 8\log 3}$$

Step 8

Find an approximate solution using a calculator.

$$x = \frac{-12\log 3 + \log 12}{6\log 12 - 8\log 3}$$
$$x \approx -1.75$$

Therefore, the value of x is approximately –1.75.

7. **Step 1**

Take the common logarithm of each side of the equation.

$$9(11)^{x+3} = 4^{6x-5}$$
$$\log\left[9(11)^{x+3}\right] = \log\left(4^{6x-5}\right)$$

Step 2

Apply the product law of logarithms.

$$\log\left[9(11)^{x+3}\right] = \log\left(4^{6x-5}\right)$$
$$\log 9 + \log\left(11^{x+3}\right) = \log\left(4^{6x-5}\right)$$

Step 3

Apply the power law of logarithms.

$$\log 9 + \log\left(11^{x+3}\right) = \log\left(4^{6x-5}\right)$$
$$\log 9 + (x+3)\log 11 = (6x-5)\log 4$$

Step 4

Apply the distributive property.

$$\log 9 + (x+3)\log 11 = (6x-5)\log 4$$
$$\log 9 + x\log 11 + 3\log 11 = 6x\log 4 - 5\log 4$$

Step 5

Isolate x.

$$\log 9 + x\log 11 + 3\log 11 = 6x\log 4 - 5\log 4$$
$$x\log 11 - 6x\log 4 = -5\log 4 - \log 9 - 3\log 11$$
$$x(\log 11 - 6\log 4) = -5\log 4 - \log 9 - 3\log 11$$
$$x = \frac{-5\log 4 - \log 9 - 3\log 11}{\log 11 - 6\log 4}$$

Step 6

Find an approximate solution using a calculator.

$$x = \frac{-5\log 4 - \log 9 - 3\log 11}{\log 11 - 6\log 4}$$
$$x \approx 2.76$$

Therefore, the value of x is approximately 2.76.

8. **Step 1**

Take the common logarithm of each side of the equation.

$$17\left(5^{x-1}\right) = 13\left(6^{3x+7}\right)$$
$$\log\left[17\left(5^{x-1}\right)\right] = \log\left[13\left(6^{3x+7}\right)\right]$$

Step 2

Apply the product law of logarithms.

$$\log\left[17\left(5^{x-1}\right)\right] = \log\left[13\left(6^{3x+7}\right)\right]$$
$$\log 17 + \log\left(5^{x-1}\right) = \log 13 + \log\left(6^{3x+7}\right)$$

Step 3

Apply the power law of logarithms.

$$\log 17 + \log\left(5^{x-1}\right) = \log 13 + \log\left(6^{3x+7}\right)$$
$$\log 17 + (x-1)\log 5 = \log 13 + (3x+7)\log 6$$

Step 4

Apply the distributive property.

$$\log 17 + (x-1)\log 5 = \log 13 + (3x+7)\log 6$$
$$\log 17 + x\log 5 - \log 5 = \log 13 + 3x\log 6 + 7\log 6$$

Step 5

Isolate x.

$$\log 17 + x\log 5 - \log 5 = \log 13 + 3x\log 6 + 7\log 6$$
$$x\log 5 - 3x\log 6 = \log 13 + 7\log 6 - \log 17 + \log 5$$
$$x(\log 5 - 3\log 6) = \log 13 + 7\log 6 - \log 17 + \log 5$$
$$x = \frac{\log 13 + 7\log 6 - \log 17 + \log 5}{\log 5 - 3\log 6}$$

Step 6

Find an approximate solution using a calculator.

$$x = \frac{\log 13 + 7\log 6 - \log 17 + \log 5}{\log 5 - 3\log 6}$$
$$x = \frac{6.02\ldots}{-1.63\ldots}$$
$$x \approx -3.69$$

Therefore, the value of x is approximately –3.69.

Lesson 6—Applications of Logarithmic Scales

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. In the formula $\frac{L_1}{L_2}=10^{\left(\frac{\text{dB}_1-\text{dB}_2}{10}\right)}$, let $\text{dB}_1=98$ and $\text{dB}_2=20$, and then solve.

$$\frac{L_1}{L_2}=10^{\left(\frac{\text{dB}_1-\text{dB}_2}{10}\right)}$$
$$\frac{L_1}{L_2}=10^{\left(\frac{98-20}{10}\right)}$$
$$\frac{L_1}{L_2}=10^{\left(\frac{78}{10}\right)}$$
$$\frac{L_1}{L_2}=10^{7.8}$$
$$\frac{L_1}{L_2}=63095734.45$$

Therefore, an orchestra is 63 095 734 times louder than a whisper.

2. **Step 1**

In the formula $\frac{I_1}{I_2}=10^{m_1-m_2}$, let $\frac{I_1}{I_2}=102$ and $m_1=5.8$.

$$\frac{I_1}{I_2}=10^{m_1-m_2}$$
$$102=10^{5.8-m_2}$$

Step 2

Convert the resulting equation to logarithmic form, and solve for m_2.

$$102=10^{5.8-m_2}$$
$$\log 102=5.8-m_2$$
$$m_2=5.8-\log 102$$
$$m_2\approx 3.79$$

The magnitude of the earthquake in City *N* was approximately 3.8.

3. **Step 1**

In the formula $\frac{\left[\text{H}^+\right]_{\text{acidic solution}}}{\left[\text{H}^+\right]_{\text{pure water}}}=10^{\text{pH}_{\text{pure water}}-\text{pH}_{\text{acidic solution}}}$, let $\frac{\left[\text{H}^+\right]_{\text{acidic solution}}}{\left[\text{H}^+\right]_{\text{pure water}}}=50$ and $\text{pH}_{\text{pure water}}=7$.

$$\frac{\left[\text{H}^+\right]_{\text{acidic solution}}}{\left[\text{H}^+\right]_{\text{pure water}}}=10^{\text{pH}_{\text{pure water}}-\text{pH}_{\text{acidic solution}}}$$
$$50=10^{7-\text{pH}_{\text{acidic solution}}}$$

Step 2

Convert the resulting equation to logarithmic form, and solve for $\text{pH}_{\text{acidic solution}}$.

$$50=10^{7-\text{pH}_{\text{acidic solution}}}$$
$$\log 50=7-\text{pH}_{\text{acidic solution}}$$
$$\text{pH}_{\text{acidic solution}}=7-\log 50$$
$$\text{pH}_{\text{acidic solution}}\approx 5.30$$

The pH of the acidic solution is approximately 5.3.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. In the formula $\frac{I_1}{I_2}=10^{m_1-m_2}$, let $m_1=7.8$ and $m_2=5.2$, and then solve.

$$\frac{I_1}{I_2}=10^{m_1-m_2}$$
$$\frac{I_1}{I_2}=10^{7.8-5.2}$$
$$\frac{I_1}{I_2}=10^{2.6}$$
$$\frac{I_1}{I_2}\approx 398.107$$

Therefore, the earthquake at point *M* is approximately 398 times more intense than the earthquake at point *N*.

2. In the formula $\text{dB}=10\log L$, substitute 10^{12} for *L*, and solve for dB.

$$\text{dB}=10\log L$$
$$\text{dB}=10\log\left(10^{12}\right)$$
$$\text{dB}=10(12)$$
$$\text{dB}=120$$

Therefore, the decibel measurement for a rock concert is 120 dB.

3. In the formula $\frac{L_1}{L_2}=10^{\left(\frac{\text{dB}_1-\text{dB}_2}{10}\right)}$, let $\text{dB}_1=120$ and $\text{dB}_2=50$, and then solve.

$$\frac{L_1}{L_2}=10^{\left(\frac{\text{dB}_1-\text{dB}_2}{10}\right)}$$
$$\frac{L_1}{L_2}=10^{\left(\frac{120-50}{10}\right)}$$
$$\frac{L_1}{L_2}=10^{\left(\frac{70}{10}\right)}$$
$$\frac{L_1}{L_2}=10^{7}$$

A rock concert is 10 000 000 or 10^7 times louder than an ordinary conversation.

4. **Step 1**

In the formula $\frac{\left[H^+\right]_{\text{battery acid}}}{\left[H^+\right]_{\text{pop}}} = 10^{\text{pH}_{\text{pop}} - \text{pH}_{\text{battery acid}}}$, let

$\frac{\left[H^+\right]_{\text{battery acid}}}{\left[H^+\right]_{\text{pop}}} = 50.3$ and $\text{pH}_{\text{battery acid}} = 0.8$.

$$\frac{\left[H^+\right]_{\text{battery acid}}}{\left[H^+\right]_{\text{pop}}} = 10^{\text{pH}_{\text{pop}} - \text{pH}_{\text{battery acid}}}$$
$$50.3 = 10^{\text{pH}_{\text{pop}} - 0.8}$$

Step 2

Convert the resulting equation to logarithmic form, and solve for pH_{pop}.

$$50.3 = 10^{\text{pH}_{\text{pop}} - 0.8}$$
$$\log 50.3 = \text{pH}_{\text{pop}} - 0.8$$
$$\log 50.3 + 0.8 = \text{pH}_{\text{pop}}$$
$$2.501 \approx \text{pH}_{\text{pop}}$$

The pH level of soda pop is approximately 2.5.

5. In the formula, $\frac{\left[H^+\right]_{\text{battery acid}}}{\left[H^+\right]_{\text{milk}}} = 10^{\text{pH}_{\text{milk}} - \text{pH}_{\text{battery acid}}}$, let

$\text{pH}_{\text{milk}} = 6.7$ and $\text{pH}_{\text{battery acid}} = 0.8$, and solve.

$$\frac{\left[H^+\right]_{\text{battery acid}}}{\left[H^+\right]_{\text{milk}}} = 10^{\text{pH}_{\text{milk}} - \text{pH}_{\text{battery acid}}}$$
$$\frac{\left[H^+\right]_{\text{battery acid}}}{\left[H^+\right]_{\text{milk}}} = 10^{6.7 - 0.8}$$
$$\frac{\left[H^+\right]_{\text{battery acid}}}{\left[H^+\right]_{\text{milk}}} = 10^{5.9}$$
$$\frac{\left[H^+\right]_{\text{battery acid}}}{\left[H^+\right]_{\text{milk}}} \approx 794\ 328.234$$

Therefore, battery acid is approximately 794 328.2 times more acidic than milk.

6. **Step 1**

In the formula $\frac{I_1}{I_2} = 10^{m_1 - m_2}$, let $\frac{I_1}{I_2} = 17$ and $m_1 = 8.8$.

$$\frac{I_1}{I_2} = 10^{m_1 - m_2}$$
$$17 = 10^{8.8 - m_2}$$

Step 2

Convert the resulting equation to logarithmic form, and solve for m_2.

$$17 = 10^{8.8 - m_2}$$
$$\log 17 = 8.8 - m_2$$
$$m_2 = 8.8 - \log 17$$
$$m_2 \approx 7.570$$

The magnitude of earthquake Y is approximately 7.6.

7. **Step 1**

In the formula $\frac{I_1}{I_2} = 10^{m_1 - m_2}$, let $\frac{I_1}{I_2} = 10$ and $m_1 = 8.8$.

$$\frac{I_1}{I_2} = 10^{m_1 - m_2}$$
$$10 = 10^{8.8 - m_2}$$

Step 2

Convert the resulting equation to logarithmic form, and solve for m_2.

$$10 = 10^{8.8 - m_2}$$
$$\log 10 = 8.8 - m_2$$
$$m_2 = 8.8 - \log 10$$
$$m_2 = 7.8$$

The magnitude of earthquake Z is 7.8.

Practice Test

ANSWERS AND SOLUTIONS

1. Let $\log_3 81 = x$. The exponential form of $\log_3 81 = x$ is $3^x = 81$.
Since $3^4 = 81$, $\log_3 81 = 4$.

2. **Step 1**

Let $2\log_5 25 = x$. Divide both sides by 2.

$$2\log_5 25 = x$$
$$\log_5 25 = \frac{x}{2}$$

Step 2

Write $\log_5 25 = \frac{x}{2}$ in exponential form.

The exponential form of $\log_5 25 = \frac{x}{2}$ is $5^{\frac{x}{2}} = 5$.

Step 3

Solve the exponential equation $5^{\frac{x}{2}} = 5$.

$$5^{\frac{x}{2}} = 25$$
$$5^{\frac{x}{2}} = 5^2$$

Equate the exponents.

$$\frac{x}{2} = 2$$
$$x = 4$$

Therefore, $2\log_5 25 = 4$.

3. **Step 1**
Apply the change of base formula.

$$\frac{\log_2 729}{\log_2 9}$$
$$= \log_9 729$$

Step 2
Let $\log_9 729 = x$, and write the equation in exponential form.

The exponential form is $9^x = 729$.

Step 3
Solve the exponential equation $9^x = 729$.

$$9^x = 729$$
$$9^x = 9^3$$

The value of x is 3.

Therefore, $\frac{\log_2 729}{\log_2 9} = 3$.

4. **Step 1**
Write the expression using base 10.

$$\log_8 121$$
$$= \frac{\log 121}{\log 8}$$

Step 2

Evaluate $\frac{\log 121}{\log 8}$ using a calculator.

$$\frac{\log 121}{\log 8} \approx 2.306$$

Therefore, $\log_8 121 = 2.3$.

5. **Step 1**
Write the expression $3\log_7 11$ using base 10.

$$3\log_7 11$$
$$= 3\left(\frac{\log 11}{\log 7}\right)$$

Step 2

Evaluate $3\left(\frac{\log 11}{\log 7}\right)$ using a calculator.

$$3\left(\frac{\log 11}{\log 7}\right) \approx 3.696$$

Therefore, $3\log_7 11 = 3.7$.

6. **Step 1**
Apply the product law of logarithms.

$$\log_4 8 + \log_4 512$$
$$= \log_4 (8 \times 512)$$
$$= \log_4 4096$$

Step 2
Evaluate $\log_4 4\,096$.

$$\log_4 4096$$
$$= \frac{\log 4096}{\log 4}$$
$$= 6$$

Therefore, $\log_4 8 + \log_4 512 = 6$.

7. **Step 1**
Apply the power law of logarithms.

$$\log_8 4608 - 2\log_8 3$$
$$= \log_8 4608 - \log_8 (3^2)$$
$$= \log_8 4608 - \log_8 9$$

Step 2
Apply the quotient law of logarithms.

$$\log_8 4608 - \log_8 9$$
$$= \log_8 \left(\frac{4608}{9}\right)$$
$$= \log_8 512$$

Step 3
Evaluate $\log_8 512$.

$$\log_8 512$$
$$= \log_8 (8^3)$$
$$= 3$$

Therefore, $\log_8 4608 - 2\log_8 3 = 3$.

8. **Step 1**
Apply the power law of logarithms.

$$3\log_9 3 + \log_9 12 - 2\log_9 2$$
$$= \log_9 (3^3) + \log_9 12 - \log_9 (2^2)$$
$$= \log_9 27 + \log_9 12 - \log_9 4$$

Step 2
Apply the power law to the first two terms of the expression, and simplify.

$$\log_9 27 + \log_9 12 - \log_9 4$$
$$= \log_9 (27 \times 12) - \log_9 4$$
$$= \log_9 324 - \log_9 4$$

Step 3
Apply the quotient law, and simplify.

$$\log_9 324 - \log_9 4$$
$$= \log_9 \left(\frac{324}{4}\right)$$
$$= \log_9 81$$

Step 4
Evaluate $\log_9 81$.

$$\log_9 81$$
$$= \log_9 \left(9^2\right)$$
$$= 2$$

Therefore, $3\log_9 3 + \log_9 12 - 2\log_9 2 = 2$.

9. **Step 1**
Describe the transformations applied to $y = \log_4 x$ to get the graph of $y = 5\log_4 x - 2$.

The function $y = 5\log_4 x - 2$ is of the form $y = a\log_4 x + k$, where $a = 5$ and $k = -2$.

Therefore, the graph $y = 5\log_4 (x) - 2$ is obtained from $y = \log_4 x$ by a vertical stretch factor of 5 and a vertical translation 2 units down.

Step 2
Apply the vertical stretch.

Some points on the graph of $y = \log_4 x$ are (0.5, –0.5), (1, 0), (4, 1), and (8, 1.5).

Multiply the y-coordinates by 5.

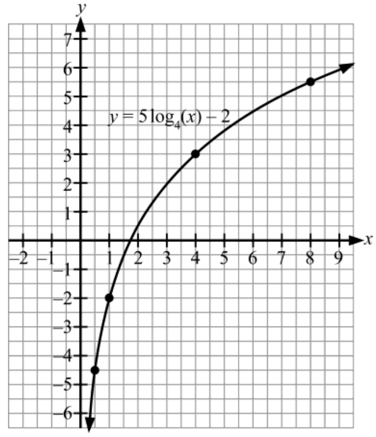
$$(0.5, -0.5 \times 5) \rightarrow (0.5, -2.5)$$
$$(1, 0 \times 5) \rightarrow (1, 0)$$
$$(4, 1 \times 5) \rightarrow (4, 5)$$
$$(8, 1.5 \times 5) \rightarrow (8, 7.5)$$

Step 3
Apply the vertical translation.

Decrease the y-coordinates from step 1 by 2.

$$(0.5, -2.5 - 2) \rightarrow (0.5, -4.5)$$
$$(1, 0 - 2) \rightarrow (1, -2)$$
$$(4, 5 - 2) \rightarrow (4, 3)$$
$$(8, 7.5 - 2) \rightarrow (8, 5.5)$$

Step 4
Plot and join the transformed points.



Step 5
Determine the vertical asymptote, and state the domain and range.

The vertical asymptote is $x = 0$ (y-axis).
The domain is $x > 0$, and the range is $y \in R$.

10. **Step 1**
Apply the horizontal stretch.

Replace x with $\frac{1}{5}x$ in the equation $y = \log_{\frac{1}{3}} x$.

$$y = \log_{\frac{1}{3}} x$$
$$y = \log_{\frac{1}{3}} \left(\frac{1}{5}x\right)$$

Step 2
Apply the horizontal translation. Replace x with $(x – 11)$ in the equation $y = \log_{\frac{1}{3}} \left(\frac{1}{5}x\right)$.

$$y = \log_{\frac{1}{3}} \left(\frac{1}{5}x\right)$$
$$y = \log_{\frac{1}{3}} \left[\frac{1}{5}(x - 11)\right]$$

Step 3
Apply the vertical translation. Replace y with $(y - (-3))$ in the equation $y = \log_{\frac{1}{3}} \left[\frac{1}{5}(x - 11)\right]$.

$$y = \log_{\frac{1}{3}} \left[\frac{1}{5}(x - 11)\right]$$
$$y - (-3) = \log_{\frac{1}{3}} \left[\frac{1}{5}(x - 11)\right]$$

Step 4
Isolate *y*.

$$y-(-3)=\log_{\frac{1}{3}}\left[\frac{1}{5}(x-11)\right]$$
$$y+3=\log_{\frac{1}{3}}\left[\frac{1}{5}(x-11)\right]$$
$$y=\log_{\frac{1}{3}}\left[\frac{1}{5}(x-11)\right]-3$$

Therefore, the equation of the transformed function is $y=\log_{\frac{1}{3}}\left[\frac{1}{5}(x-11)\right]-3$.

11. Step 1
Determine the equation of the vertical asymptote.

Since the graph of $y=\log_{\frac{1}{3}} x$ is translated 11 units right, the vertical asymptote becomes $x=0+11=11$.

Step 2
State the domain.
The horizontal translation 11 units right results in a domain of $x>(0+11)$ or $x>11$.

Step 3
State the range.
The transformations have no effect on the range.
Therefore, the range remains as $y \in R$.

12. Step 1
Determine the function $f^{-1}(x)$.

Let $y=14+9^{x+3}$. Interchange *x* and *y*.

$$y=14+9^{x+3}$$
$$x=14+9^{y+3}$$

Step 2
Solve for *y*.

$$x=14+9^{y+3}$$
$$x-14=9^{y+3}$$
$$\log(x-14)=\log\left(9^{y+3}\right)$$
$$\log(x-14)=(y+3)\log 9$$
$$\frac{\log(x-14)}{\log 9}=y+3$$
$$\frac{\log(x-14)}{\log 9}-3=y$$

Therefore, the inverse function is $y=\frac{\log(x-14)}{\log 9}-3$, which can also be written as $f^{-1}(x)=\frac{\log(x-14)}{\log 9}-3$.

Step 3
Determine the value of $f^{-1}(16)$.

$$f^{-1}(x)=\frac{\log(x-14)}{\log 9}-3$$
$$f^{-1}(16)=\frac{\log(16-14)}{\log 9}-3$$
$$f^{-1}(16)=\frac{\log 2}{\log 9}-3$$
$$f^{-1}(16)\approx -2.684$$

Therefore, the value of $f^{-1}(16)$ to the nearest hundredth is –2.68.

13. Step 1
Apply the power law of logarithms.

$$\log_{\sqrt{2}}(n+1)+4\log_{\sqrt{2}} 3$$
$$=\log_{\sqrt{2}}(n+1)+\log_{\sqrt{2}}\left(3^4\right)$$
$$=\log_{\sqrt{2}}(n+1)+\log_{\sqrt{2}} 81$$

Step 2
Apply the product law of logarithms.

$$=\log_{\sqrt{2}}(n+1)+\log_{\sqrt{2}} 81$$
$$=\log_{\sqrt{2}}\left[81(n+1)\right]$$
$$=\log_{\sqrt{2}}(81n+81)$$

Therefore,
$\log_{\sqrt{2}}(n+1)+4\log_{\sqrt{2}} 3=\log_{\sqrt{2}}(81n+81)$.

14. Step 1
Apply the power law of logarithms.

$$3\log_7(5m)-9\log_7 m$$
$$=\log_7\left[(5m)^3\right]-\log_7\left(m^9\right)$$

Step 2
Apply the quotient law of logarithms.

$$=\log_7\left[(5m)^3\right]-\log_7\left(m^9\right)$$
$$=\log_7\left[\frac{(5m)^3}{m^9}\right]$$
$$=\log_7\left[\frac{5^3m^3}{m^9}\right]$$
$$=\log_7\left(\frac{125}{m^6}\right)$$

Therefore, $3\log_7(5m)-9\log_7 m=\log_7\left(\frac{125}{m^6}\right)$.

15. Step 1
Apply the power law of logarithms.

$$4\log_5 m - \log_5 n + \frac{1}{2}\log_5 p$$
$$= \log_5\left(m^4\right) - \log_5 n + \log_5\left(p^{\frac{1}{2}}\right)$$

Step 2
Apply the quotient law to the first two terms of the expression.

$$\log_5\left(m^4\right) - \log_5 n + \log_5\left(p^{\frac{1}{2}}\right)$$
$$= \log_5\left(\frac{m^4}{n}\right) + \log_5\left(p^{\frac{1}{2}}\right)$$

Step 3
Apply the product law of logarithms.

$$\log_5\left(\frac{m^4}{n}\right) + \log_5\left(p^{\frac{1}{2}}\right)$$
$$= \log_5\left(\frac{m^4 p^{\frac{1}{2}}}{n}\right)$$

Therefore,

$$4\log_5 m - \log_5 n + \frac{1}{2}\log_5 p$$
$$= \log_5\left(\frac{m^4 p^{\frac{1}{2}}}{n}\right).$$

16. Step 1
Write the equation in exponential form.

$$\log_3\left[\log_5(25x)\right] = 2$$
$$3^2 = \log_5(25x)$$
$$9 = \log_5(25x)$$

Step 2
Write the equation $9 = \log_5(25x)$ in exponential form.

$$9 = \log_5(25x)$$
$$5^9 = 25x$$

Step 3
Solve for *x*.

$$5^9 = 25x$$
$$\frac{5^9}{25} = x$$
$$78125 = x$$

Step 4
Verify the solution.

$x = 78125$	
LHS	RHS
$\log_3\left[\log_5\left[25(78125)\right]\right]$ $= \log_3\left[\log_5\left[(5^2)(5^7)\right]\right]$ $= \log_3\left[\log_5(5^9)\right]$ $= \log_3(9)$ $= 2$	2

Since the LHS =RHS, the solution $x = 78125$ is verified.

17. Step 1
Apply the product law of logarithms.

$$\log_3 x + \log_3(x-2) = 1$$
$$\log_3\left[x(x-2)\right] = 1$$
$$\log_3\left(x^2 - 2x\right) = 1$$

Step 2
Write the equation in exponential form.

The exponential form of the equation $\log_3\left(x^2 - 2x\right) = 1$ is $3^1 = x^2 - 2x$ or $3 = x^2 - 2x$.

Step 3
Solve for *x*.

$$3 = x^2 - 2x$$
$$0 = x^2 - 2x - 3$$
$$0 = (x+1)(x-3)$$
$$x+1 = 0 \qquad x-3 = 0$$
$$x = -1 \qquad x = 3$$

Step 4
Verify the solutions.

$x = -1$	
LHS	RHS
$\log_3(-1) + \log_3(-1-2)$ $= \log_3(-1) + \log_3(-3)$	1

The expressions $\log_3(-1)$ and $\log_3(-3)$ cannot be evaluated because the logarithm of a negative number is undefined. Therefore, $x=-1$ is an extraneous root.

$x=3$	
LHS	RHS
$\log_3 3+\log_3(3-2)$ $=\log_3 3+\log_3 1$ $=1+0$ $=1$	1

Since the LHS =RHS, the solution $x=3$ is verified.

18. Step 1
Write the logarithmic expressions on the same side of the equation.

$$\log(2x)=6-\log_3(2x)$$
$$\log(2x)+\log_3(2x)=6$$

Step 2
Apply the change of base formula using base 10.

$$\log(2x)+\log_3(2x)=6$$
$$\log(2x)+\frac{\log(2x)}{\log 3}=6$$

Step 3
Remove the common factor of $\log(2x)$.

$$\log(2x)+\frac{\log(2x)}{\log 3}=6$$
$$\log(2x)\left[1+\frac{1}{\log 3}\right]=6$$

Step 4
Isolate log $(2x)$.

Divide both sides by $\left[1+\frac{1}{\log 3}\right]$.

$$\log(2x)\left[1+\frac{1}{\log 3}\right]=6$$
$$\log(2x)=\frac{6}{\left[1+\frac{1}{\log 3}\right]}$$

Step 5
Evaluate $\frac{6}{\left[1+\frac{1}{\log 3}\right]}$ using a calculator.

$$\log(2x)=\frac{6}{\left[1+\frac{1}{\log 3}\right]}$$
$$\log(2x)=\frac{6}{3.09590\ldots}$$
$$\log(2x)=1.93804\ldots$$

Step 6
Solve for *x*.

The exponential form is $10^{1.93804}=2x$.

$$10^{1.93804\ldots}=2x$$
$$\frac{10^{1.93804\ldots}}{2}=x$$
$$43.3525\ldots=x$$

Step 7
Verify the solution.

$x=43.3525\ldots$	
LHS	RHS
$\log\left[2(43.3525\ldots)\right]$ ≈ 1.9380	$6-\log_3\left[2(43.3525\ldots)\right]$ $=6-\frac{\log\left[2(43.3525\ldots)\right]}{\log 3}$ ≈ 1.9380

Therefore, the solution is approximately $x=43.4$.

19. Step 1
Isolate the power.

$$5^{x+5}-76=-12$$
$$5^{x+5}=64$$

Step 2
Write the equation in logarithmic form.

$$5^{x+5}=64$$
$$\log_5 64=x+5$$

Step 3
Apply the change of base formula to the expression $\log_5 64$.

$$\log_5 64=x+5$$
$$\frac{\log 64}{\log 5}=x+5$$

Step 4
Isolate *x*.

$$\frac{\log 64}{\log 5} = x + 5$$
$$\log_5 64 - 5 = x$$

Step 5
Find an approximate solution using a calculator.

$$x = \frac{\log 64}{\log 5} - 5$$
$$x \approx -2.4$$

Therefore, the value of *x* is approximately –2.4.

20. **Step 1**
Take the common logarithm of each side of the equation.

$$3(7)^{x-2} = 4(11)^{13-x}$$
$$\log\left[3(7)^{x-2}\right] = \log\left[4(11)^{13-x}\right]$$

Step 2
Apply the product law of logarithms.

$$\log\left[3(7)^{x-2}\right] = \log\left[4(11)^{13-x}\right]$$
$$\log 3 + \log\left(7^{x-2}\right) = \log 4 + \log\left(11^{13-x}\right)$$

Step 3
Apply the power law of logarithms.

$$\log 3 + \log\left(7^{x-2}\right) = \log 4 + \log\left(11^{13-x}\right)$$
$$\log 3 + (x-2)\log 7 = \log 4 + (13-x)\log 11$$

Step 4
Apply the distributive property.

$$\log 3 + (x-2)\log 7 = \log 4 + (13-x)\log 11$$
$$\log 3 + x\log 7 - 2\log 7 = \log 4 + 13\log 11 - x\log 11$$

Step 5
Isolate *x*.

$$\log 3 + x\log 7 - 2\log 7 = \log 4 + 13\log 11 - x\log 11$$
$$x\log 7 + x\log 11 = \log 4 + 13\log 11 - \log 3 + 2\log 7$$
$$x(\log 7 + \log 11) = \log 4 + 13\log 11 - \log 3 + 2\log 7$$
$$x = \frac{\log 4 + 13\log 11 - \log 3 + 2\log 7}{\log 7 + \log 11}$$

Step 6
Find an approximate solution using a calculator.

$$x = \frac{\log 4 + 13\log 11 - \log 3 + 2\log 7}{\log 7 + \log 11}$$
$$x = \frac{15.3532\ldots}{1.88649\ldots}$$
$$x \approx 8.1$$

Therefore, the value of *x* is approximately 8.1.

21. **Step 1**
Substitute 20 for P_o and 16 for *G* in the equation $G = 10\log\left(\frac{P_o}{P_i}\right)$.

$$G = 10\log\left(\frac{P_o}{P_i}\right)$$
$$16 = 10\log\left(\frac{20}{P_i}\right)$$

Step 2
Divide both sides of the equation by 10.

$$16 = 10\log\left(\frac{20}{P_i}\right)$$
$$\frac{16}{10} = \log\left(\frac{20}{P_i}\right)$$

Step 3
Write the equation $\frac{16}{10} = \log\left(\frac{20}{P_i}\right)$ in exponential form.

The equation in exponential form is $10^{\frac{16}{10}} = \frac{20}{P_i}$.

Step 4
Solve for P_i to the nearest tenth.

$$10^{\frac{16}{10}} = \frac{20}{P_i}$$
$$P_i = \frac{20}{10^{\frac{16}{10}}}$$
$$P_i = \frac{20}{39.8107\ldots}$$
$$P_i \approx 0.5$$

The input power is approximately 0.5 W.

22. **Step 1**
Substitute $965355I_r$ for *I* in the formula $m = \log\left(\frac{I}{I_r}\right)$.

$$m = \log\left(\frac{I}{I_r}\right)$$
$$m = \log\left(\frac{965355I_r}{I_r}\right)$$

Step 2
Determine the value of *m*.

$m = \log\left(\frac{965355 I_r}{I_r}\right)$
$m = \log 965355$
$m \approx 6.0$

The magnitude of the earthquake in City P was approximately 6.0 on the Richter scale.

23. The formula $\frac{I_1}{I_2} = 10^{m_1 - m_2}$ can be used to compare an earthquake with an intensity of I_1 with an earthquake with an intensity of I_2.

In the formula $\frac{I_1}{I_2} = 10^{m_1 - m_2}$, let $m_1 = 6.0$ and $m_2 = 3.4$, and then solve.

$\frac{I_1}{I_2} = 10^{m_1 - m_2}$
$\frac{I_1}{I_2} = 10^{6.0 - 3.4}$
$\frac{I_1}{I_2} = 10^{2.6}$
$\frac{I_1}{I_2} \approx 398.107$

Therefore, the earthquake in City P is approximately 398 times more intense than the earthquake in City Q.

TRIGONOMETRY

Lesson 1—Understanding Radian Measure

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Multiply 135° by $\frac{\pi \text{ rad}}{180°}$.

$135° \times \frac{\pi \text{ rad}}{180°}$
$= \frac{135°\pi}{180°} \text{rad}$
$= \frac{3\pi}{4} \text{rad}$

Converted to radians, 135° is $\frac{3\pi}{4}$ rad.

2. Multiply 0.85 rad by $\frac{180°}{\pi \text{ rad}}$.

$0.85 \text{ rad} \times \frac{180°}{\pi \text{ rad}}$
$= \frac{153°}{\pi}$
$\approx 48.7°$

Converted to degrees, 0.85 rad is approximately 48.7°.

3. **Step 1**
In the formula $a = r\theta$, substitute 101.2 for *a* and 18.5 for *r*.

$a = r\theta$
$101.2 = 18.5\theta$

Step 2
Solve for *θ*.

$101.2 = 18.5\theta$
$\frac{101.2}{18.5} = \theta$
$5.5 \approx \theta$

Therefore, the measure of the angle is approximately 5.5 rad.

4. **Step 1**
Determine the radian measure of the angle.

$135° \times \frac{\pi \text{ rad}}{180°}$
$= \frac{135\pi}{180} \text{rad}$
$= \frac{3\pi}{4} \text{rad}$

Step 2
Apply the formula $a = r\theta$ to find the radius.

$$\begin{aligned} a &= r\theta \\ 40.4 &= r\left(\frac{3\pi}{4}\right) \\ \frac{40.4(4)}{3\pi} &= r \\ 17.1 &\approx r \end{aligned}$$

Therefore, the radius is approximately 17.1 cm.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. Multiply $\frac{17\pi}{12}$ rad by $\frac{180°}{\pi \text{ rad}}$.

$$\frac{17\pi}{12}\text{rad} \times \frac{180°}{\pi \text{ rad}}$$
$$= 255°$$

Converted to degrees, $\frac{17\pi}{12}$ rad is 255°.

2. Multiply $\frac{2\pi}{3}$ rad by $\frac{180°}{\pi \text{ rad}}$.

$$\frac{2\pi}{3}\text{rad} \times \frac{180°}{\pi \text{ rad}}$$
$$= 120°$$

Converted to degrees, $\frac{2\pi}{3}$ rad is 120°.

3. Multiply $\frac{\pi}{8}$ rad by $\frac{180°}{\pi \text{ rad}}$.

$$\frac{\pi}{8}\text{rad} \times \frac{180°}{\pi \text{ rad}}$$
$$= 22.5°$$

Converted to degrees, $\frac{\pi}{8}$ rad is 22.5°.

4. Multiply 3π rad by $\frac{180°}{\pi \text{ rad}}$.

$$3\pi \text{ rad} \times \frac{180°}{\pi \text{ rad}}$$
$$= 540°$$

Converted to degrees, 3π rad is 540°.

5. Multiply $-\frac{9\pi}{2}$ rad by $\frac{180°}{\pi \text{ rad}}$.

$$-\frac{9\pi}{2}\text{rad} \times \frac{180°}{\pi \text{ rad}}$$
$$= -810°$$

Converted to degrees, $-\frac{9\pi}{2}$ rad is –810°.

6. Multiply 103° by $\frac{\pi \text{ rad}}{180°}$.

$$103° \times \frac{\pi \text{ rad}}{180°}$$
$$= \frac{103\pi}{180}\text{rad}$$

Converted to radians, 103° is $\frac{103\pi}{180}$ rad.

7. Multiply 40° by $\frac{\pi \text{ rad}}{180°}$.

$$40° \times \frac{\pi \text{ rad}}{180°}$$
$$= \frac{40\pi}{180}\text{rad}$$
$$= \frac{2\pi}{9}\text{rad}$$

Converted to radians, 40° is $\frac{2\pi}{9}$ rad.

8. Multiply 210° by $\frac{\pi \text{ rad}}{180°}$.

$$210° \times \frac{\pi \text{ rad}}{180°}$$
$$= \frac{210\pi}{180}\text{rad}$$
$$= \frac{7\pi}{6}\text{rad}$$

Converted to radians, 210° is $\frac{7\pi}{6}$ rad.

9. Multiply 315° by $\frac{\pi \text{ rad}}{180°}$.

$$315° \times \frac{\pi \text{ rad}}{180°}$$
$$= \frac{315\pi}{180}\text{rad}$$
$$= \frac{7\pi}{4}\text{rad}$$

Converted to radians, 315° is $\frac{7\pi}{4}$ rad.

10. Multiply –900° by $\frac{\pi \text{ rad}}{180°}$.

$$\begin{aligned}&-900°\times\frac{\pi \text{ rad}}{180°}\\&=\frac{900\pi}{180}\text{rad}\\&=-5\pi \text{ rad}\end{aligned}$$

Converted to radians, –900° is -5π rad.

11. Step 1

In the formula $a = r\theta$, substitute 12 for *a* and $\frac{\pi}{3}$ for *θ*.

$$\begin{aligned}a&=r\theta\\12&=r\left(\frac{\pi}{3}\right)\end{aligned}$$

Step 2
Solve for *r*.

$$\begin{aligned}12&=r\left(\frac{\pi}{3}\right)\\12(3)&=\pi r\\\frac{12(3)}{\pi}&=r\\11.5&\approx r\end{aligned}$$

Therefore, the radius of the circle is approximately 11.5 m.

12. Step 1
In the formula $a = r\theta$, substitute 130 for *a* and 40 for *r*.

$$\begin{aligned}a&=r\theta\\130&=40\theta\end{aligned}$$

Step 2
Solve for *θ*.

$$\begin{aligned}130&=40\theta\\\frac{130}{40}&=\theta\\3.25\text{ rad}&=\theta\end{aligned}$$

Step 3
Determine the degree measure of the angle.

$$\begin{aligned}&3.25\text{ rad}\times\frac{180°}{\pi \text{ rad}}=3.25\times\frac{180°}{\pi}\\&\approx 186.2°\end{aligned}$$

Therefore, the measure of the angle is approximately 186.2°.

13. Step 1
Determine the radian measure of the angle.

$$\begin{aligned}&57°\times\frac{\pi \text{ rad}}{180°}\\&=\frac{57\pi}{180}\text{rad}\\&=\frac{19\pi}{60}\text{rad}\end{aligned}$$

Step 2
Apply the formula $a = r\theta$ to find the radius.

$$\begin{aligned}a&=r\theta\\34&=r\left(\frac{19\pi}{60}\right)\\34(60)&=19\pi r\\\frac{34(60)}{19\pi}&=r\\34.2&\approx r\end{aligned}$$

Therefore, the radius is approximately 34.2 mm.

14. Step 1
Apply the formula $a = r\theta$ to determine the value of *θ*.

$$\begin{aligned}a&=r\theta\\3r&=r\theta\\3\text{ rad}&=\theta\end{aligned}$$

Step 2
Determine the degree measure of the angle.

$$\begin{aligned}&3\text{ rad}\times\frac{180°}{\pi \text{ rad}}\\&=3\times\frac{180°}{\pi}\\&\approx 171.9°\end{aligned}$$

Therefore the angle subtended by Tyler's walking path is approximately 171.9°.

15. Step 1
In the formula $a = r\theta$, substitute $(x-10)$ for *a*, 12 for *r*, and 2.4 for θ.

$$\begin{aligned}a&=r\theta\\x-10&=12(2.4)\end{aligned}$$

Step 2
Solve for *x*.

$$\begin{aligned}x-10&=12(2.4)\\x&=12(2.4)+10\\x&=28.8+10\\x&=38.8\end{aligned}$$

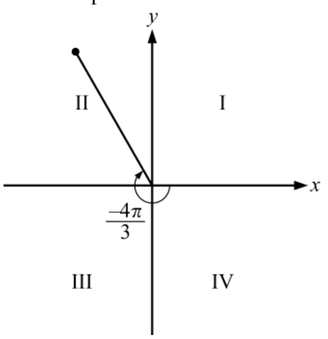
Therefore, the value of *x* is 38.8.

Lesson 2—Angles in Standard Position

CLASS EXERCISES
ANSWERS AND SOLUTIONS

1. A negative angle means the angle rotates in a clockwise direction. If $-\pi$ rad is equal to half a revolution in the clockwise direction, then $-\frac{4\pi}{3}$ rad is equal to more than half a revolution. You can rewrite $-\frac{4\pi}{3}$ as $-\pi+\left(-\frac{\pi}{3}\right)$, which results in half a revolution in the clockwise direction and then $\frac{1}{3}$ of half a revolution more.

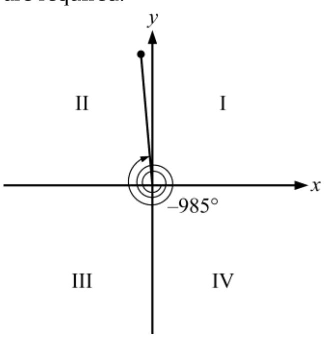
This sketch shows an angle of $-\frac{4\pi}{3}$ rad in standard position.



Therefore, $-\frac{4\pi}{3}$ rad terminates in quadrant II.

2. A negative angle means that the terminal arm rotates clockwise. A full rotation in the clockwise direction is equal to –360°.

Two full rotations in the clockwise direction equal –720°. Therefore, to reach –985°, a rotation of –720° and an additional $-985°-(-720°)=-265°$ are required.



Therefore, an angle of –985° terminates in quadrant II.

3. The general formula that gives all coterminal angles of any angle θ is of the form $(\theta+2\pi n)\text{rad}$, where θ is in radians, and n is any integer.

Therefore, all angles of the form $\left(\frac{7\pi}{6}+2\pi n\right)\text{rad}$ are coterminal with $\frac{7\pi}{6}$, where n is any integer.

4. For any angle θ, measured in degrees, all angles of the form $(\theta+360n)°$, $n\in I$, will be coterminal with angle θ. There is an infinite number of positive and negative coterminal angles with the angle 48°. In this case, the simplest calculation is given.

Step 1
Substitute 48° for θ and 1 for n in the expression $\theta+360°n$.

$$\begin{aligned}&\theta+360°n\\&=48°+360°(1)\\&=408°\end{aligned}$$

If $n=1$, the angle that is coterminal with 48° is 408°.

Step 2
Substitute 48° for θ and –1 for n in the expression $\theta+360°n$.

$$\begin{aligned}&\theta+360°n\\&=48°+360°(-1)\\&=48°-360°\\&=-312°\end{aligned}$$

If $n=-1$, the angle that is coterminal with 48° is –312°.

5. Find the length, r, using the Pythagorean theorem, $x^2+y^2=r^2$.

$$\begin{aligned}x^2+y^2&=r^2\\(-2)^2+5^2&=r^2\\4+25&=r^2\\29&=r^2\\\sqrt{29}&=r\end{aligned}$$

To find the exact values of, $\cos\theta$, and $\tan\theta$, substitute the known information.

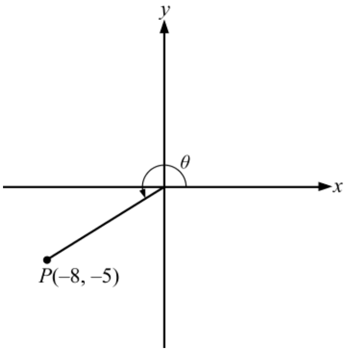
$\sin\theta=\frac{y}{r}$ $\quad\cos\theta=\frac{x}{r}$ $\quad\tan\theta=\frac{y}{x}$

$\sin\theta=\frac{5}{\sqrt{29}}$ $\quad\cos\theta=-\frac{2}{\sqrt{29}}$ $\quad\tan\theta=-\frac{5}{2}$

6. **Step 1**
Sketch a diagram representing θ.

It is given that $\tan\theta = \frac{5}{8}$, and the terminal arm is in quadrant III.

If $\tan\theta = \frac{y}{x}$, then possible values of x and y are $x = -8$ and $y = -5$.



Step 2
Determine the value of $\sin\theta$.

Since $\sin\theta = \frac{y}{r}$, the value of r must be determined using the Pythagorean theorem.

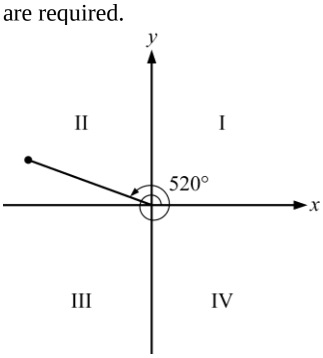
$$\begin{aligned} r^2 &= x^2 + y^2 \\ r^2 &= (-8)^2 + (-5)^2 \\ r^2 &= 64 + 25 \\ r^2 &= 89 \\ r &= \sqrt{89} \end{aligned}$$

Therefore, $\sin\theta = \frac{-5}{\sqrt{89}}$, or $\sin\theta = -\frac{5}{\sqrt{89}}$.

PRACTICE EXERCISES
ANSWERS AND SOLUTIONS

1. A positive angle means that the terminal arm rotates counterclockwise.

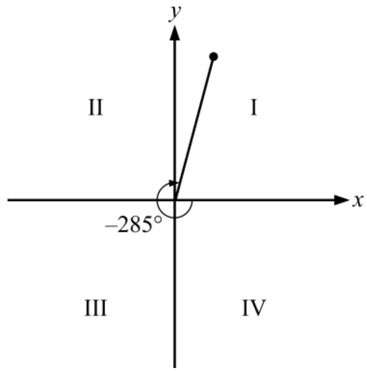
A full rotation is equal to 360º. To reach 520º, a rotation of 360º and an additional 520° – 360° = 160° are required.



Therefore, an angle of 520º terminates in quadrant II.

2. A negative angle means that the terminal arm rotates clockwise.

To reach –285º, a rotation of –270º and an additional –285° – (–270)° = –15° are required.



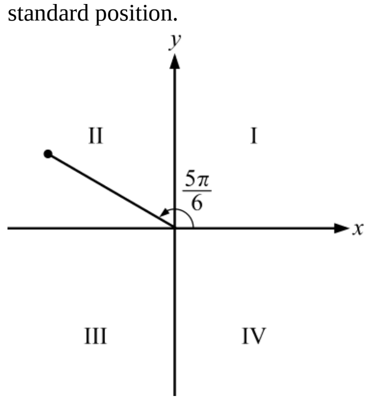
Therefore, an angle of –285º terminates in quadrant I.

3. A positive angle means the angle rotates in a counterclockwise direction.

If $\frac{\pi}{2}$ rad is equal to a quarter of a revolution, then $\frac{3\pi}{6}$ rad is also equal to a quarter of a revolution.

To reach $\frac{5\pi}{6}$ rad, a rotation of $\frac{3\pi}{6}$ rad and an additional $\frac{5\pi}{6} - \frac{3\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$ rad are required.

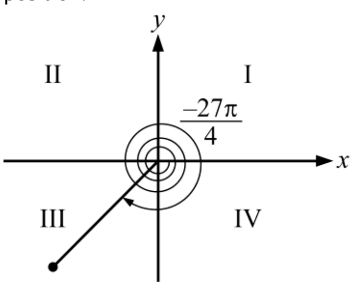
This sketch shows an angle of $\frac{5\pi}{6}$ rad in standard position.



Therefore, $\frac{5\pi}{6}$ rad terminates in quadrant II.

4. A negative angle means that the terminal arm rotates in a clockwise direction. If one revolution in the clockwise direction is equal to -2π rad, then $-\frac{8\pi}{4}$ rad also equals one revolution in the clockwise direction. Therefore, $-\frac{24\pi}{4}$ rad is equivalent to three revolutions in the clockwise direction. To reach $-\frac{27\pi}{4}$ rad, a rotation of $-\frac{24\pi}{4}$ rad and an additional $-\frac{27\pi}{4}-\left(-\frac{24\pi}{4}\right)=-\frac{3\pi}{4}\text{rad}$ are required.

This sketch shows an angle of $-\frac{27\pi}{4}$ rad in standard position.

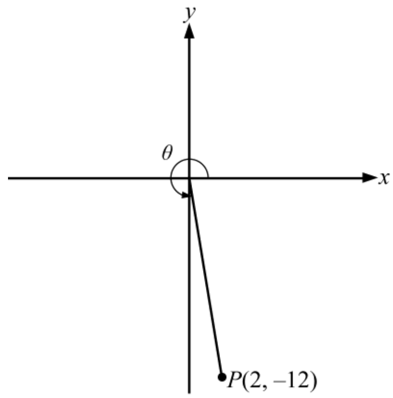


Therefore, $-\frac{27\pi}{4}$ rad terminates in quadrant III.

5. The general formula that gives all coterminal angles of θ is of the form $(\theta+2\pi n)$ rad, where θ is in radians, and n is any integer. Therefore, angles that are coterminal with $\frac{3\pi}{2}$ rad are given by the formula $\left(\frac{3\pi}{2}+2\pi n\right)$ rad, where $n \in I$.

6. The general formula that gives all coterminal angles of θ is of the form $(\theta+360n)°$, where θ is in degrees, and n is any integer. Therefore, angles that are coterminal with $-65°$ are given by the formula $(-65+360n)°$, or $(360n-65)°$, where $n \in I$.

7. Point P is given as $P(2,-12)$, so $x=2$ and $y=-12$.



Find the length, r, using the Pythagorean theorem, $x^2+y^2=r^2$.

$$\begin{aligned} x^2+y^2 &= r^2 \\ 2^2+(-12)^2 &= r^2 \\ 4+144 &= r^2 \\ 148 &= r^2 \\ \sqrt{148} &= r \\ 2\sqrt{37} &= r \end{aligned}$$

To find the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$, substitute the known information.

$$\sin\theta=\frac{y}{r} \qquad \cos\theta=\frac{x}{r} \qquad \tan\theta=\frac{y}{x}$$

$$\sin\theta=\frac{-12}{2\sqrt{37}} \qquad \cos\theta=\frac{2}{2\sqrt{37}} \qquad \tan\theta=\frac{-12}{2}$$

$$\sin\theta=-\frac{6}{\sqrt{37}} \qquad \cos\theta=\frac{1}{\sqrt{37}} \qquad \tan\theta=-6$$

8. From the given diagram, $x=-4$ and $y=10$.
Find the length, r, using the Pythagorean theorem, $x^2+y^2=r^2$.

$$\begin{aligned} x^2+y^2 &= r^2 \\ (-4)^2+10^2 &= r^2 \\ 16+100 &= r^2 \\ 116 &= r^2 \\ \sqrt{116} &= r \\ 2\sqrt{29} &= r \end{aligned}$$

To find the exact values of $\csc\theta$, $\sec\theta$, and $\cot\theta$, substitute the known information.

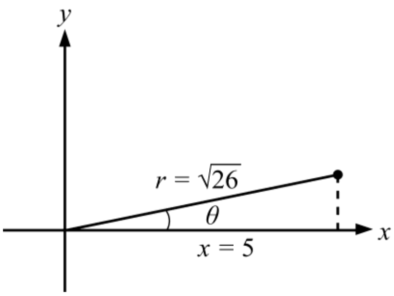
$$\csc\theta=\frac{r}{y} \qquad \sec\theta=\frac{r}{x} \qquad \cot\theta=\frac{y}{x}$$

$$\csc\theta=\frac{2\sqrt{29}}{10} \qquad \sec\theta=\frac{2\sqrt{29}}{-4} \qquad \cot\theta=\frac{10}{-4}$$

$$\csc\theta=\frac{\sqrt{29}}{5} \qquad \sec\theta=-\frac{\sqrt{29}}{2} \qquad \cot\theta=-\frac{5}{2}$$

9. **Step 1**
Sketch a diagram representing θ.

It is given that $\sec\theta = \frac{\sqrt{26}}{5}$, and the terminal arm is in quadrant I. If $\sec\theta = \frac{r}{x}$, then possible values of x and r are $x = 5$ and $r = \sqrt{26}$.



Step 2
Determine the value of $\tan\theta$.

Since $\tan\theta = \frac{y}{x}$, the value of y must be determined using the Pythagorean theorem.

$$\begin{aligned} r^2 &= x^2 + y^2 \\ \left(\sqrt{26}\right)^2 &= 5^2 + y^2 \\ 26 &= 25 + y^2 \\ 1 &= y^2 \\ \pm 1 &= y \end{aligned}$$

Since the tangent ratio is positive in quadrant I, $\tan\theta = \frac{1}{5}$.

10. Since $\cot\theta = \frac{x}{y}$ and it is given that $r = \sqrt{34}$ and $y = 5$, determine the value of x using the Pythagorean theorem.

$$\begin{aligned} r^2 &= x^2 + y^2 \\ \left(\sqrt{34}\right)^2 &= x^2 + 5^2 \\ 34 &= x^2 + 25 \\ 9 &= x^2 \\ 3 &= x \end{aligned}$$

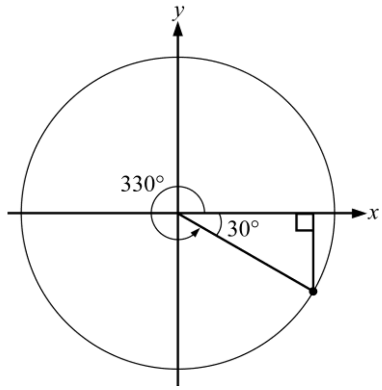
Therefore, $\cot\theta = \frac{3}{5}$.

Lesson 3—The Unit Circle

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Define the angle 330° in the unit circle using a special triangle.

The angle 330° corresponds to a special triangle with a reference angle of 30° in quadrant IV.



Step 2
Find the exact value of $\cos 330°$.

In a 30-60-90 special triangle, the side adjacent to 30° has a length of $\frac{\sqrt{3}}{2}$. Since the 330° is in quadrant IV, it follows that the exact value of $\cos 330°$ is $\frac{\sqrt{3}}{2}$.

Step 3
Find the exact value of $\sin 330°$.

In a 30-60-90 special triangle, the side opposite to 30° has a length of $\frac{1}{2}$. Since the angle is in quadrant IV, it follows that the exact value of $\sin 330°$ is $-\frac{1}{2}$.

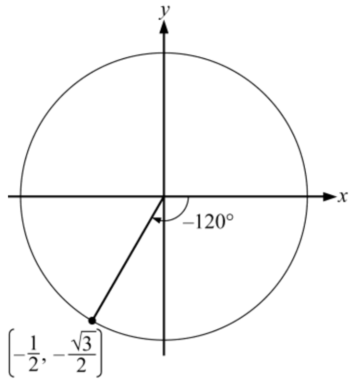
Step 4
Find the exact value of $\tan 330°$.

The tangent ratio of the side lengths of the special triangle is $\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$.

Since the angle is in quadrant IV, it follows that the exact value of $\tan 330°$ is $-\frac{1}{\sqrt{3}}$.

2. **Step 1**
Locate the terminal arm of –120° on the unit circle.

The angle –120° terminates in the quadrant III and has a reference angle of 60°. Therefore, the coordinates on the terminal arm are $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$.



Step 2
Find the exact value of csc (–120°).
Since $\csc\theta=\frac{1}{y}$ on the unit circle and the *y*-coordinate of the ordered pair $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$ is $-\frac{\sqrt{3}}{2}$, it follows that the exact value of $\csc(-120°)$ is $-\frac{2}{\sqrt{3}}$.

Step 3
Find the exact value of sec (–120°).
Since $\sec\theta=\frac{1}{x}$ on the unit circle and the *x*-coordinate of the ordered pair $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$ is $-\frac{1}{2}$, it follows that the exact value of $\sec(-120°)$ is –2.

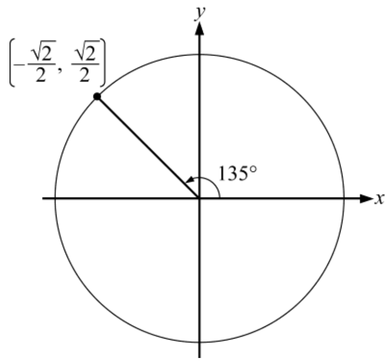
Step 4
Find the exact value of cot(–120°).
In the unit circle, $\cot\theta=\frac{x}{y}$. Therefore, the exact value of cot(–120°) is determined as follows:

$$\cot(-120°) = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Locate the terminal arm of 135° on the unit circle.

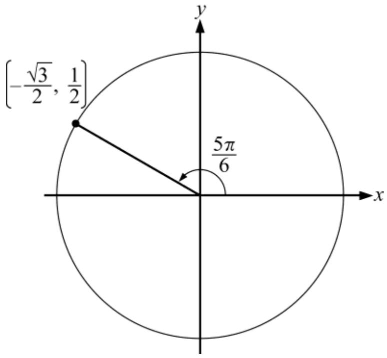
The angle 135° terminates in the quadrant II and has a reference angle of 45º. Therefore, the coordinates on the terminal arm are $\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$.



Step 2
Find the exact value of sin 135°.
Since $\sin\theta=y$ and the *y*-coordinate of the ordered pair $\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$ is $\frac{\sqrt{2}}{2}$, it follows that the exact value of $\sin 135°$ is $\frac{\sqrt{2}}{2}$.

2. **Step 1**
Locate the terminal arm of $\frac{5\pi}{6}$ rad on the unit circle.
The angle $\frac{5\pi}{6}$ rad terminates in the quadrant II and has a reference angle of $\frac{\pi}{6}$ rad.

Therefore, the coordinates on the terminal arm are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.



Step 2

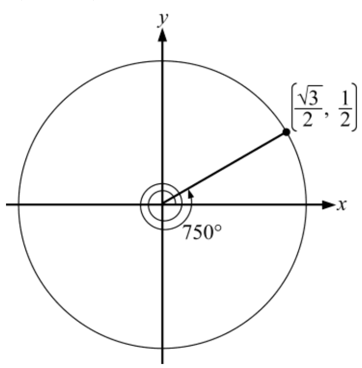
Find the exact value of $\sin\left(\frac{5\pi}{6}\right)$.

Since $\sin\theta = y$ and the *y*-coordinate of the ordered pair $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ is $\frac{1}{2}$, it follows that the exact value of $\sin\left(\frac{5\pi}{6}\right)$ is $\frac{1}{2}$.

3. **Step 1**

Locate the terminal arm of 750º on the unit circle.
The angle 750º terminates in the quadrant I and has a reference angle of 30º.

Therefore, the coordinates on the terminal arm are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.



Step 2

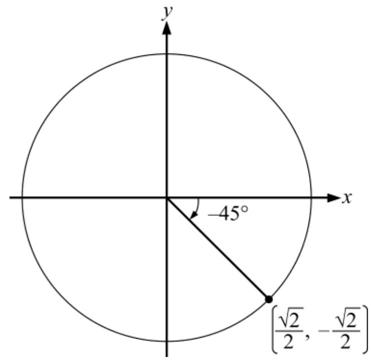
Find the exact value of $\csc 750°$.

Since $\csc\theta = \frac{1}{y}$ and the *y*-coordinate of the ordered pair $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ is $\frac{1}{2}$, it follows that the exact value of $\csc 750°$ is 2.

4. **Step 1**

Locate the terminal arm of –45° on the unit circle.

The angle –45° terminates in the quadrant IV and has a reference angle of 45º. Therefore, the coordinates on the terminal arm are $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.



Step 2

Find the exact value of $\cos(-45°)$.

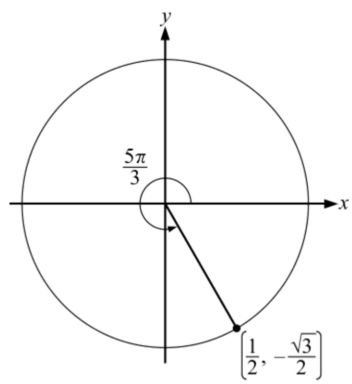
Since $\cos\theta = x$ and the *y*-coordinate of the ordered pair $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ is $\frac{\sqrt{2}}{2}$, it follows that the exact value of $\cos(-45°)$ is $\frac{\sqrt{2}}{2}$.

5. **Step 1**

Locate the terminal arm of $\frac{5\pi}{3}$ rad on the unit circle.

The angle $\frac{5\pi}{3}$ rad terminates in the quadrant IV and has a reference angle of $\frac{\pi}{3}$.

Therefore, the coordinates on the terminal arm are $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.



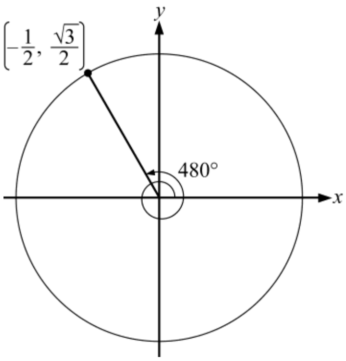
Step 2

Find the exact value of $\sec\left(\frac{5\pi}{3}\right)$.

Since $\sec\theta = \frac{1}{x}$ on the unit circle and the *x*-coordinate of the ordered pair $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ is $\frac{1}{2}$, it follows that the exact value of $\sec\left(\frac{5\pi}{3}\right)$ is 2.

6. **Step 1**
Locate the terminal arm of 480º on the unit circle.

The angle 480º terminates in the quadrant II and has a reference angle of 60º. Therefore, the coordinates on the terminal arm are $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.



Step 2
Find the exact value of $\cot 480°$.

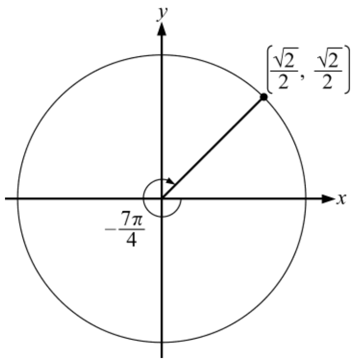
In the unit circle, $\cot\theta = \frac{x}{y}$. Therefore, the exact value of $\cot 480°$ is determined as follows:

$$\begin{aligned}&\cot 480° \\ &= \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} \\ &= -\frac{1}{\sqrt{3}}\end{aligned}$$

7. **Step 1**

Locate the terminal arm of $-\frac{7\pi}{4}$ rad on the unit circle.

The angle $-\frac{7\pi}{4}$ rad terminates in the quadrant I and has a reference angle of $\frac{\pi}{4}$. Therefore, the coordinates on the terminal arm are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.



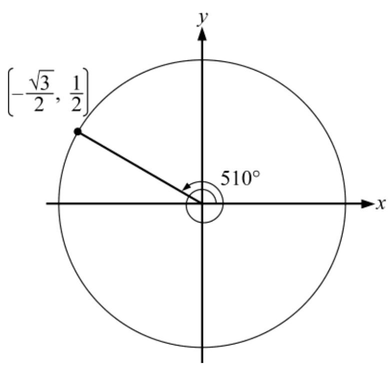
Step 2

Find the exact value of $\cos\left(-\frac{7\pi}{4}\right)$.

Since $\cos\theta = x$ and the *x*-coordinate of the ordered pair $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ is $\frac{\sqrt{2}}{2}$, it follows that the exact value of $\cos\left(-\frac{7\pi}{4}\right)$ is $\frac{\sqrt{2}}{2}$.

8. **Step 1**
Locate the terminal arm of 510º on the unit circle.

The angle 510º terminates in the quadrant II and has a reference angle of 30º. Therefore, the coordinates on the terminal arm are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.



Step 2
Find the exact value of $\tan 510°$.

In the unit circle, $\tan\theta = \frac{y}{x}$. Therefore, the exact value of $\tan 510°$ is determined as follows:

$$\begin{aligned}&\tan 510°\\&=\frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}}\\&=-\frac{1}{\sqrt{3}}\end{aligned}$$

Lesson 4—Graphing Trigonometric Functions

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. The amplitude is $|a|$. The graph shows that the amplitude is 6. Since the graph is reflected in the *x*-axis, *a* is negative. Therefore the value of *a* is –6.

2. The equation $y=\sin\left(-\frac{1}{5}x\right)$ is of the form $y=\sin bx$.

Therefore, the period is determined as follows:

$$\begin{aligned}\text{period}&=\frac{360°}{|b|}\\\text{period}&=\frac{360°}{\left|-\frac{1}{5}\right|}\\\text{period}&=\frac{360°}{\frac{1}{5}}\\\text{period}&=1\ 800°\end{aligned}$$

Therefore, the period of the function $y=\sin\left(-\frac{1}{5}x\right)$ is 1 800º.

3. If the graph of $y=\sin x$ is translated $\frac{5\pi}{6}$ rad to the right, the value of *c* is $\frac{5\pi}{6}$, and the new equation is $y=\sin\left(x-\frac{5\pi}{6}\right)$.

Since the *x*-intercepts of $y=\sin x$ are given by $x=\pi n$ rad, where *n* is any integer, the *x*-intercepts of $y=\sin\left(x-\frac{5\pi}{6}\right)$ increase by $\frac{5\pi}{6}$ rad and are given by $x=\left(\pi n+\frac{5\pi}{6}\right)\text{rad}$.

4. The equation of the horizontal midline axis is $y=d$. The graph shows that the equation of the horizontal midline axis is $y=6$. Therefore, the value of *d* is 6.

5. **Step 1**
Rewrite the function in the form $y=a\cos\left[b(x-c)\right]+d$.

Factor out 3 from the expression in the brackets.

$$\begin{aligned}y&=-2\cos(3x+\pi)+6\\y&=-2\cos\left[3\left(x+\frac{\pi}{3}\right)\right]+6\end{aligned}$$

Step 2
Determine the amplitude.

The value of *a* is –2. Therefore, the amplitude is $|-2|=2$.

Step 3
Determine the period.

The value of *b* is 3. Therefore, the period is $\frac{2\pi}{|b|}=\frac{2\pi}{3}$.

Step 4
Determine the phase shift.

The value of *c* is $-\frac{\pi}{3}$. Therefore, the phase shift is $\frac{\pi}{3}$ rad to the left.

Step 5
Determine the horizontal midline axis.

The value of *d* is 6. Therefore, the equation of horizontal midline axis is $y=6$.

Step 6
Determine the range.

Since the horizontal midline axis is 6 and the amplitude is 2, the maximum value is $6+2=8$. Similarly, the minimum value is $6-2=4$. Therefore, the range is $4\le y\le 8$.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**

Factor out $\frac{1}{3}$ from each term in the brackets.

$$y=\sin\left(\frac{1}{3}x+\frac{\pi}{4}\right)$$
$$y=\sin\left[\frac{1}{3}\left(x+\frac{3\pi}{4}\right)\right]$$

Step 2
Determine how to obtain the graph of $y=\sin\left[\frac{1}{3}\left(x+\frac{3\pi}{4}\right)\right]$ from the graph of $y=\sin x$.

The equation $y=\sin\left[\frac{1}{3}\left(x+\frac{3\pi}{4}\right)\right]$ is of the form $y=\sin\left[b\left(x-c\right)\right]$, where $b=\frac{1}{3}$ and $c=-\frac{3\pi}{4}$.

Therefore, the graph of $y=\sin\left[\frac{1}{3}\left(x+\frac{3\pi}{4}\right)\right]$ is obtained from the graph of $y=\sin x$ by a horizontal stretch factor of 3 and a phase shift $\frac{3\pi}{4}$ rad to the left.

2. **Step 1**
Determine how to obtain the graph of $y=-4\sin x+1$ from the graph of $y=\sin x$.

The equation $y=-4\sin x+1$ is of the form $y=a\sin x+d$, where $a=-4$ and $d=1$. Therefore, the graph of $y=-4\sin x+1$ is obtained from $y=\sin x$ by a vertical stretch factor of 4, a reflection in the x-axis, and a vertical translation 1 unit up.

Step 2
Apply the vertical stretch.

Points on the graph of $y=\cos x$ are (0, 0), $\left(\frac{\pi}{2},1\right)$, ($\pi$, 0), $\left(\frac{3\pi}{2},-1\right)$, and (2$\pi$, 0).

Multiply the y-coordinates by 4.

$$\left(0,0\times4\right)\rightarrow\left(0,0\right)$$
$$\left(\frac{\pi}{2},1\times4\right)\rightarrow\left(\frac{\pi}{2},4\right)$$
$$\left(\pi,0\times4\right)\rightarrow\left(\pi,0\right)$$
$$\left(\frac{3\pi}{2},-1\times4\right)\rightarrow\left(\frac{3\pi}{2},-4\right)$$
$$\left(2\pi,0\times4\right)\rightarrow\left(2\pi,0\right)$$

Step 3
Apply the vertical reflection.

Multiply the y-coordinates in step 2 by –1.

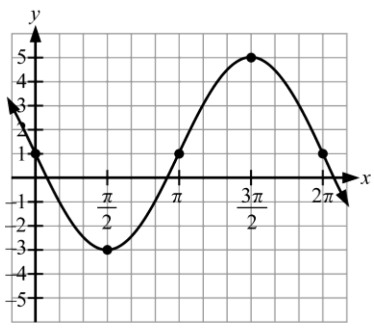
$$\left(0,0\times-1\right)\rightarrow\left(0,0\right)$$
$$\left(\frac{\pi}{2},4\times-1\right)\rightarrow\left(\frac{\pi}{2},-4\right)$$
$$\left(\pi,0\times-1\right)\rightarrow\left(\pi,0\right)$$
$$\left(\frac{3\pi}{2},-4\times-1\right)\rightarrow\left(\frac{3\pi}{2},4\right)$$
$$\left(2\pi,0\times-1\right)\rightarrow\left(2\pi,0\right)$$

Step 4
Apply the vertical translation.

Increase the y-coordinates in step 3 by 1.

$$\left(0,0+1\right)\rightarrow\left(0,1\right)$$
$$\left(\frac{\pi}{2},-4+1\right)\rightarrow\left(\frac{\pi}{2},-3\right)$$
$$\left(\pi,0+1\right)\rightarrow\left(\pi,1\right)$$
$$\left(\frac{3\pi}{2},4+1\right)\rightarrow\left(\frac{3\pi}{2},5\right)$$
$$\left(2\pi,0+1\right)\rightarrow\left(2\pi,1\right)$$

Step 5
Plot and join the transformed points.



3. The first maximum of the graph of $y=\sin x$ is (90°, 1). The closest maximum on the graph of $y=\cos x$ to the point (90°, 1) is (0°, 1) on the right. Therefore, the graph of $y=\sin x$ is obtained from the graph of $y=\cos x$ by a phase shift $\left|90°-0°\right|=90°$ to the right.

4. The function $y = -12\sin x + 2$ is of the form $y = a\sin x + d$, where $a = -12$ and $d = 2$.

Therefore, the graph of the function has a horizontal midline axis at $y = 2$ and an amplitude of $|-12| = 12$.

The maximum value is $2 + 12 = 14$, and the minimum value is $2 - 12 = -10$.

Therefore, the range is $-10 \le y \le 14$, and the domain is $x \in R$.

5. The function $y = -\frac{2}{5}\sin(x + 15°) + 4$ is of the form $y = a\sin(x - c) + d$, where $a = -\frac{2}{5}$, $c = -15°$, and $d = 4$.

Step 1
Determine the amplitude.

The value of *a* is $-\frac{2}{5}$. Therefore, the amplitude is $\left|-\frac{2}{5}\right| = \frac{2}{5}$.

Step 2
Determine the period.
The value of *b* is 1. Therefore, the period is $\frac{360°}{|1|} = 360°$.

Step 3
Determine the phase shift.

The value of *c* is –15º. Therefore, the phase shift is 15º left.

Step 4
Determine the horizontal midline axis.

The value of *d* is 4. Therefore, the equation of horizontal midline axis is $y = 4$.

Step 5
Determine the range.

Since the horizontal midline axis is 4 and the amplitude is $\frac{2}{5}$, the maximum value is $4 + \frac{2}{5} = \frac{22}{5}$. Similarly, the minimum value is $4 - \frac{2}{5} = \frac{18}{5}$. Therefore, the range is $\frac{22}{5} \le y \le \frac{18}{5}$.

6. **Step 1**
Rewrite the function in form $y = a\cos[b(x - c)] + d$.
Factor out 2 from each term in the brackets.

$$y = 3\cos\left(2x + \frac{4\pi}{3}\right) - 13$$
$$y = 3\cos\left[2\left(x + \frac{2\pi}{3}\right)\right] - 13$$

The equation $y = 3\cos\left[2\left(x + \frac{2\pi}{3}\right)\right] - 13$ is of the form $y = a\cos[b(x - c)] + d$, where $a = 3$, $b = 2$, $c = -\frac{2\pi}{3}$, and $d = -13$.

Step 2
Determine the amplitude.

The value of *a* is 3. Therefore, the amplitude is 3.

Step 3
Determine the period.

The value of *b* is 2. Therefore, the period is $\frac{360°}{|2|} = 180°$.

Step 4
Determine the phase shift.

The value of *c* is $-\frac{2\pi}{3}$. Therefore, the phase shift is left $\frac{2\pi}{3}$ rad.

Step 5
Determine the horizontal midline axis.
The value of *d* is –13. Therefore, the equation of horizontal midline axis is $y = -13$.

Step 6
Determine the range.

Since the horizontal midline axis is –13 and the amplitude is 3, the maximum value is $-13 + 3 = -10$. Similarly, the minimum value is $-13 - 3 = -16$. Therefore, the range is $-10 \le y \le -16$.

7. **Step 1**
Apply the vertical stretch.

Replace *y* with $\frac{1}{3}y$ in the equation $y = \cos x$.

$$y = \cos x$$
$$\frac{1}{3}y = \cos x$$

Step 2
Apply the horizontal phase shift.

Replace *x* with $\left(x+\frac{\pi}{3}\right)$ in the equation $\frac{1}{3}y=\cos x$.

$$\frac{1}{3}y=\cos x$$
$$\frac{1}{3}y=\cos\left(x+\frac{\pi}{3}\right)$$

Step 3
Apply the vertical translation.

Replace *y* with $(y-2)$ in the equation $\frac{1}{3}y=\cos\left(x+\frac{\pi}{3}\right)$.

$$\frac{1}{3}y=\cos\left(x+\frac{\pi}{3}\right)$$
$$\frac{1}{3}(y-2)=\cos\left(x+\frac{\pi}{3}\right)$$

Step 4
Solve for *y*.

$$\frac{1}{3}(y-2)=\cos\left(x+\frac{\pi}{3}\right)$$
$$y-2=3\cos\left(x+\frac{\pi}{3}\right)$$
$$y=3\cos\left(x+\frac{\pi}{3}\right)+2$$

Therefore, the equation of the transformed equation is $y=3\cos\left(x+\frac{\pi}{3}\right)+2$.

8. **Step 1**
Determine the amplitude.

The value of *a* is 3. Therefore, the amplitude is 3.

Step 2
Determine the horizontal midline axis.

The value of *d* is 2. Therefore, the equation of horizontal midline axis is $y=2$.

Step 3
Determine the period.

The value of *b* is 1. Therefore, the period is $\frac{2\pi}{|1|}=2\pi$.

Step 4
Determine the domain and range.

The domain is $x\in R$.

Since the horizontal midline axis is 2 and the amplitude is 3, the maximum value is $2+3=5$. Similarly, the minimum value is $2-3=-1$. Therefore, the range is $-1\le y\le 5$.

9. **Step 1**
Determine the value of *a*.

The graph has a maximum of 8 and a minimum of 2, so the amplitude is $\frac{8-2}{2}=3$. Therefore, the value of *a* is 3.

Step 2
Determine the value of *b*.

The value of *b* is determined using the formula $|b|=\frac{2\pi}{\text{period}}$.

Since the period of the graph is 8, the value of *b* is determined as follows:

$$|b|=\frac{2\pi}{\text{period}}$$
$$|b|=\frac{2\pi}{8}$$
$$|b|=\frac{\pi}{4}$$

Step 3
Determine the value of *d*.

The graph has a maximum of 8 and a minimum of 2, so the horizontal midline axis is $y=\frac{8+2}{2}=5$. Therefore, the value of *d* is 5.

Step 4
Determine the function that represents the graph.
The function that represents the graph is

$$y=3\sin\left(\frac{\pi}{4}x\right)+5.$$

Lesson 5—Solving Problems Involving Sinusoidal Functions

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. a) Step 1
Determine the domain of the graph.

The graph shows the movement of the tide from 0 s to 32 s. Therefore, the domain is $0 \le t \le 32$.

Step 2
Determine the range of the graph.

The range is the height values between the maximum and minimum values of the graph.

The maximum value is 8 m, and the minimum value is 2 m. Therefore, the range is $2 \le h \le 8$.

b) The first maximum on the graph of the function is located at (7,8). Therefore, it
takes 7 s for the tidal wave to reach its maximum height.

c) The graph shows a period of 26 s.

d) Step 1
Determine the value of *a*.

The graph has a maximum of 8 m and a minimum of 2 m, so the amplitude is $\frac{8-2}{2} = 3$.

Therefore, the value of *a* is 3.

Step 2
Determine the value of *b*.

The value of *b* is determined using the formula $|b| = \frac{2\pi}{\text{period}}$.

Since the period of the graph is 26 s, the value of *b* is determined as follows:

$$|b| = \frac{2\pi}{\text{period}}$$
$$|b| = \frac{2\pi}{26}$$
$$|b| = \frac{\pi}{13}$$
$$b = \frac{\pi}{13}$$

Step 3
Determine the value of *d*.

The horizontal midline axis is located at $y = \frac{8+2}{2} = 5$.

Therefore, the value of *d* is 5.

Step 4
Determine the equation.

Substituting the values of *a*, *b*, and *d* into $h(t) = a\sin bt + d$, the equation can be expressed as $h(t) = 3\sin\left(\frac{\pi}{13}t\right) + 5$.

2. a) Step 1
Clear all lists in the calculator.

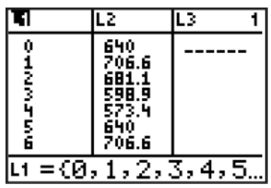
Press 2ND + . Select 4:ClrAllLists, and press ENTER to clear any previously entered data.

Step 2
Enter the data values into the calculator.

Press STAT, and select 1:Edit….

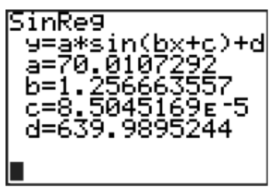
Enter the values for time in L1 and the values for frequency in L2.

After you have entered all the data values, you should see the following window.



Step 3
Perform the sinusoidal regression.

Press STAT ▷ to highlight CALC.
Select C:SinReg, and press ENTER to obtain the following window.



Step 4
Interpret the information, and write the equation of the trigonometric function.

For the data entered, $a \approx 70.01$, $b \approx 1.26$, $c \approx 8.50$, and $d \approx 639.99$. Therefore, the trigonometric equation of the function that best approximates the data is $y = 70.01\sin(1.26x + 8.50) + 639.99$, where y is frequency of the siren after x seconds.

b) An equivalent form of
$y = 70.01\sin(1.26x + 8.50) + 639.99$ is
$y = 70.01\sin(1.26(x + 6.75)) + 639.99$.

The value of b in this function is 1.26.

Therefore, the period is calculated as follows.

$$\text{period} = \frac{2\pi}{|b|}$$
$$\text{period} = \frac{2\pi}{|1.26|}$$
$$\text{period} \approx 4.99$$

The period is approximately 5.0 s.

c) Use the function
$y = 70.01\sin(1.26x + 8.50) + 639.99$ to find the value of y when $x = 7$. Make sure the calculator is in radian mode.

$$y = 70.01\sin(1.26x + 8.50) + 639.99$$
$$y = 70.01\sin(1.26(7) + 8.50) + 639.99$$
$$y \approx 570.0$$

Therefore, the frequency of the siren at 7 s is 570.0 Hz.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. The graph starts to repeat itself when $x = 360°$. Therefore, the period is 360°.

2. **Step 1**
Determine the measurement of θ after $1\frac{1}{2}$ revolutions.

After making $1\frac{1}{2}$ revolutions, the rock has passed through $360° + \frac{1}{2}(360°) = 540°$.

Step 2
Determine the height of the rock after a 540° rotation.

According to the graph, after a 540° rotation, the height of the rock is approximately 28 cm. Therefore, the height of the rock after $1\frac{1}{2}$ revolutions is 28 cm.

3. **Step 1**
Determine the maximum height of the piston head.

In the function $h = 5\sin(157.08t) + 15$, the amplitude is 5, and horizontal midline axis is $y = 15$. Therefore, the maximum height of the piston head is $15 + 5 = 20\text{ cm}$.

Step 2
Determine the minimum height of the piston head.

The minimum value of the function is determined by subtracting the amplitude from the value of the horizontal midline axis. Therefore, the minimum height of the piston head is $15 - 5 = 10\text{ cm}$.

4. The time it takes for the piston to go up and down through one cycle within the cylinder is equal to the period of the function. The value of b in the function $h = 5\sin(157.08t) + 15$ is 157.08.

Therefore, the period can be calculated as follows.

$$\text{period} = \frac{2\pi}{|b|}$$
$$\text{period} = \frac{2\pi}{|157.08|}$$
$$\text{period} = \frac{2\pi}{|157.08|}$$
$$\text{period} \approx 0.04$$

Therefore, it takes approximately 0.04 s for the piston to go up and down through one cycle within the cylinder.

5. One cycle takes 0.04 s. The number of cycles made in 1 h, or $60 \times 60 = 3600\text{ s}$, can be found as follows:

$$\frac{3600\text{ s}}{x} = \frac{0.04\text{ s}}{1\text{ cycle}}$$
$$\frac{(3600\text{ s})(1\text{ cycle})}{0.04\text{ s}} = x$$
$$90\,000\text{ cycles} = x$$

The piston makes 90 000 complete cycles in one hour.

6. **Step 1**
Determine the maximum height of the rider.

In the function $h(t) = 4\sin\left(\frac{\pi}{2}(t-6)\right)+6$, the amplitude is 4, and horizontal midline axis is $y = 6$.

Therefore, the maximum height of the ferris wheel is $6+4=10\text{ m}$.

Step 2
Determine the minimum height of the rider.
The minimum value of the function is determined by subtracting the amplitude from the value of the horizontal midline axis. Therefore, the minimum height of the ferris wheel is $6-4=2\text{ m}$.

7. One complete revolution is equivalent to the period for the given function.

In the function $h(t) = 4\sin\left(\frac{\pi}{2}(t-6)\right)+6$,

the value of b is $\frac{\pi}{12}$.

Therefore, the period is calculated as follows:

$$\begin{aligned}
\text{period} &= \frac{2\pi}{|b|}\\
\text{period} &= \frac{2\pi}{\left|\frac{\pi}{12}\right|}\\
\text{period} &= 2\pi\times\frac{12}{\pi}\\
\text{period} &= 24
\end{aligned}$$

It takes 24 s for the ferris wheel to make one complete revolution.

8. To determine the height of the rider after 13 s, substitute 13 for t in the given function, and solve for $h(t)$.

$$\begin{aligned}
h(t) &= 4\sin\left(\frac{\pi}{2}(t-6)\right)+6\\
h(13) &= 4\sin\left(\frac{\pi}{2}(13-6)\right)+6\\
h(13) &= 4\sin\left(\frac{\pi}{2}(7)\right)+6\\
h(13) &= 4\sin\left(\frac{7\pi}{2}\right)+6\\
h(13) &= 2
\end{aligned}$$

Therefore, the height of the rider after 13 s is 2 m.

9. **Step 1**
Clear all lists in the calculator.

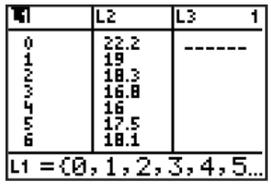
Press $\boxed{\text{2nd}}$ $\boxed{+}$. Select 4:ClrAllLists, and press $\boxed{\text{ENTER}}$ to clear any previously entered data.

Step 2
Enter the data values into the calculator.

Press $\boxed{\text{STAT}}$, and select 1:Edit….

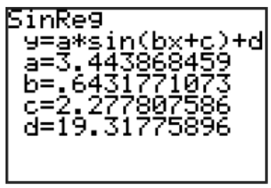
Enter the values for time in L1 and the values for pressure in L2.

After you have entered all the data values, you should obtain the following window.



Step 3
Perform the sinusoidal regression.

Press $\boxed{\text{STAT}}$ $\boxed{\triangleright}$ to highlight CALC.
Select C:SinReg, and press $\boxed{\text{ENTER}}$ to obtain the following window.



Step 4
Interpret the information, and write the equation of the trigonometric function.

For the data entered, $a \approx 3.42$, $b \approx 0.64$, $c \approx 2.29$, and $d \approx 19.31$. Therefore, the equation of the trigonometric function that best approximates the data is $y = 3.44\sin(0.64x+2.29)+19.31$, where y is the pressure in PSI over a period of time, x, in hours.

10. To determine the water pressure after 17 h, substitute 17 for x in the function $y = 3.44\sin(0.64x+2.29)+19.31$, and solve for y.

Make sure your calculator is in radian mode.

$$\begin{aligned}
y &= 3.44\sin(0.64x+2.29)+19.31\\
y &= 3.44\sin(0.64(17)+2.29)+19.31\\
y &= 3.44\sin(13.17)+19.31\\
y &\approx 21.3
\end{aligned}$$

The pressure after 17 h is 21.3 PS1.

Lesson 6—Solving Trigonometric Equations Algebraically

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Step 1

Isolate the trigonometric ratio in the equation.

$$4\cos^2\theta = 3$$
$$\cos^2\theta = \frac{3}{4}$$
$$\sqrt{\cos^2\theta} = \sqrt{\frac{3}{4}}$$
$$\cos\theta = \pm\frac{\sqrt{3}}{2}$$

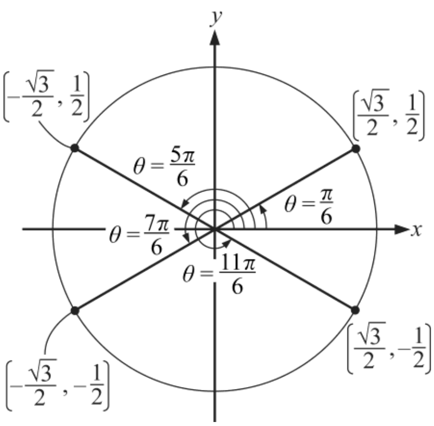
Step 2

Determine the exact values of θ in the equation $\cos\theta = \pm\frac{\sqrt{3}}{2}$, where $0 \le \theta \le 2\pi$.

Since $\cos\theta = x$ and it is given that $\cos\theta = \pm\frac{\sqrt{3}}{2}$, determine the angles on the unit circle where the x-coordinate of its point on the terminal arm is equal to $-\frac{\sqrt{3}}{2}$ or $\frac{\sqrt{3}}{2}$.

The cosine ratio is $\frac{\sqrt{3}}{2}$ in quadrants I and IV.

The cosine ratio is $-\frac{\sqrt{3}}{2}$ in quadrants II and III.



The exact values of θ in the equation $4\cos^2\theta = 3$ are $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, and $\frac{11\pi}{6}$.

2. Step 1

Bring all terms to one side of the equation.

$$\sqrt{3}\tan^2\theta - \tan\theta = -\sqrt{3}\tan x + 1$$
$$\sqrt{3}\tan^2\theta + \sqrt{3}\tan x - \tan\theta - 1 = 0$$

Step 2

Factor the trigonometric equation, and apply the zero product property to each factor.

$$\sqrt{3}\tan^2\theta + \sqrt{3}\tan x - \tan\theta - 1 = 0$$
$$\sqrt{3}\tan\theta(\tan\theta + 1) - 1(\tan + 1) = 0$$
$$(\tan\theta + 1)(\sqrt{3}\tan\theta - 1) = 0$$

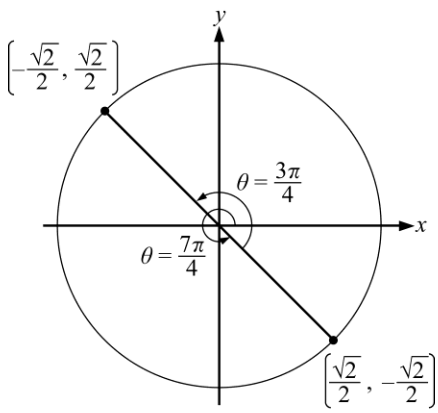
$$\tan\theta + 1 = 0 \qquad \sqrt{3}\tan\theta - 1 = 0$$
$$\tan\theta = -1 \qquad \sqrt{3}\tan\theta = 1$$
$$\tan\theta = \frac{1}{\sqrt{3}}$$

Step 3

Determine the exact values of θ in the equation $\tan\theta = -1$, where $0 < \theta < 2\pi$.

Identify angles on the unit circle where the ratio $\frac{y}{x}$ of its point is equal to –1.

The tangent ratio is –1 in quadrants II and IV.

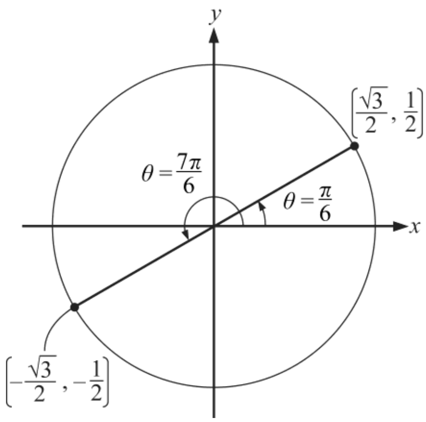


The exact values of θ in the equation $\tan\theta = -1$ are $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

Step 4
Determine the exact values of θ in the equation $\tan\theta = \frac{1}{\sqrt{3}}$, where $0 < \theta < 2\pi$.

Identify angles on the unit circle where the ratio $\frac{y}{x}$ of its point is equal to $\frac{1}{\sqrt{3}}$.

The tangent ratio is $\frac{1}{\sqrt{3}}$ in quadrants I and III.



The exact values of θ in the equation $\tan\theta = \frac{1}{\sqrt{3}}$ are $\frac{\pi}{6}$ and $\frac{7\pi}{6}$.

Therefore, the exact values of θ in the equation $\sqrt{3}\tan^2\theta - \tan\theta = -\sqrt{3}\tan x + 1$, where $0 < \theta < 2\pi$, are $\frac{\pi}{6}$, $\frac{3\pi}{4}$, $\frac{7\pi}{6}$, and $\frac{7\pi}{4}$.

3. Step 1

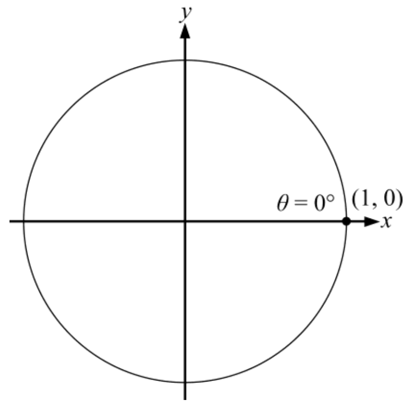
Factor the trigonometric equation, and apply the zero product property to each factor.

$$\tan^2\left(\frac{1}{2}\theta\right) + \tan\left(\frac{1}{2}\theta\right) = 0$$
$$\tan\left(\frac{1}{2}\theta\right)\left[\tan\left(\frac{1}{2}\theta\right) + 1\right] = 0$$
$$\tan\left(\frac{1}{2}\theta\right) = 0 \qquad \tan\left(\frac{1}{2}\theta\right) + 1 = 0$$
$$\tan\left(\frac{1}{2}\theta\right) = -1$$

Step 2
Let $x = \frac{1}{2}\theta$ so that $\tan x = 0$, $\tan x = -1$, and the domain is redefined as follows:
$$0° \le \theta < 360°$$
$$0° \le \frac{1}{2}\theta < 180°$$
$$0° \le x < 180°$$

Step 3
Determine the exact values of x in the equation $\tan x = 0$, where $0° \le x < 180°$.

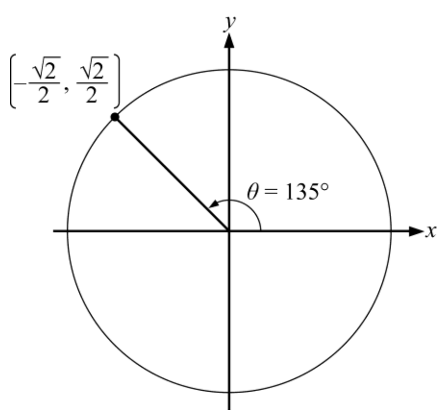
Identify angles on the unit circle where the ratio $\frac{y}{x}$ of its point is equal to 0.



The exact value of x, where $0° \le x < 180°$, is 0º.

Step 4
Determine the exact values of x in the equation $\tan x = -1$, where $0° \le x < 180°$.

Identify angles on the unit circle where the ratio $\frac{y}{x}$ of its point is equal to –1.



The exact value of x in the equation $\tan x = -1$, where $0° \le x < 180°$, is 135º.

Step 5
Determine the exact values of θ within the given domain $0° \le \theta < 360°$.

The values of x are 0º and 135º.

The exact values of θ are determined by multiplying each solution for x by 2. Therefore, the solutions for θ in the equation $\tan^2\left(\frac{1}{2}\theta\right)+\tan\left(\frac{1}{2}\theta\right)=0$ are 0º and 270º.

4. **Step 1**
Isolate the trigonometric function.

$$\sqrt{3}\tan\theta+1=0$$
$$\sqrt{3}\tan\theta=-1$$
$$\tan\theta=-\frac{1}{\sqrt{3}}$$

Step 2
Determine the solutions for θ in the equation $\tan\theta=-\frac{1}{\sqrt{3}}$ within one revolution of the unit circle.

According to the unit circle, the values of θ in the equation $\tan\theta=-\frac{1}{\sqrt{3}}$, where $0\le\theta<2\pi$, are $\frac{5\pi}{6}$ and $\frac{11\pi}{6}$.

Step 3
Determine the general solution.

The solutions are angles coterminal with $\frac{5\pi}{6}$ and $\frac{11\pi}{6}$.

The angles $\frac{5\pi}{6}$ and $\frac{11\pi}{6}$ also differ by π. Therefore, the general solution to equation $\sqrt{3}\tan\theta+1=0$ can be expressed as $\theta=\frac{5\pi}{6}+\pi n$, where n is any integer.

5. **Step 1**
Isolate the trigonometric function.

$$3\sin\theta-1=0$$
$$3\sin\theta=1$$
$$\sin\theta=\frac{1}{3}$$

Step 2
Determine the solutions for $\sin\theta=\frac{1}{3}$, where $0\le\theta<2\pi$.

Find the reference angle for $\sin\theta=\frac{1}{3}$ using the inverse sine function on a calculator.

$$\theta_{ref}=\sin^{-1}\left(\frac{1}{3}\right)$$
$$\theta_{ref}\approx 0.34$$

According to the CAST rule, sine is positive in quadrants I and II. Therefore, the reference angle $\theta_{ref}\approx 0.34$ is a solution.

Determine the measure of the angle with a terminal arm in quadrant II by subtracting the reference angle from π.

$$\theta\approx\pi-0.34$$
$$\theta\approx 2.80$$

The approximate values of θ in the trigonometric equation $3\sin\theta-1$, where $0\le\theta<2\pi$, are 0.34 and 2.80.

6. **Step 1**
Isolate the trigonometric function.

$$7\sin\theta-5=0$$
$$7\sin\theta=5$$
$$\sin\theta=\frac{5}{7}$$

Step 2
Determine the solutions for $\sin\theta=\frac{5}{7}$ within one revolution of the unit circle.

To solve the equation $\sin\theta=\frac{5}{7}$, find the reference angle, θ_{ref}, using a calculator.

$$\sin\theta_{ref}=\frac{5}{7}$$
$$\theta_{ref}=\sin^{-1}\left(\frac{5}{7}\right)$$
$$\theta_{ref}=\sin^{-1}\left(\frac{5}{7}\right)$$
$$\theta_{ref}\approx 45.6°$$

According to the CAST rule, sine is positive in quadrants I and II. Therefore, the reference angle $\theta_{ref}\approx 45.6°$ is a solution.

Determine the measure of the angle in quadrant II by subtracting the reference angle from 180º.

$$\theta=180°-45.6°$$
$$\theta\approx 134.4°$$

Step 3
Determine the general solution for θ.

The solutions are angles coterminal with 45.6º and 134.4º. Therefore, the general solution for the equation $7\sin\theta-5=0$ is $\theta=(45.6+360n)°$ and $(134.4+360n)°$, where n is any integer.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**

Express $\csc\theta$ as $\frac{1}{\sin\theta}$.

$$\csc\theta=\sin\theta$$
$$\frac{1}{\sin\theta}=\sin\theta$$

Step 2

Isolate the trigonometric ratio in the equation.

$$\frac{1}{\sin\theta}=\sin\theta$$
$$1=\sin^2\theta$$
$$\pm\sqrt{1}=\sin\theta$$
$$\pm1=\sin\theta$$

Step 3

Determine the exact values of θ in the equation $\pm1=\sin\theta$, where $0°\le\theta<360°$.

According to the unit circle, $\sin\theta=1$ when $\theta=90°$, and $\sin\theta=-1$ when $\theta=270°$. Therefore, the exact values of θ in the equation $\csc\theta=\sin\theta$ are 90º and 270º.

2. **Step 1**

Factor the trigonometric equation, and apply the zero product property to each factor.

$$\csc^2\theta+\csc\theta-2=0$$
$$(\csc\theta+2)(\csc\theta-1)=0$$

$$\csc\theta+2=0 \qquad \csc\theta-1=0$$
$$\csc\theta=-2 \qquad \csc\theta=1$$

Step 2

Rewrite the equations $\csc\theta=1$ and $\csc\theta=-2$ in terms of $\sin\theta$.

$$\csc\theta=-2 \qquad \csc\theta=1$$
$$\frac{1}{\sin\theta}=-2 \qquad \frac{1}{\sin\theta}=1$$
$$-\frac{1}{2}=\sin\theta \qquad 1=\sin\theta$$

Step 3

Determine the exact values of θ, where $0°\le\theta<360°$.

According to the unit circle, $\sin\theta=-\frac{1}{2}$ when $\theta=210°$ and 330º. Also, $\sin\theta=1$ when $\theta=90°$. Therefore, the exact values of θ in the equation $\csc^2\theta+\csc\theta-2=0$ are 90º, 210º, and 330º.

3. **Step 1**

Bring all terms to one side of the equation.

$$2\sin\theta\cos\theta=\cos\theta$$
$$2\sin\theta\cos\theta-\cos\theta=0$$

Step 2

Factor the trigonometric equation, and apply the zero product property to each factor.

$$2\sin\theta\cos\theta-\cos\theta=0$$
$$\cos\theta(2\sin\theta-1)=0$$

$$\cos\theta=0 \qquad \begin{aligned}2\sin\theta-1-&=0\\2\sin\theta&=1\\\sin\theta&=\frac{1}{2}\end{aligned}$$

Step 3

Determine the exact values of θ, where $0°\le\theta<360°$.

According to the unit circle, $\cos\theta=0$ when $\theta=90°$ and 270º. Also, $\sin\theta=\frac{1}{2}$ when $\theta=30°$ and 150º. Therefore, the exact values of θ in the equation $2\sin\theta\cos\theta=\cos\theta$ are 30º, 90º 150º, and 270º.

4. **Step 1**

Factor the trigonometric equation, and apply the zero product property to each factor.

$$2\cos^2(2\theta)=\sqrt{3}\cos(2\theta)$$
$$2\cos^2(2\theta)-\sqrt{3}\cos(2\theta)=0$$
$$\cos(2\theta)\left[2\cos(2\theta)-\sqrt{3}\right]=0$$

$$\cos(2\theta)=0 \qquad \begin{aligned}2\cos(2\theta)-\sqrt{3}&=0\\2\cos(2\theta)&=\sqrt{3}\\\cos(2\theta)&=\frac{\sqrt{3}}{2}\end{aligned}$$

Step 2

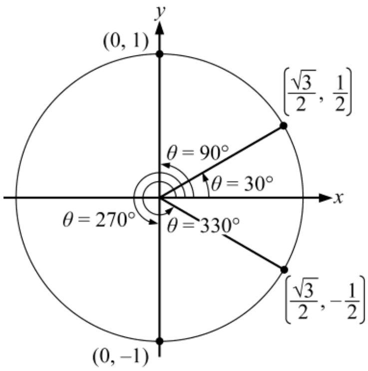
Let $x=2\theta$ so that $\cos x=0$, $\cos x=\frac{\sqrt{3}}{2}$, and the domain is redefined as follows:

$$0°\le\theta<360°$$
$$0°\le2\theta<720°$$
$$0°\le x<720°$$

Step 3
Determine the exact values of *x* in the equations

$\cos x = 0$ and $\cos x = \frac{\sqrt{3}}{2}$, where $0° \le x < 720°$.

Since the value of $\cos x$ corresponds to the *x*-coordinates of points on the unit circle, determine the locations where the *x*-coordinate of a point is equal to 0 or $\frac{\sqrt{3}}{2}$.



The exact values of *x* in equation $\cos x = 0$, where $0° \le 2\theta < 720°$, are 90°, 270°, 450°, and 630°.

The exact values of *x* in equation $\cos x = \frac{\sqrt{3}}{2}$, where $0 \le 2\theta < 720$, are 30 , 330º, 390º, and 690º.

Therefore, the solutions for *x* are 30°, 90°, 270°, 330°, 390°, 450°, 630°, and 690°.

Step 4
Determine the exact values of *θ* within the domain $0° \le \theta < 360°$.

The exact values of *θ* are determined by dividing each solution for *x* by 2. Therefore, the solutions for *θ* are 15°, 45°, 135°, 165°, 195°, 225°, 315°, and 345°.

5. **Step 1**
Isolate the trigonometric ratio in the equation.

$$4\cos\theta + 2 = 0$$
$$4\cos\theta = -2$$
$$\cos\theta = -\frac{2}{4}$$
$$\cos\theta = -\frac{1}{2}$$

Step 2
Determine the exact values of *θ*, where $0 \le \theta < 2\pi$.

According to the unit circle, $\cos\theta = -\frac{1}{2}$ when $\theta = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$. Therefore, the exact values of *θ* in the equation $4\cos\theta + 2 = 0$ are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

6. **Step 1**
Rewrite the equation in terms of $\tan\theta$.

$$\cot^2\theta = 1$$
$$\frac{1}{\tan^2\theta} = 1$$

Step 2
Isolate the trigonometric ratio in the equation.

$$\frac{1}{\tan^2\theta} = 1$$
$$1 = \tan^2\theta$$
$$\pm\sqrt{1} = \tan\theta$$
$$\pm 1 = \tan\theta$$

Step 3
Determine the exact values of *θ*, where $0 \le \theta < 2\pi$.

According to the unit circle, $\tan\theta = 1$ when $\theta = \frac{\pi}{4}$ and $\frac{5\pi}{4}$. Also, $\tan\theta = -1$ when $\theta = \frac{3\pi}{4}$ and $\frac{7\pi}{4}$.
Therefore, the exact values of *θ* in the equation $\cot^2\theta = 1$ are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$.

7. **Step 1**
Subtract $\sin\theta$ from both sides of the equation.

$$2\sin\theta = \sin\theta$$
$$2\sin^2\theta - \sin\theta = 0$$

Step 2
Factor the trigonometric equation, and apply the zero product property to each factor.

$$2\sin^2\theta - \sin\theta = 0$$
$$\sin\theta(2\sin\theta - 1) = 0$$

$$\sin\theta = 0 \qquad 2\sin\theta - 1 = 0$$
$$2\sin\theta = 1$$
$$\sin\theta = \frac{1}{2}$$

Step 3
Determine the exact values of θ, where $0 \le \theta < 2\pi$.

According to unit circle $\sin\theta = 0$ when $\theta = 0\,\text{rad}$ and π rad. Also, $\sin\theta = \frac{1}{2}$ when $\theta = \frac{\pi}{6}\,\text{rad}$ and $\frac{5\pi}{6}$ rad. Therefore, the exact values of θ in the equation $2\sin^2\theta = \sin\theta$ are $0, \frac{\pi}{6}, \pi$, and $\frac{5\pi}{6}$.

8. **Step 1**
Factor the trigonometric equation, and apply the zero product property to each factor.

$$\sec^2\left(\frac{1}{2}\theta\right) + 3\sec\left(\frac{1}{2}\theta\right) + 2 = 0$$
$$\left(\sec\left(\frac{1}{2}\theta\right) + 1\right)\left(\sec\left(\frac{1}{2}\theta\right) + 2\right) = 0$$
$$\sec\left(\frac{1}{2}\theta\right) + 1 = 0 \qquad \sec\left(\frac{1}{2}\theta\right) + 2 = 0$$
$$\sec\left(\frac{1}{2}\theta\right) = -1 \qquad \sec\left(\frac{1}{2}\theta\right) = -2$$

Step 2
Let $x = \frac{1}{2}\theta$ so that the equations become $\sec x = -1$ and $\sec x = -2$. The domain is redefined as follows:

$$0 \le \theta < 2\pi$$
$$0 \le \frac{1}{2}\theta < \pi$$
$$0 \le x < \pi$$

Step 3
Determine the exact values of x in the equation $\sec x = -1$ and $\sec x = -2$, where $0 \le x < \pi$.

According to the unit circle, the exact value of x in the equation $\sec x = -2$ is $\frac{2\pi}{3}$. There is no solution within the domain of $0 \le x < \pi$ for the equation $\sec x = -1$.

Step 4
Determine the exact values of θ within the domain $0 \le \theta < 2\pi$.

The exact values of θ are determined by multiplying each solution for x by 2.

Therefore, the solution for θ is $\frac{4\pi}{3}$.

9. **Step 1**
Bring all terms to one side of the equation.

$$\tan^2\theta - 5 = 3 + 2\tan\theta$$
$$\tan^2\theta - 2\tan\theta - 8 = 0$$
$$\tan\theta - 4 = 0 \qquad \tan\theta + 2 = 0$$
$$\tan\theta = 4 \qquad \tan\theta = -2$$

Step 2
Factor the trigonometric equation, and apply the zero product property to each of the factors.

$$\tan^2\theta - 2\tan\theta - 8 = 0$$
$$(\tan\theta - 4)(\tan\theta + 2) = 0$$

Step 3
Determine the solutions for $\tan\theta = 4$, where $0° \le \theta < 360°$.

Find the reference angle for $\tan\theta = 4$ using the inverse tan function on a calculator.

$$\theta_{ref} = \tan^{-1} 4$$
$$\theta_{ref} \approx 76.0°$$

According to the CAST rule, tan is positive in quadrants I and III. Therefore, the reference angle $\theta_{ref} \approx 76.0°$ is a solution.

Determine the measure of the angle with a terminal arm in quadrant III by adding the reference angle to 180º.

$$\theta \approx 76.0° + 180°$$
$$\theta \approx 256.0°$$

Step 4
Determine the solutions for $\tan\theta = -2$, where $0° \le \theta < 360°$.

Find the reference angle for $\tan\theta = -2$ using the inverse tan function on a calculator.

$$\theta_{ref} = \tan^{-1}(-2)$$
$$\theta_{ref} \approx -63.4°$$

Since the reference angle is an acute positive angle, $\theta_{ref} \approx 63.4°$.

According to the CAST rule, tan is negative in quadrants II and IV. Determine the measure of the angle with a terminal arm in quadrant II by subtracting the reference angle from 180º.

$$\theta \approx 180° - 63.4°$$
$$\theta \approx 116.6°$$

Determine the measure of the angle with a terminal arm in quadrant IV by subtracting the reference angle from 360º.

$\theta \approx 360° - 63.4°°$

$\theta \approx 296.6°$

Rounded to the nearest tenth of a degree, the approximate values of θ in the trigonometric equation $\tan^2\theta - 5 = 3 + 2\tan\theta$, where $0° \le \theta < 360°$, are 76.0º, 116.6º, 256.0º,
and 296.6º.

10. **Step 1**
Bring all terms to one side of the equation.

$$3\cos^2\theta + \cos\theta = 5\cos\theta - 2\cos^2\theta$$
$$5\cos^2\theta - 4\cos\theta = 0$$

Step 2
Factor the trigonometric equation, and apply the zero product property to each of the factors.

$$5\cos^2\theta - 4\cos\theta = 0$$
$$\cos\theta(5\cos\theta - 4) = 0$$
$$\cos\theta = 0 \qquad 5\cos\theta - 4 = 0$$
$$5\cos\theta = 4$$
$$\cos\theta = \frac{4}{5}$$

Step 3
Determine the solutions for $\cos\theta = 0$, where $0° \le \theta < 360°$.

According to the unit circle, the value of θ in the equation $\cos\theta = 0$, where $0° \le \theta < 360°$, are 90.0º and 270.0º.

Step 4
Determine the solutions for $\cos\theta = \frac{4}{5}$, where $0° \le \theta < 360°$.

Find the reference angle for $\cos\theta = \frac{4}{5}$ using the inverse cosine function on a calculator.

$$\theta_{ref} = \cos^{-1}\left(\frac{4}{5}\right)$$
$$\theta_{ref} \approx 36.9°$$

According to the CAST rule, cosine is positive in quadrants I and IV. Therefore, the reference angle $\theta_{ref} \approx 36.9°$ is a solution.

Determine the measure of the angle with a terminal arm in quadrant IV by subtracting the reference angle from 360º.

$\theta \approx 360° - 36.9°$

$\theta \approx 323.1°$

Rounded to the nearest tenth of a degree, the approximate values of θ in the trigonometric equation $3\cos^2\theta + \cos\theta = 5\cos\theta - 2\cos^2\theta$, where $0° \le \theta < 360°$, are 36.9º, 90.0º, 270.0º, and 323.1º.

11. **Step 1**
Rewrite the equation in terms of a primary trigonometric ratio.

Let $\sec\theta = \frac{1}{\cos\theta}$.

$$\sqrt{3}\sec\theta - 2 = 0$$
$$\sqrt{3}\left(\frac{1}{\cos\theta}\right) - 2 = 0$$

Step 2
Isolate the trigonometric ratio in the equation.

$$\sqrt{3}\left(\frac{1}{\cos\theta}\right) - 2 = 0$$
$$\sqrt{3}\left(\frac{1}{\cos\theta}\right) = 2$$
$$\frac{1}{\cos\theta} = \frac{2}{\sqrt{3}}$$
$$\cos\theta = \frac{\sqrt{3}}{2}$$

Step 3
Determine the exact values of θ within one revolution.

According to the unit circle, the values of θ in the equation $\cos\theta = \frac{\sqrt{3}}{2}$, where $0° \le \theta < 360°$, are 30º and 330º.

Step 4
Determine the general solution for θ.

The solutions are angles coterminal 30º and 330º. Therefore, the general solution is $\theta = (30 + 360n)°$ and $(330 + 360n)°$, where n is any integer.

12. **Step 1**
Add cosθ to both sides of the equation.

$$\sin\theta - \cos\theta = 0$$
$$\sin\theta = \cos\theta$$

Step 2
Determine the values of θ in the equation $\sin\theta = \cos\theta$ within one revolution of the unit circle.

The values of θ in the equation $\sin\theta = \cos\theta$ correspond to points on the unit circle where the x-coordinate is equal to the y-coordinate. This occurs when θ is equal to 45º and 225º.

Step 3
Determine the general solution for θ.

The solutions are angles coterminal 45º and 225º. The angles 45º and 225º differ by 180º. Therefore, the general solution can be expressed as $(45 + 180n)°$, where n is any integer.

13. Step 1
Isolate the trigonometric function.

$$\sin\theta - \frac{1}{2} = 0$$
$$\sin\theta = \frac{1}{2}$$

Step 2

Determine the values of θ in the equation $\sin\theta = \frac{1}{2}$, where $0 \le \theta < 2\pi$.

According to the unit circle, the values of θ in the equation $\sin\theta = \frac{1}{2}$, where $0 \le \theta \le 2\pi$, are $\frac{\pi}{6}$ rad and $\frac{5\pi}{6}$ rad.

Step 3
Determine the general solution of θ.

The solutions are angles coterminal $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

Therefore, the general solution to the equation $\sin\theta - \frac{1}{2} = 0$ is $\theta = \frac{\pi}{6} + 2\pi n$ and $\frac{5\pi}{6} + 2\pi n$, where n is any integer.

14. Step 1
Rewrite the equation in terms of a primary trigonometric ratio. Let $\cot\theta = \frac{1}{\tan\theta}$.

$$3\tan\theta = \cot\theta$$
$$3\tan\theta = \frac{1}{\tan\theta}$$

Step 2
Isolate the primary trigonometric function in the equation.

$$3\tan\theta = \frac{1}{\tan\theta}$$
$$3\tan^2\theta = 1$$
$$\tan^2\theta = \frac{1}{3}$$
$$\tan\theta = \pm\frac{1}{\sqrt{3}}$$

Step 3

Determine the values of θ in the equation $\tan\theta = \frac{1}{\sqrt{3}}$, where $0 \le \theta < 2\pi$.

According to the unit circle, the values of θ in the equation $\tan\theta = \frac{1}{\sqrt{3}}$, where $0 \le \theta < 2\pi$, are $\frac{\pi}{6}$ rad and $\frac{7\pi}{6}$ rad.

Step 4

Determine the values of θ in the equation $\tan\theta = -\frac{1}{\sqrt{3}}$, where $0 \le \theta < 2\pi$.

According to the unit circle, the values of θ in the equation $\tan\theta = -\frac{1}{\sqrt{3}}$, where $0 \le \theta < 2\pi$, are $\frac{5\pi}{6}$ rad and $\frac{11\pi}{6}$ rad.

Step 5
Determine the general solution for θ.

The solutions are coterminal with the angles $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, and $\frac{11\pi}{6}$. The angles $\frac{\pi}{6}$ and $\frac{7\pi}{6}$ differ by π, and the angles $\frac{5\pi}{6}$ and $\frac{11\pi}{6}$ also differ by π.

Therefore, the general solution to equation $3\tan\theta = \cot\theta$ can be written as $\theta = \frac{\pi}{6} + \pi n$ and $\frac{5\pi}{6} + \pi n$, where n is any integer.

Lesson 7—Solving Trigonometric Equations by Graphing

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Bring all terms to one side of the equation.

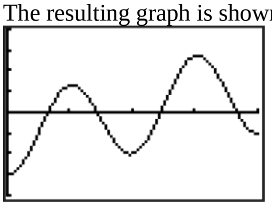
$$4\sin^2\theta - \cos\left(\frac{\theta}{2}\right) = 2$$
$$4\sin^2\theta - \cos\left(\frac{\theta}{2}\right) - 2 = 0$$

Step 2
Graph the related function using a TI-83 or similar calculator.

Press $\boxed{\text{Y=}}$, enter the equation as $Y_1 = 4\sin(X)^2 - \cos(X/2) - 2$, and press $\boxed{\text{GRAPH}}$.

An appropriate window setting is *y*:[0, 360, 90] and *y*:[–4, 4, 1].

The resulting graph is shown.

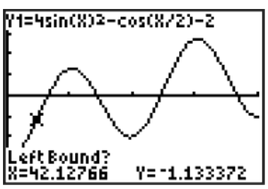


There are four solution values (*x*-intercepts) in the domain $0° < \theta \le 360°$.

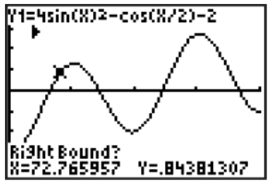
Step 3
Determine the zeros of the function.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 2:zero.

When asked for a left bound, position the cursor just left of the first zero, and press $\boxed{\text{ENTER}}$.

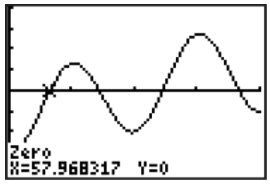


When asked for a right bound, position the cursor just right of the same zero, and press $\boxed{\text{ENTER}}$.

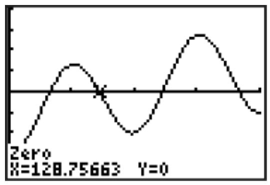


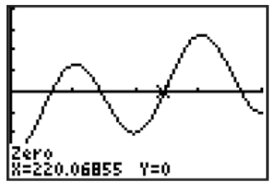
At the "Guess?" prompt, press $\boxed{\text{ENTER}}$.

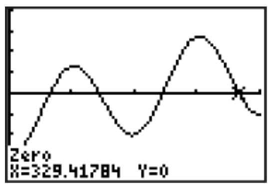
The results are the coordinates of the zero of the function.



Repeat the same process to obtain the other zeros.







The zeros of the function are 57.97, 128.76, 220.07, and 329.42.

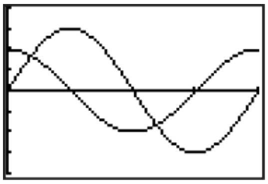
Therefore, the values of θ in the equation $4\sin^2\theta - \cos\left(\frac{\theta}{2}\right) = 2$, where $0 < \theta \le 2\pi$, are approximately 57.97º, 128.76º, 220.07º, and 329.42º.

2. **Step 1**
Graph each side of the equation $3\sin x = 2\cos(x-1)$.

Press $\boxed{\text{Y=}}$, enter the equation as two separate functions ($Y_1 = 3\sin(X)$ and $Y_2 = 2\cos(X-1)$), and press $\boxed{\text{GRAPH}}$.

An appropriate window setting is x:[0, 360, 90] and y:[–4, 4, 1].

The resulting graph is shown.

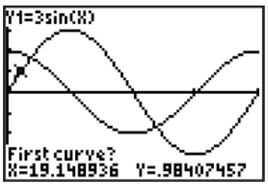


There are two solution values (points of intersection) in the domain $0° < x \le 360°$.

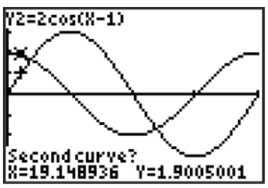
Step 2
Find the points of intersection.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 5:intersect.

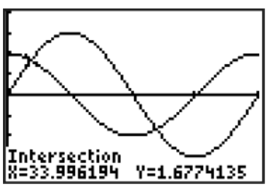
When asked for a first curve, position the cursor just left or right of the first intersection point, and press $\boxed{\text{ENTER}}$.



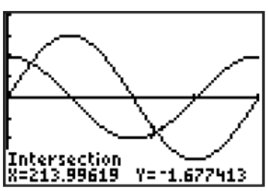
When asked for a second curve, position the cursor just left or right of the first intersection point, and press $\boxed{\text{ENTER}}$.



At the "Guess?" prompt, press $\boxed{\text{ENTER}}$.



Repeat the same process to obtain the other point of intersection.



The solutions to the equation $3\sin x = 2\cos(x-1)$, where $0° < x \le 360°$, are approximately 34.00º and 214.00º.

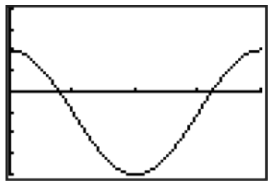
PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Graph the related function using a TI-83 or similar calculator.

Press $\boxed{\text{Y=}}$, enter the equation as $Y_1 = 3\cos(X) - 1$, and press $\boxed{\text{GRAPH}}$.

An appropriate window setting is x:[0, 360,90] and y:[–4, 4, 1].

The resulting graph is shown.



There are two solution values (x-intercepts) in the domain $0° < x \le 360°$.

Step 2
Determine the zeros of the function.

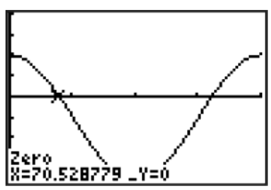
Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 2:zero.

When asked for a left bound, position the cursor just left of the first zero, and press $\boxed{\text{ENTER}}$.

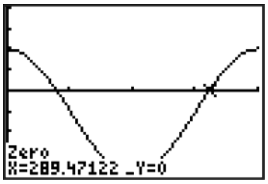
When asked for a right bound, position the cursor just right of the same zero, and press $\boxed{\text{ENTER}}$.

At the "Guess?" prompt, press $\boxed{\text{ENTER}}$.

The results are the coordinates of the first zero of the function.



Repeat the same process to obtain the second zero.



The zeros of the function are 70.5º and 289.5º.

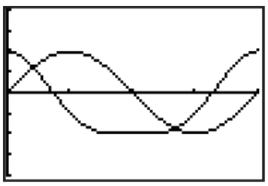
Therefore, the values of x in the equation $3\cos x - 1 = 0$, where $0° < x \leq 360°$, are approximately 70.5º and 289.5º.

2. **Step 1**
Graph each side of the equation $2\sin(x) = 2\cos(x) - \sin^2 x$.

Press $\boxed{\text{Y=}}$, enter the equation as two separate functions ($Y_1 = 2\sin(X)$ and $Y_2 = 2\cos(X) - \sin(X)^2$), and press $\boxed{\text{GRAPH}}$.

An appropriate window setting is x:[0, 360, 90] and y:[–4, 4, 1].

The resulting graph is shown.



There are two solution values (points of intersection) in the domain $0° < x \leq 360°$.

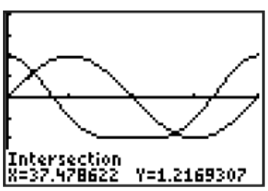
Step 2
Find the points of intersection.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 5:intersect.

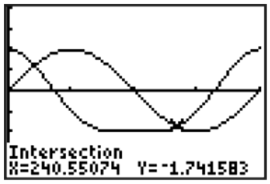
When asked for a first curve, position the cursor just left or right of the first intersection point, and press $\boxed{\text{ENTER}}$.

When asked for a second curve, position the cursor just left or right of the first intersection point, and press $\boxed{\text{ENTER}}$.

At the "Guess?" prompt, press $\boxed{\text{ENTER}}$.



Repeat the same process to obtain the other point of intersection.

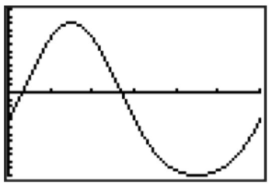


The solutions to the equation $2\sin(x) = 2\cos(x) - \sin^2 x$, where $0° < x \leq 360°$, are approximately 37.5º and 240.6º.

3. **Step 1**
Graph the related function using a TI-83 or similar calculator.
Press $\boxed{\text{Y=}}$, enter the equation as $Y_1 = 3\sin(X)^2 + 11\sin(X) - 4$, and press $\boxed{\text{GRAPH}}$.
Make sure your calculator is in radian mode.

An appropriate window setting is $x:\left[0, 2\pi, \frac{\pi}{3}\right]$ and y:[–12, 12, 1].

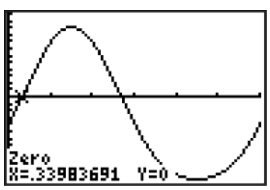
The resulting graph is shown.



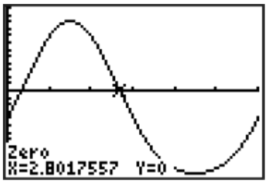
There are two solution values (x-intercepts) in the domain $0 < \theta \leq 2\pi$.

Step 3
Determine the zeros of the function.
Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 2:zero.

When asked for a left bound, position the cursor just left of the first zero, and press $\boxed{\text{ENTER}}$.
When asked for a right bound, position the cursor just right of the same zero, and press $\boxed{\text{ENTER}}$.
At the "Guess?" prompt, press $\boxed{\text{ENTER}}$.
The results are the coordinates of the first zero of the function.



Repeat the same process to obtain the second zero.



The zeros of the function are 0.3 and 2.8.

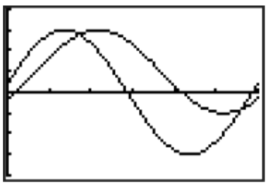
Therefore, the values of *x* in the equation $3\sin^2 x + 11\sin x - 4 = 0$, where $0 < \theta \le 2\pi$, are approximately 0.3 rad and 2.8 rad.

4. **Step 1**
Graph each side of the equation $3\sin(3-x) = 2\cos(4+x) + 1$.

Press $\boxed{\text{Y=}}$, enter the equation as two separate functions ($\text{Y}_1 = 3\sin(3-\text{X})$ and $\text{Y}_2 = 2\cos(4+\text{X})+1$), and press $\boxed{\text{GRAPH}}$.

An appropriate window setting is $x:\left[0, 2\pi, \frac{\pi}{3}\right]$ and *y*:[–4, 4, 1].

The resulting graph is shown.



There are two solution values (points of intersection) in the domain $0 < \theta \le 2\pi$.

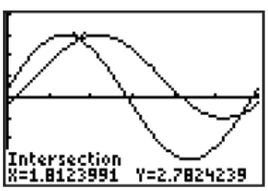
Step 2
Find the points of intersection.

Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, and select 5:intersect.

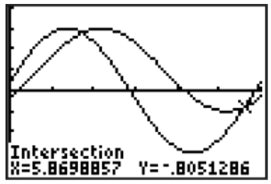
When asked for a first curve, position the cursor just left or right of the first intersection point, and press $\boxed{\text{ENTER}}$.

When asked for a second curve, position the cursor just left or right of the intersection point, and press $\boxed{\text{ENTER}}$.

At the “Guess?” prompt, press $\boxed{\text{ENTER}}$.



Repeat the same process to obtain the other point of intersection.



The solutions to the equation $3\sin(3-x) = 2\cos(4+x) + 1$, where $0 < x \le 2\pi$, are approximately 1.8 rad and 5.9 rad.

Lesson 8—Reciprocal, Quotient, and Pythagorean Identities

CLASS EXERCISES
ANSWERS AND SOLUTIONS

1.

Step	LHS	RHS
1. Apply the reciprocal identities.	$\frac{\sec x}{\sin x}-\frac{\sin x}{\cos x}$ $=\frac{\frac{1}{\cos x}}{\sin x}-\frac{\sin x}{\cos x}$ $=\frac{1}{\cos x\sin x}-\frac{\sin x}{\cos x}$	$\cot x$ $=\frac{\cos x}{\sin x}$
2. Rewrite the LHS as a single fraction.	$=\frac{1}{\cos x\sin x}-\frac{\sin^2 x}{\cos x\sin x}$ $=\frac{1-\sin^2 x}{\cos x\sin x}$	
3. Apply the Pythagorean identity to the LHS.	$=\frac{\cos^2 x}{\cos x\sin x}$	
4. Divide out common factors.	$=\frac{\cos x\cos x}{\cos x\sin x}$ $=\frac{\cancel{\cos x}\cos x}{\cancel{\cos x}\sin x}$ $=\frac{\cos x}{\sin x}$	

Since LHS = RHS, the proof is complete.

2. a)

Steps	LHS	RHS
1. Replace x with $\frac{\pi}{6}$.	$\csc x-\sin x$ $=\csc\left(\frac{\pi}{6}\right)-\sin\left(\frac{\pi}{6}\right)$	$\cot x\cos x$ $=\cot\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)$
2. Rewrite the equation in terms of primary trigonometric ratios.	$=\frac{1}{\sin\left(\frac{\pi}{6}\right)}-\sin\left(\frac{\pi}{6}\right)$	$=\frac{\cos\left(\frac{\pi}{6}\right)}{\tan\left(\frac{\pi}{6}\right)}$
3. Evaluate each side of the equation.	$=\frac{1}{\frac{1}{2}}-\frac{1}{2}$	$=\frac{\frac{\sqrt{3}}{2}}{\frac{1}{\sqrt{3}}}$
4. Simplify.	$=2-\frac{1}{2}$ $=\frac{3}{2}$	$=\frac{3}{2}$

Since LHS = RHS, the identity $\csc x-\sin x=\cot x\cos x$ is true for $x=\frac{\pi}{6}$.

b)

Steps	LHS	RHS
1. Apply the reciprocal and quotient identities.	$\csc x - \sin x$ $= \dfrac{1}{\sin x} - \sin x$	$\cot x \cos x$ $= \dfrac{\cos x}{\sin x}\cos x$ $= \dfrac{\cos^2 x}{\sin x}$
2. Rewrite LHS as a single fraction.	$= \dfrac{1}{\sin x} - \dfrac{\sin^2 x}{\sin x}$ $= \dfrac{1-\sin^2 x}{\sin x}$	
3. Apply the Pythagorean identity.	$= \dfrac{\cos^2 x}{\sin x}$	

Since LHS = RHS, the proof is complete.

3. **Step 1**
Apply the reciprocal identity and quotient identities.

$$\frac{1+\sin x}{1-\sin x} = (\sec x + \tan x)^2$$

$$\frac{1+\sin x}{1-\sin x} = \left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)^2$$

Step 2
State the non-permissible values.

The left-hand side is undefined when $\sin x = 1$, and the right-hand side is undefined when $\cos x = 0$.

The general solution to the equation $\sin x = 1$ is $x = \left(\frac{\pi}{2} + 2\pi n\right)\text{rad}$, and the general solution to the equation $\cos x = 0$ is $x = \left(\frac{\pi}{2} + \pi n\right)\text{rad}$. Therefore, the non-permissible values are $x = \left(\frac{\pi}{2} + \pi n\right)\text{rad}$, where n is any integer.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1.

Steps	LHS	RHS
1. Apply the reciprocal and quotient identities.	$\sec x$ $= \dfrac{1}{\cos x}$	$\dfrac{\csc x}{\cot x}$ $= \dfrac{\frac{1}{\sin x}}{\frac{\cos x}{\sin x}}$
2. Simplify.		$= \left(\dfrac{1}{\sin x}\right)\left(\dfrac{\sin x}{\cos x}\right)$ $= \left(\dfrac{1}{\cancel{\sin x}}\right)\left(\dfrac{\cancel{\sin x}}{\cos x}\right)$ $= \dfrac{1}{\cos x}$

Since LHS = RHS, the proof is complete.

2.

Steps	LHS	RHS
1. Apply the quotient and Pythagorean identities.	$1-\sin x\cos x\tan x=1-\sin x\cos x\left(\dfrac{\sin x}{\cos x}\right)$	$\cos^2 x$ $=1-\sin^2 x$
2. Simplify.	$=1-\sin x\cancel{\cos x}\left(\dfrac{\sin x}{\cancel{\cos x}}\right)$ $=1-\sin x\sin x$ $=1-\sin^2 x$	

Since LHS = RHS, the proof is complete.

3.

Steps	LHS	RHS
1. Apply the reciprocal and quotient identities.	$\dfrac{\csc x}{\tan x\cot x}$ $=\dfrac{\frac{1}{\sin x}}{\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}}$	$\cos x$
2. Write the denominator as a single fraction.	$=\dfrac{\frac{1}{\sin x}}{\frac{\sin^2 x}{\cos x\sin x}+\frac{\cos^2 x}{\sin x\cos x}}$ $=\dfrac{\frac{1}{\sin x}}{\frac{\sin^2 x+\cos^2 x}{\cos x\sin x}}$	
3. Apply the Pythagorean identity.	$=\dfrac{\frac{1}{\sin x}}{\frac{1}{\cos x\sin x}}$	
4. Simplify	$=\dfrac{\cos x\sin x}{\sin x}$ $=\cos x$	

Since LHS = RHS, the proof is complete.

4.

Steps	LHS	RHS
1. Rewrite as a single fraction.	$\dfrac{1}{1-\sin x}+\dfrac{1}{1+\sin x}$ $=\dfrac{1+\sin x+1-\sin x}{(1-\sin x)(1+\sin x)}$	$2\sec^2 x$
2. Simplify.	$=\dfrac{1+1+\sin x-\sin x}{1+\sin x-\sin x-\sin^2 x}$ $=\dfrac{2}{1-\sin^2 x}$	
3. Apply the Pythagorean identity.	$=\dfrac{2}{\cos^2 x}$	
4. Apply the reciprocal identity.	$=2\sec^2 x$	

Since LHS = RHS, the proof is complete.

5.

Steps	**LHS**	**RHS**
1. Multiply the numerator and the denominator by $(1+\cos x)$.	$\frac{\sin x}{1-\cos x}$ $=\frac{\sin x(1+\cos x)}{(1-\cos x)(1+\cos x)}$	$\csc x+\cot x$
2. Expand the denominator.	$=\frac{\sin x(1+\cos x)}{1-\cos^2 x}$	
3. Apply the Pythagorean identity to the denominator.	$=\frac{\sin x(1+\cos x)}{\sin^2 x}$	
4. Divide out common factors.	$=\frac{\sin x(1+\cos x)}{\sin^2 x}$ $=\frac{\sin x(1+\cos x)}{\sin x\sin x}$ $=\frac{1+\cos x}{\sin x}$	
5. Rewrite as two separate fractions.	$=\frac{1}{\sin x}+\frac{\cos x}{\sin x}$	
6. Apply the reciprocal and quotient identities.	$=\csc x+\cot x$	

Since LHS = RHS, the proof is complete.

6.

Steps	**LHS**	**RHS**
1. Replace x with 45°.	$\frac{(\sin 45°)^3}{\cos 45°-(\cos 45°)^3}$	$\tan 45°$
2. Evaluate each side.	$=\frac{(\sin 45°)^3}{\cos 45°-(\cos 45°)^3}$ $=\frac{\left(\frac{\sqrt{2}}{2}\right)^3}{\frac{\sqrt{2}}{2}-\left(\frac{\sqrt{2}}{2}\right)^3}$	= 1
3. Simplify the LHS.	$=\frac{\frac{2\sqrt{2}}{8}}{\frac{\sqrt{2}}{2}-\frac{2\sqrt{2}}{8}}=\frac{\frac{2\sqrt{2}}{8}}{\frac{4\sqrt{2}}{8}-\frac{2\sqrt{2}}{8}}$ $=\frac{\frac{2\sqrt{2}}{8}}{\frac{2\sqrt{2}}{8}}$ $=1$	

Since LHS = RHS, the identity is true for x = 45°.

7.

Steps	LHS	RHS
1. Factor.	$\dfrac{\sin^3 x}{\cos x - \cos^3 x}$ $= \dfrac{\sin x\left(\sin^2 x\right)}{\cos x\left(1-\cos^2 x\right)}$	$\tan x$
2. Apply the Pythagorean identity.	$= \dfrac{\sin x\left(\sin^2 x\right)}{\cos x\left(\sin^2 x\right)}$	
3. Divide out common factors.	$= \dfrac{\sin x\left(\cancel{\sin^2 x}\right)}{\cos x\left(\cancel{\sin^2 x}\right)}$ $= \dfrac{\sin x}{\cos x}$	
4. Apply the quotient identity.	$= \tan x$	

Since LHS = RHS, the proof is complete.

8. **Step 1**
Apply the quotient identity.

$$\frac{\sin^3 x}{\cos x - \cos^3 x} = \tan x$$
$$\frac{\sin^3 x}{\cos x - \cos^3 x} = \frac{\sin x}{\cos x}$$

Step 2
State the non-permissible values.

The right-hand side of the identity is undefined when $\cos x = 0$, and the left-hand side is undefined when $\cos x - \cos^3 x = 0$.

Solve for *x* in the equation $\cos x - \cos^3 x = 0$.

$$\cos x - \cos^3 x = 0$$
$$\cos x\left(1-\cos^2 x\right) = 0$$

$$\cos x = 0 \qquad 1-\cos^2 x = 0$$
$$1 = \cos^2 x$$
$$\pm 1 = \cos x$$

Therefore, the non-permissible values are $x = 90° + 180°n$ and $x = 180°n$, where *n* is any integer.

Lesson 9—Sum, Difference, and Double-Angle Identities

CLASS EXERCISES
ANSWERS AND SOLUTIONS

1. **Step 1**
Find two special angles whose sum or difference is 105º.
The special angles 60º and 45º have a sum of 105º.

Step 2
Apply the identity $\cos(A+B)=\cos A\cos B-\sin A\sin B$.

Replace A with 60° and B with 45°.

$$\begin{aligned}&\cos 105^\circ\\&=\cos(60^\circ+45^\circ)\\&=\cos 60^\circ\cos 45^\circ-\sin 60^\circ\sin 45^\circ\end{aligned}$$

Step 3
Evaluate.

$$\begin{aligned}&\cos 60^\circ\cos 45^\circ-\sin 60^\circ\sin 45^\circ\\&=\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\&=\frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4}\\&=\frac{\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

The exact value of $\cos 105^\circ$ is $\frac{\sqrt{2}-\sqrt{6}}{4}$.

2. The expression $\cos(2x)\cos(7x)+\sin(2x)\sin(7x)$ has the same arrangement as the right side of the identity $\cos(A-B)=\cos A\cos B+\sin A\sin B$. Comparing the two configurations makes $A=2x$ and $B=2x$.

The expression can be rewritten as follows:

$$\begin{aligned}&\cos(2x)\cos(7x)+\sin(2x)\sin(7x)=\cos(2x-7x)\\&\cos(2x)\cos(7x)+\sin(2x)\sin(7x)=\cos(-5x)\end{aligned}$$

Therefore, the simplified form of $\cos(2x)\cos(7x)+\sin(2x)\sin(7x)$ is $\cos(-5x)$.

3. Apply the double-angle identity $\cos(2A)=2\cos^2 A-1$.

Replace $\cos A$ with $-\frac{5}{6}$, and evaluate.

$$\begin{aligned}&\cos(2A)\\&=2\left(-\frac{5}{6}\right)^2-1\\&=2\left(\frac{25}{36}\right)-1\\&=\frac{25}{18}-1\\&=\frac{7}{18}\end{aligned}$$

The exact value of $\cos(2A)$ is $\frac{7}{18}$.

4. Simplify the left-hand side of the equation to make it equal to the right-hand side.

Steps	LHS	RHS
1. Apply the double-angle identity.	$\frac{1+\cos(2x)}{\sin(2x)}$ $=\frac{1+\cos^2 x-\sin^2 x}{2\cos x\sin x}$	$\cot x$
2. Apply the Pythagorean identities.	$=\frac{1-\sin^2 x+\cos^2 x}{2\cos x\sin x}$ $=\frac{\cos^2 x+\cos^2 x}{2\cos x\sin x}$ $=\frac{2\cos^2 x}{2\cos x\sin x}$	
3. Divide out common factors.	$=\frac{2\cos x\cos x}{2\cos x\sin x}$ $=\frac{\cancel{2}\,\cancel{\cos x}\,\cos x}{\cancel{2}\,\cancel{\cos x}\,\sin x}$ $=\frac{\cos x}{\sin x}$	
4. Apply the quotient identity.	$=\cot x$	

Since LHS = RHS, the proof is complete.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Find two special angles whose sum or difference is 105º.

The special angles 45º and 60º have a sum of 105º.

Step 2

Apply the identity $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$.

Replace A with 45º and B with 60º.

$$\tan 105°$$
$$=\tan(45°+60°)$$
$$=\frac{\tan 45°+\tan 60°}{1-\tan 45°\tan 60°}$$

Step 3
Evaluate.

$$\frac{\tan 45°+\tan 60°}{1-\tan 45°\tan 60°}$$
$$=\frac{1+\sqrt{3}}{1-1(\sqrt{3})}$$
$$=\frac{1+\sqrt{3}}{1-\sqrt{3}}$$

The exact value of $\tan 105°$ is $\frac{1+\sqrt{3}}{1-\sqrt{3}}$.

2. **Step 1**

Find two special angles whose sum or difference is $\frac{\pi}{12}$ rad.

The special angles $\frac{\pi}{4}$ rad and $\frac{\pi}{6}$ rad have a difference of $\frac{\pi}{12}$ rad.

Step 2
Apply the identity
$\cos(A-B)=\cos A\cos B+\sin A\sin B$.

Replace A with $\frac{\pi}{4}$ and B with $\frac{\pi}{6}$.

$$\cos\left(\frac{\pi}{12}\right)$$
$$=\cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$$
$$=\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)+\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$$

Step 3
Evaluate.

$$\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)+\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$$
$$=\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$
$$=\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}$$
$$=\frac{\sqrt{6}+\sqrt{2}}{4}$$

The exact value of $\cos\left(\frac{\pi}{12}\right)$ is $\frac{\sqrt{6}+\sqrt{2}}{4}$.

3. **Step 1**
Determine a positive angle that is coterminal with –165º.

A positive angle that is coterminal with –165º is $-165°+360°=195°$.

Step 2
Find two special angles whose sum or difference is 195º.

The special angles 45º and 150º have a sum of 195º.

Step 3
Apply the identity $\sin(A+B)=\sin A\cos B+\cos A\sin B$.

Replace A with 45° and B with 150°.

$$\sin 195°$$
$$=\sin(45°+150°)$$
$$=\sin 45°\cos 150°+\cos 45°\sin 150°$$

Step 4
Evaluate.

$$\sin 45°\cos 150°+\cos 45°\sin 150°$$
$$=\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$
$$=-\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}$$
$$=\frac{-\sqrt{6}+\sqrt{2}}{2}$$
$$=\frac{\sqrt{2}-\sqrt{6}}{2}$$

The exact value of $\sin(-165°)$ is $\frac{\sqrt{2}-\sqrt{6}}{2}$.

4. The expression $\sin(5x)\cos x-\cos(5x)\sin x$ has the same arrangement as the right side of the identity $\sin(A-B)=\sin A\cos B-\cos A\sin B$. Comparing the two configuration makes $A=5x$ and $B=x$.

The expression can be rewritten as follows:

$$\sin(5x)\cos x-\cos(5x)\sin x=\sin(5x-x)$$
$$\sin(5x)\cos x-\cos(5x)\sin x=\sin(4x)$$

Therefore, the simplified form of $\sin(5x)\cos x-\cos(5x)\sin x$ is $\sin(4x)$.

5. **Step 1**
Apply the identity $\cos(A-B)=\cos A\cos B+\sin A\sin B$.

Since $\cos\left(\frac{\pi}{3}-\theta\right)$, it follows that $A=\frac{\pi}{3}$ and $B=\theta$.
Therefore, the expression can be rewritten as

$$\cos\left(\frac{\pi}{3}-\theta\right)=\cos\left(\frac{\pi}{3}\right)\cos\theta+\sin\left(\frac{\pi}{3}\right)\sin\theta.$$

Step 2
Determine the value of $\sin\theta$.
Using the Pythagorean identity $\sin^2\theta+\cos^2\theta=1$, replace $\cos\theta$ with $-\frac{5}{13}$, and solve for $\sin\theta$.

$$\sin^2\theta+\cos^2\theta=1$$
$$\sin^2\theta+\left(-\frac{5}{13}\right)^2=1$$
$$\sin^2\theta+\frac{25}{169}=1$$
$$\sin^2\theta=1-\frac{25}{169}$$
$$\sin^2\theta=\frac{144}{169}$$
$$\sin\theta=\pm\sqrt{\frac{144}{169}}$$
$$\sin\theta=\pm\frac{12}{13}$$

Since the sine ratio is negative in quadrant III, $\sin\theta=-\frac{12}{13}$.

Step 3

Determine the value of $\cos\left(\frac{\pi}{3}-\theta\right)$.

In the expression $\cos\left(\frac{\pi}{3}\right)\cos\theta+\sin\left(\frac{\pi}{3}\right)\sin\theta$,

replace $\cos\theta$ with $-\frac{5}{13}$ and $\sin\theta$ with $-\frac{12}{13}$.

$$\begin{aligned}&\cos\left(\frac{\pi}{3}\right)\left(-\frac{5}{13}\right)+\sin\left(\frac{\pi}{3}\right)\left(-\frac{12}{13}\right)\\&=\frac{1}{2}\left(-\frac{5}{13}\right)+\frac{\sqrt{3}}{2}\left(-\frac{12}{13}\right)\\&=-\frac{5}{26}-\frac{12\sqrt{3}}{26}\\&=\frac{-5-12\sqrt{3}}{26}\end{aligned}$$

Therefore, the exact value of $\cos\left(\frac{\pi}{3}-\theta\right)$ is

$\frac{-5-12\sqrt{3}}{26}$.

6. Step 1

Find the value of $\tan\theta$.

Apply Pythagorean identity $1+\tan^2\theta=\sec^2\theta$.

Using the Pythagorean identity $1+\tan^2\theta=\sec^2\theta$,

replace $\sec\theta$ with $\frac{11}{6}$, and solve for $\tan\theta$.

$$\begin{aligned}1+\tan^2\theta&=\sec^2\theta\\\tan^2\theta&=\sec^2\theta-1\\\tan^2\theta&=\left(\frac{11}{6}\right)^2-1\\\tan^2\theta&=\frac{121}{36}-1\\\tan^2\theta&=\frac{85}{36}\\\tan\theta&=\pm\sqrt{\frac{85}{36}}\\\tan\theta&=\pm\frac{\sqrt{85}}{6}\end{aligned}$$

Since θ is an angle in quadrant I, $\tan\theta$ is positive.

Therefore, $\tan\theta=\frac{\sqrt{85}}{6}$.

Step 2

Find the exact value of $\tan(2\theta)$.

Apply the identity $\tan(2\theta)=\frac{2\tan\theta}{1-\tan^2\theta}$.

Substitute $\frac{\sqrt{85}}{6}$ for $\tan\theta$, and simplify.

$$\begin{aligned}\tan(2\theta)&=\frac{2\tan\theta}{1-\tan^2\theta}\\\tan(2\theta)&=\frac{2\left(\frac{\sqrt{85}}{6}\right)}{1-\left(\frac{\sqrt{85}}{6}\right)^2}\\\tan(2\theta)&=\frac{\frac{\sqrt{85}}{3}}{1-\frac{85}{36}}\\\tan(2\theta)&=\frac{\frac{\sqrt{85}}{3}}{-\frac{49}{36}}\\\tan(2\theta)&=\frac{\sqrt{85}}{3}\times-\frac{36}{49}\\\tan(2\theta)&=-\frac{12\sqrt{85}}{49}\end{aligned}$$

Therefore, $\tan(2\theta)=-\frac{12\sqrt{85}}{49}$.

7. Simplify the right-hand side of the equation to make it equal to the left-hand side.

Steps	LHS	RHS
1. Apply the double-angle identity.	$\frac{1}{2}\sec x$	$\frac{\sin x}{\sin(2x)}$ $= \frac{\sin x}{2\sin x\cos x}$
2. Divide out common factors.		$= \frac{\cancel{\sin x}}{2\cancel{\sin x}\cos x}$ $= \frac{1}{2\cos x}$
3. Apply the reciprocal identity.		$= \frac{1}{2}\sec x$

Since LHS = RHS, the proof is complete.

8. Simplify the left-hand side of the equation to make it equal to the right-hand side.

Steps	LHS	RHS
1. Factor the numerator.	$-\sec(2x)$	$\frac{1}{\sin^2 x-\cos^2 x}$
2. Apply the reciprocal identity.	$= -\frac{1}{\cos(2x)}$	
3. Apply the double-angle identity.	$= -\frac{1}{\cos^2 x-\sin^2 x}$	
4. Multiply the denominator by –1.	$= \frac{1}{-\cos^2 x+\sin^2 x}$ $= \frac{1}{\sin^2 x-\cos^2 x}$	

Since LHS = RHS, the proof is complete.

9. Simplify the right-hand side of the equation to make it equal to the left-hand side.

Steps	LHS	RHS
1. Apply the double-angle identity.	$\cos^2 x$	$\frac{1+\cos(2x)}{2}$ $= \frac{1+2\cos^2 x-1}{2}$
2. Simplify.		$= \frac{2\cos^2 x}{2}$ $= \cos^2 x$

Since LHS = RHS, the proof is complete.

10. Simplify the right-hand side of the equation to make it equal to the left-hand side.

Steps	LHS	RHS
1. Apply the reciprocal identity.	$\cos(2x)$	$\dfrac{1-\tan^2 x}{1+\tan^2 x}$ $=\dfrac{1-\dfrac{\sin^2 x}{\cos^2 x}}{1+\dfrac{\sin^2 x}{\cos^2 x}}$
2. Add terms in the numerator and denominator.		$=\dfrac{\dfrac{\cos^2 x-\sin^2 x}{\cos^2 x}}{\dfrac{\cos^2 x+\sin^2 x}{\cos^2 x}}$
3. Rewrite as single fraction.		$=\dfrac{\cos^2 x-\sin^2 x}{\cos^2 x}\left(\dfrac{\cos^2 x}{\cos^2 x+\sin^2 x}\right)$ $=\dfrac{\cos^2 x-\sin^2 x}{\cancel{\cos^2 x}}\left(\dfrac{\cancel{\cos^2 x}}{\cos^2 x+\sin^2 x}\right)$ $=\dfrac{\cos^2 x-\sin^2 x}{\cos^2 x+\sin^2 x}$
4. Apply the Pythagorean identity.		$=\dfrac{\cos^2 x-\sin^2 x}{1}$ $=\cos^2 x-\sin^2 x$
5. Apply the double-angle identity.		$=\cos(2x)$

Since LHS = RHS, the proof is complete.

11. Simplify the left-hand side of the equation to make it equal to the right-hand side.

Steps	LHS	RHS
1. Apply the difference identity for cosine.	$\cos y\cos(x-y)-\sin y\sin(x-y)$ $=\cos[y+(x-y)]$	$\cos x$
2. Simplify.	$=\cos x$	

Since LHS = RHS, the proof is complete.

12. Simplify the right-hand side of the equation to make it equal to the left-hand side.

Steps	LHS	RHS
1. Apply the quotient identity.	$\tan(2x)$	$\dfrac{2\tan x}{1-\tan^2 x}$ $=\dfrac{2\left(\dfrac{\sin x}{\cos x}\right)}{1-\left(\dfrac{\sin x}{\cos x}\right)^2}$ $=\dfrac{\dfrac{2\sin x}{\cos x}}{1-\dfrac{\sin^2 x}{\cos^2 x}}$
2. Combine terms in the denominator.		$=\dfrac{\dfrac{2\sin x}{\cos x}}{\dfrac{\cos^2 x-\sin^2 x}{\cos^2 x}}$
3. Rewrite as a single fraction.		$=\dfrac{2\sin x}{\cos x}\left(\dfrac{\cos^2 x}{\cos^2 x-\sin^2 x}\right)$ $=\dfrac{2\sin x}{\cancel{\cos x}}\left(\dfrac{\cancel{\cos x}\cos x}{\cos^2 x-\sin^2 x}\right)$ $=\dfrac{2\sin x\cos x}{\cos^2 x-\sin^2 x}$
4. Apply the double-angle identity.		$=\dfrac{\sin(2x)}{\cos(2x)}$
5. Apply the quotient identity.		$=\tan(2x)$

Since LHS = RHS, the proof is complete.

Practice Test

ANSWERS AND SOLUTIONS

1. Multiply 450° by $\frac{\pi \text{ rad}}{180°}$.

$450° \times \frac{\pi \text{ rad}}{180°}$
$= \frac{450\pi}{180} \text{rad}$
$= \frac{5\pi}{2} \text{rad}$

Converted to radians, 450° is $\frac{5\pi}{2}$ rad.

2. Multiply $-\frac{19\pi}{4}$ rad by $\frac{180°}{\pi \text{ rad}}$.

$-\frac{19\pi}{4} \text{rad} \times \frac{180°}{\pi \text{ rad}}$
$= -\frac{3420°}{4}$
$= -855°$

Converted to radians, $-\frac{19\pi}{4}$ rad is $-855°$.

3. **Step 1**
Determine the angle of the major arc.

Since the major arc takes up more than half of the circumference of a circle, the major arc is $360° - 113° = 247°$.

Step 2
Convert the angle of major arc into radians.

$247° \times \frac{\pi \text{ rad}}{180°}$
$= \frac{247\pi}{180} \text{rad}$

Step 3
Determine the length of the major arc.

Substitute $\frac{247\pi}{180}$ rad for θ and 7 for r into the equation $a = r\theta$.

$a = r\theta$
$a = (7)\left(\frac{247\pi}{180}\right)$
$a \approx 30.2$

Therefore, to the nearest tenth of a centimetre, the major arc length is 30.2 cm.

4. **Step 1**
Determine the measure of angle θ.

$\frac{3\pi}{2} + \frac{9\pi}{20}$
$= \frac{30\pi}{20} + \frac{9\pi}{20}$
$= \frac{39\pi}{20} \text{rad}$

Step 2
Determine which of the given angles is coterminal with angle θ.

The general formula that gives all coterminal angles of θ is of the form $(\theta + 2\pi n)$ rad, where n is any integer.

Therefore, angles that are coterminal with $\frac{39\pi}{20}$ rad are given by the formula $\left(\frac{39\pi}{20} + 2\pi n\right)$ rad, where n is any integer.

5. Find the length, r, using the Pythagorean theorem, $x^2 + y^2 = r^2$.

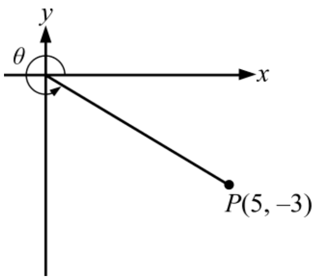
$3^2 + (-6)^2 = r^2$
$9 + 36 = r^2$
$45 = r^2$
$\sqrt{45} = r$
$3\sqrt{5} = r$

To find the exact values of cscθ, secθ, and cotθ, substitute the known information.

$\csc\theta = \frac{r}{y}$	$\sec\theta = \frac{r}{x}$	$\cot\theta = \frac{y}{x}$
$\csc\theta = \frac{3\sqrt{5}}{-6}$	$\sec\theta = \frac{3\sqrt{5}}{3}$	$\cot\theta = \frac{3}{-6}$
$\csc\theta = -\frac{\sqrt{5}}{2}$	$\sec\theta = \sqrt{5}$	$\cot\theta = -\frac{1}{2}$

6. **Step 1**
Sketch a diagram representing *θ*.

It is given that $\cot\theta = -\frac{5}{3}$, and the terminal arm is in quadrant IV. If $\cot\theta = \frac{x}{y}$, possible values of *x* and *y* are $x = 5$ and $y = -3$.



Step 2
Determine the value of sin*θ*.

Since $\sin\theta = \frac{y}{r}$, the value of *r* must be determined using the Pythagorean theorem.

$$\begin{aligned} x^2 + y^2 &= r^2 \\ 5^2 + (-3)^2 &= r^2 \\ 25 + 9 &= r^2 \\ 34 &= r^2 \\ \sqrt{34} &= r \end{aligned}$$

Therefore, $\sin\theta = \frac{-3}{\sqrt{34}}$.

Step 3
Determine the value of cos*θ*.

Since $\cos\theta = \frac{x}{r}$, $\cos\theta = \frac{5}{\sqrt{34}}$.

7. **Step 1**
Locate the terminal arm of $\frac{11\pi}{6}$ rad on the unit circle.

The angle $\frac{11\pi}{6}$ rad terminates in the quadrant IV and has a reference angle of $\frac{\pi}{6}$. Therefore, the coordinates on the terminal arm are $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

Step 2
Find the exact value of $\cos\left(\frac{11\pi}{6}\right)$.

Since $\sin\theta = x$ and the *x*-coordinate of the ordered pair $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ is $\frac{\sqrt{3}}{2}$, it follows that the exact value of $\cos\left(\frac{11\pi}{6}\right)$ is $\frac{\sqrt{3}}{2}$.

8. **Step 1**
Locate the terminal arm of 240º on the unit circle.

The angle 240º terminates in the quadrant III and has a reference angle of 60º. Therefore, the coordinates on the terminal arm are $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

Step 2
Find the exact value of $\csc 240°$.

Since $\csc\theta = \frac{1}{y}$ and the *y*-coordinate of the ordered pair $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ is $-\frac{\sqrt{3}}{2}$, it follows that the exact value of $\csc 240°$ is $-\frac{2}{\sqrt{3}}$.

9. **Step 1**
Locate the terminal arm of $-\frac{11\pi}{3}$ rad on the unit circle.

The angle $-\frac{11\pi}{3}$ rad terminates in the quadrant I and has a reference angle of $\frac{\pi}{3}$. Therefore, the coordinates on the terminal arm are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Step 2

Find the exact value of $\cot\left(-\frac{11\pi}{3}\right)$.

In the unit circle, $\cot\theta = \frac{x}{y}$. Therefore, the exact value of $\cot\left(-\frac{11\pi}{3}\right)$ is determined as follows:

$$\begin{aligned}&\cot\left(-\frac{11\pi}{3}\right)\\&=\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\\&=\frac{\sqrt{3}}{2}\times\frac{2}{1}\\&=\sqrt{3}\end{aligned}$$

10. Step 1
Determine the value of *a*.

The amplitude of the graph is 3 units.
Therefore, the value of *a* is 3.

Step 2
Determine the value of *b*.

The period is 540º, so the value of *b* is $\frac{360°}{540°}=\frac{2}{3}$.

Step 3
Determine the value of *c*.

The phase shift is 45º to the left, so the value of *c* is –45º.

Step 4
Determine the value of *d*.

The horizontal midline axis is $y=-2$, which means the value of *d* is –2.

Step 5
Determine the equation of the graph.

The equation of the given graph is of the form $y=a\cos\left[b\left(x-c\right)\right]+d$. The equation of the graph is $y=3\cos\left[\frac{2}{3}\left(x+45\right)\right]-2$.

11. Step 1
Apply the vertical stretch.

Replace *y* with $\frac{1}{4}y$ in the equation $y=\sin x$.

$$\begin{aligned}y&=\sin x\\\frac{1}{4}y&=\sin x\end{aligned}$$

Step 2
Apply the horizontal stretch.

Replace *x* with $\frac{1}{3}x$ in the equation $\frac{1}{4}y=\sin x$.

$$\begin{aligned}\frac{1}{4}y&=\sin x\\\frac{1}{4}y&=\sin\left(\frac{1}{3}x\right)\end{aligned}$$

Step 3
Apply the reflection.

Replace *y* with –*y* in the equation $\frac{1}{4}y=\sin\left(\frac{1}{3}x\right)$.

$$\begin{aligned}\frac{1}{4}y&=\sin\left(\frac{1}{3}x\right)\\\frac{1}{4}(-y)&=\sin\left(\frac{1}{3}x\right)\end{aligned}$$

Step 4
Apply the phase shift.

Replace *x* with $\left(x-\frac{\pi}{6}\right)$ in the equation

$\frac{1}{4}(-y)=\sin\left(\frac{1}{3}x\right)$.

$$\begin{aligned}\frac{1}{4}(-y)&=\sin\left(\frac{1}{3}x\right)\\\frac{1}{4}(-y)&=\sin\left[\frac{1}{3}\left(x-\frac{\pi}{6}\right)\right]\end{aligned}$$

Step 5
Solve for *y*.

$$\begin{aligned}\frac{1}{4}(-y)&=\sin\left[\frac{1}{3}\left(x-\frac{\pi}{6}\right)\right]\\-y&=4\sin\left[\frac{1}{3}\left(x-\frac{\pi}{6}\right)\right]\\y&=-4\sin\left[\frac{1}{3}\left(x-\frac{\pi}{6}\right)\right]\end{aligned}$$

Therefore, the equation of the transformed equation is $y=-4\sin\left[\frac{1}{3}\left(x-\frac{\pi}{6}\right)\right]$.

12. Step 1
Determine the amplitude.
The value of *a* is –4. Therefore, the amplitude is $|-4| = 4$.

Step 2
Determine the horizontal midline axis.
The value of *d* is 0. Therefore, the equation of horizontal midline axis is $y = 0$.

Step 3
Determine the period.
The value of *b* is $\frac{1}{3}$. Therefore, the period is $\frac{2\pi}{\left|\frac{1}{3}\right|} = 6\pi$.

Step 4
Determine the domain and range.
The domain is $x \in R$.

Since the horizontal midline axis is $y = 0$ and the amplitude is 4, the maximum value is $0 + 4 = 4$. Similarly, the minimum value is $0 - 4 = -4$. Therefore, the range is $-4 \le y \le 4$.

13. Step 1
Isolate the trigonometric ratio in the equation.

$$2\cos(2\theta) = \sqrt{3}$$
$$\cos(2\theta) = \frac{\sqrt{3}}{2}$$

Step 2
Determine a new domain.

The variable θ has a coefficient of 2.
Since $0 \le \theta \le 360°$, the new domain is $0° \le 2\theta \le 720°$.

Thus, the values for 2θ can be any value from 0 up to and including 720º.

Step 3
Determine the exact values of 2θ, where $0° \le 2\theta \le 720°$.

According to the unit circle, the exact values of 2θ in equation $\cos(2\theta) = \frac{\sqrt{3}}{2}$, where $0° \le 2\theta < 720°$, are 30º, 330º, 390º, and 690º.

Step 4
Determine the exact values of θ within the given domain $0 \le \theta \le 360°$. The exact values of θ are determined by dividing each solution for 2θ by 2.
Therefore, the solutions for θ are 15º, 165º, 195º, and 345º.

14. Step 1
Factor the trigonometric equation, and apply the zero product property to each factor.

$$2\sin^2\theta + 3\sin\theta + 1 = 0$$
$$(2\sin\theta + 1)(\sin\theta + 1) = 0$$

$$2\sin\theta + 1 = 0 \qquad \sin\theta + 1 = 0$$
$$\sin\theta = -\frac{1}{2} \qquad \sin\theta = -1$$

Step 2
Determine the exact values of θ, where $0° \le \theta \le 360°$. According to the unit circle, $\sin\theta = -\frac{1}{2}$ when θ = 210º and 330º. Also, $\sin\theta = -1$ when θ = 270º. Therefore, the exact values of θ in the equation $2\sin^2\theta + 3\sin\theta + 1 = 0$ are 210º, 270º, and 330º.

15. Step 1
Rewrite the equation in terms of a primary trigonometric ratio.

Let $\csc\theta = \frac{1}{\sin\theta}$.

$$\sqrt{3}\csc\theta - 2 = 0$$
$$\sqrt{3}\left(\frac{1}{\sin\theta}\right) - 2 = 0$$

Step 2
Isolate the trigonometric ratio in the equation.

$$\sqrt{3}\left(\frac{1}{\sin\theta}\right) - 2 = 0$$
$$\sqrt{3}\left(\frac{1}{\sin\theta}\right) = 2$$
$$\frac{1}{\sin\theta} = \frac{2}{\sqrt{3}}$$
$$\sin\theta = \frac{\sqrt{3}}{2}$$

Step 3
Determine the exact values of θ, where $0 \le \theta \le 2\pi$.

According to the unit circle, the values of θ in the equation $\sin\theta = \frac{\sqrt{3}}{2}$, where $0 \le \theta \le 2\pi$, are $\theta = \frac{\pi}{3}$ rad and $\frac{2\pi}{3}$ rad.

16. Step 1
Bring all terms to one side of the equation.

$$\tan^2\theta = \tan\theta$$
$$\tan^2\theta - \tan\theta = 0$$

Step 2
Factor the trigonometric equation, and apply the zero product property to each of the factors.

$$\tan^2\theta - \tan\theta = 0$$
$$\tan\theta(\tan\theta - 1) = 0$$
$$\tan\theta = 0 \qquad \tan\theta - 1 = 0$$
$$\tan\theta = 1$$

Step 3
Determine the exact values of θ, where $0 \le \theta \le 2\pi$.

According to the unit circle, $\tan\theta = 0$ when $\theta = 0, \pi$, and 2π. Also, $\tan\theta = 1$ when $\theta = \frac{\pi}{4}$ and $\frac{5\pi}{4}$.

Therefore, the exact values of θ in the equation $\tan^2\theta = \tan\theta$ are 0, $\frac{\pi}{4}$, π, $\frac{5\pi}{4}$, and 2π.

17. Step 1
Factor the trigonometric equation, and apply the zero product property to each of the factors.

$$2\sin^2\theta + \sin\theta = 0$$
$$\sin\theta(2\sin\theta + 1) = 0$$
$$\sin\theta = 0 \qquad 2\sin\theta + 1 = 0$$
$$2\sin\theta = -1$$
$$\sin\theta = -\frac{1}{2}$$

Step 2
Determine the values of θ, where $0 \le \theta \le 2\pi$.

According to the unit circle, $\sin\theta = 0$ when $\theta = 0$ and π.

Also, $\sin\theta = -\frac{1}{2}$ when $\theta = \frac{7\pi}{6}$ rad and $\frac{11\pi}{6}$ rad.

Therefore, the exact values of θ in the equation $2\sin^2\theta - \sin\theta = 0$ are 0, π, $\frac{7\pi}{6}$, and $\frac{11\pi}{6}$.

18. The difference identity for cosine is $\cos(A-B) = \cos A\cos B + \sin A\sin B$.

Determine the value of cos*A*, cos*B*, and sin*B*.

Step 1
Determine the value of cos*A*.

It is given that $\sin A = -\frac{4}{5}$, and angle *A* is in quadrant III. Therefore, $y = -4$ and $r = 5$.
Use the Pythagorean theorem to find the value of *x*.

$$x^2 + y^2 = r^2$$
$$x^2 + (-4)^2 = 5^2$$
$$x^2 + 16 = 25$$
$$x^2 = 9$$
$$x = \pm 3$$

Since angle *A* in quadrant III, $x = -3$.

Therefore, the exact value of cos*A* is $-\frac{3}{5}$.

Step 2
Determine the value of cos*B*.

It is given that $\tan B = \frac{12}{5}$, and angle *B* is in quadrant III. Therefore, $x = -5$ and $y = -12$. Use the Pythagorean theorem to find the value of *r*.

$$x^2 + y^2 = r^2$$
$$(-5)^2 + (-12)^2 = r^2$$
$$25 + 144 = r^2$$
$$169 = r^2$$
$$r = 13$$

Therefore, the exact value of cos*B* is $-\frac{5}{13}$.

Step 3
Determine the value of sin*B*.

It is given that $\tan B = \frac{12}{5}$, $\cos B = \frac{5}{13}$, and angle *B* is in quadrant III. Therefore, the exact value of sin*B* is $-\frac{12}{13}$.

Step 4
Apply the identity $\cos(A-B) = \cos A\cos B + \sin A\sin B$ to find the value of $\cos(A-B)$.

Substitute $-\frac{3}{5}$ for cos *A*, $-\frac{5}{13}$ for cos*B*, $-\frac{4}{5}$ for sin*A*, and $-\frac{12}{13}$ for sin*B*.

$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$
$$\cos(A-B)=\left(-\frac{3}{5}\right)\left(-\frac{5}{13}\right)+\left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right)$$
$$\cos(A-B)=\frac{15}{65}+\frac{48}{65}$$
$$\cos(A-B)=\frac{63}{65}$$

Therefore, the exact value of $\cos(A-B)$ is $\frac{63}{65}$.

19. Simplify the right-hand side of the equation to make it equal to the left-hand side.

Steps	**LHS**	**RHS**
1. Write a single fraction.	$2\csc\theta$	$\frac{\sin\theta}{1-\cos\theta}+\frac{\sin\theta}{1+\cos\theta}$ $=\frac{\sin\theta(1+\cos\theta)+\sin\theta(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$
2. Simplify.		$=\frac{\sin\theta(1+\cos\theta+1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$ $=\frac{\sin\theta(2)}{1-\cos^2\theta}$ $=\frac{2\sin\theta}{1-\cos^2\theta}$
3. Apply the Pythagorean identity.		$=\frac{2\sin\theta}{\sin^2\theta}$
4. Divide out common factors.		$=\frac{2\cancel{\sin\theta}}{\sin\theta\cancel{\sin\theta}}$ $=\frac{2}{\sin\theta}$
5. Apply the reciprocal identity.		$=2\csc\theta$

Since LHS = RHS, the proof is complete.

20. Simplify the left-hand side of the equation to make it equal to the right-hand side.

Steps	LHS	RHS
1. Apply the reciprocal identity.	$\dfrac{1-\tan\theta}{1-\cot\theta}$ $=\dfrac{1-\tan\theta}{1-\dfrac{1}{\tan\theta}}$	$-\tan\theta$
2. Combine the terms in the denominator.	$=\dfrac{1-\tan\theta}{\dfrac{\tan\theta-1}{\tan\theta}}$	
3. Write as a single fraction.	$=1-\tan\theta\times\dfrac{\tan\theta}{\tan\theta-1}$ $=\dfrac{\tan\theta(1-\tan\theta)}{\tan\theta-1}$	

21. Simplify the left-hand side of the equation to make it equal to the right-hand side.

Steps	LHS	RHS
1. Apply the sum identity.	$\dfrac{\sin(x+y)}{\cos x\cos y}$ $=\dfrac{\sin x\cos y+\cos x\sin y}{\cos x\cos y}$	$\tan x+\tan y$
2. Split into two fractions.	$=\dfrac{\sin x\cos y}{\cos x\cos y}+\dfrac{\cos x\sin y}{\cos x\cos y}$	
3. Divide out common factors.	$=\dfrac{\sin x\cancel{\cos y}}{\cos x\cancel{\cos y}}+\dfrac{\cancel{\cos x}\sin y}{\cancel{\cos x}\cos y}$ $=\dfrac{\sin x}{\cos x}+\dfrac{\sin y}{\cos y}$	
4. Apply the reciprocal identity.	$=\dfrac{\sin x}{\cos x}+\dfrac{\sin y}{\cos y}$ $=\tan x+\tan y$	

Since LHS = RHS, the proof is complete.

PERMUTATIONS AND COMBINATIONS

Lesson 1—The Fundamental Counting Principle

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
List the stages of the task.

The task is to form a three-person executive from a committee of 12 people.

This task has three stages:

1. Select the chairperson. There are 12 possible selections for the chairperson.
2. Select the vice chairperson. There are 11 people remaining. Therefore, there are 11 possible selections for the vice chairperson.
3. Select the secretary. There are 10 people remaining. Therefore, there are 10 possible selections for the secretary.

Step 2
Determine the number of possible three-person executives.

Apply the fundamental counting principle. Multiply the number of possible selections for each position in the executive.
$12 \times 11 \times 10 = 1\,320$

Therefore, there are 1 320 possible three-person executives.

2. **Step 1**
List the stages of the task.

The task is to make a five-character licence plate that fits the given restrictions. This task has five stages:

1. Select the first character. There are 21 consonants in the alphabet, so there are 21 choices for the first character.
2. Select the second character. This letter must be a vowel. There are 5 vowels, so there are 5 choices for the second character.
3. Select the third character. Since the third character must be a consonant, there are 21 choices for the third character.
4. Select the fourth character. An odd digit must be selected. Since there are 5 odd digits (1, 3, 5, 7, and 9), there are 5 choices for the fourth character.
5. Select the fifth character. From the remaining 4 odd digits, there are 4 choices for the fifth character.

Step 2
Determine the number of possible licence plates for this particular province.

Apply the fundamental counting principle.

Multiply the number of choices for each character on the licence plate.
$21 \times 5 \times 21 \times 5 \times 4 = 44\,100$

There are 44 100 possible licence plates for this particular province.

3. Rewrite the factorials as products, and then cancel out the common factors in the numerator and denominator.

$$\begin{aligned}&\frac{9!}{3!}\\&=\frac{9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{3\times 2\times 1}\\&=9\times 8\times 7\times 6\times 5\times 4\\&=60\,480\end{aligned}$$

Therefore, $\frac{9!}{3!}=60\,480$.

4. **Step 1**
Rewrite the denominator as $(n-12)(n-13)(n-14)(n-15)!$

$$\frac{(n-15)!}{(n-12)!}=\frac{(n-15)!}{(n-12)(n-13)(n-14)(n-15)!}$$

Step 2
Cancel out the common factors in the numerator and denominator.

$$\begin{aligned}&\frac{(n-15)!}{(n-12)(n-13)(n-14)(n-15)!}\\&=\frac{1}{(n-12)(n-13)(n-14)}\end{aligned}$$

Therefore, $\frac{(n-15)!}{(n-12)!}=\frac{1}{(n-12)(n-13)(n-14)}$.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Determine the number of choices for skis and boots.

Since rental package contains deluxe skis only, the number of choices for skis is 1. The number of choices for boots is 3.

Step 2
Use the fundamental counting principle to find the number of different rental packages.

The total number of different packages is found by multiplying the number of choices for skis by the number of choices for boots.
$1 \times 3 = 3$

There are three different rental packages of one pair of deluxe skis and one pair of boots.

2. **Step 1**
Determine the choices of skis and boots.

The number of choices for skis is 3. The number of choices for boots is 3.

Step 2
Use the fundamental counting principle to find the number of different rental packages.

The total number of different packages is found by multiplying the number of choices for skis by the number of choices for boots.
$3 \times 3 = 9$

There are 9 different rental packages consisting of one pair of skis and one pair of boots.

3. **Step 1**
List the stages of selecting a four-digit bank machine code.

Creating a four-digit bank machine code consists of four stages:

1. Assign the first digit. There are 10 ways to assign the first digit.
2. Assign the second digit. There are 9 digits left, so there are 9 ways to assign the second digit.
3. Assign the third digit. With 8 digits remaining, there are 8 ways to assign the third digit.
4. Assign the fourth digit. Since 7 digits remain, there are 7 ways to assign the fourth digit.

Step 2
Apply the fundamental counting principle.
$10 \times 9 \times 8 \times 7 = 5\,040$

There are 5 040 possible four-digit bank codes in which the digits are all different.

4. The seating arrangement could either start with a boy or a girl. Therefore, determine the number of possibilities for which the row starts with a girl, and then determine the number of possibilities for which the row starts with a boy. The sum of these possibilities gives the total number of seating arrangements.

Step 1
Determine the number of choices for each position in the row if it starts with a girl.

Since there are 4 girls, there are 4 ways to fill the first seat. Since boys and girls are to sit in alternate positions, there should be a boy in the second place. Since there are 4 boys, there are 4 ways to fill the second seat. Similarly, there are 3 ways to fill the third and fourth seats, 2 ways to fill in the fifth and sixth seats, and 1 way to fill the seventh and eighth seats.

Step 2
Apply the fundamental counting principle to find the total number of seating arrangements starting with a girl.
$4 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1 = 576$

The number of possible seating arrangements starting with a girl is 576.

Step 3
Determine the number of choices for each position in the row if it starts with a boy.

There will also be 576 seating arrangements with a boy in first place.

Step 4
Determine the total number of possible seating arrangements.

The total number of possible seating arrangements is $576 + 576 = 1152$.

5. **Step 1**
Determine the number of possible choices for each digit.

This task involves five stages:

1. Assign the first digit. The five-digit number can begin with a 7 or an 8. Therefore, there are 2 ways to assign the first digit.
2. Assign the second digit. There are 10 digits between 0 and 9. Therefore, there are 10 ways to assign the second digit.
3. Assign the third digit. There are 10 ways to assign the third digit.
4. Assign the fourth digit. There are 10 ways to assign the fourth digit.
5. Assign the fifth digit. The five-digit number has to be even. Any number ending in 0, 2, 4, 6, or 8 is even. Therefore, there are 5 ways to assign the last digit.

Step 2
Determine the number of even five-digit whole numbers that begin with a 7 or an 8.

Apply the fundamental counting principle.
$2 \times 10 \times 10 \times 10 \times 5 = 10\,000$

Therefore, there are 10 000 different even five-digit whole numbers that begin with a 7 or an 8.

6. Step 1
Determine how many three-digit numbers start with the digit 3.

The first digit must be a 3. The remaining two digits can be any of the digits between 0 and 9.

Therefore, by the fundamental counting principle, there are $1\times10\times10=100$ three-digit numbers that start with the digit 3.

Step 2
Determine how many three-digit numbers have no 3s.

Since the first digit cannot be a 3 or a 0, there are 8 ways to assign the first digit. The last two positions could be any digit between 0 and 9 except for 3, so there are 9 ways to assign the second and third digits. By the fundamental counting principle, there are $8\times9\times9=648$ three-digit numbers that have no 3s.

Step 3
Determine the total number of possible three-digit numbers that either start with the digit 3 or have no 3s.

The total can be found by adding the two possibilities together.
$100+648=748$

There are 748 three-digit numbers that either start with the digit 3 or have no 3s.

7. Step 1
Rewrite the denominator as $9\times8\times7\times6!$.
$$\frac{6!}{9!}=\frac{6!}{9\times8\times7\times6!}$$

Step 2
Cancel out the common factors in the numerator and denominator, and then evaluate.
$$\begin{aligned}&\frac{6!}{9\times8\times7\times6!}\\&=\frac{1}{9\times8\times7}\\&=\frac{1}{504}\end{aligned}$$
Therefore, $\frac{6!}{9!}=\frac{1}{504}$.

8. Step 1
Rewrite 26! as $26\times25\times24!$.
$$\frac{26!}{24!}-17=\frac{26\times25\times24!}{24!}-17$$

Step 2
Cancel out the common factors in the numerator and denominator, and then evaluate.
$$\begin{aligned}&\frac{26\times25\times24!}{24!}-17\\&=(26\times25)-17\\&=650-17\\&=633\end{aligned}$$
Therefore, $\frac{26!}{24!}-17=633$.

9. Step 1
Rewrite the numerator as $(n+3)(n+2)(n+1)n!$.
$$\frac{(n+3)!}{n!}=\frac{(n+3)(n+2)(n+1)n!}{n!}$$

Step 2
Cancel out the common factors in the numerator and denominator.
$$\begin{aligned}&\frac{(n+3)(n+2)(n+1)n!}{n!}\\&=(n+3)(n+2)(n+1)\end{aligned}$$
Therefore, $\frac{(n+3)!}{n!}=(n+3)(n+2)(n+1)$.

10. Step 1
Rewrite $(n+1)!$ as $(n+1)n(n-1)!$ and $n!$ as $n(n-1)!$.

$$\frac{n!(n+1)!}{(n-1)!(n-1)!}=\frac{n(n-1)!(n+1)n(n-1)!}{(n-1)!(n-1)!}$$

Step 2
Cancel out the common factors in the numerator and denominator.
$$\begin{aligned}&\frac{n(n-1)!(n+1)n(n-1)!}{(n-1)!(n-1)!}\\&=n(n+1)n\end{aligned}$$

Step 3
Expand.
$$\begin{aligned}&n(n+1)n\\&=(n^2+n)n\\&=n^3+n^2\end{aligned}$$

Therefore, $\frac{n!(n+1)!}{(n-1)!(n-1)!}=n^3+n^2$.

Lesson 2—Permutations

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. Since there are 11 different players, and all players are to be assigned a different position, the order matters. Assuming that each of the 11 players can play any of the 11 positions, the number of permutations of the 11 players is $11\times10\times9\times8\times7\times6\times5\times4\times3\times2\times1=39\ 916\ 800$.

Therefore, there are 39 916 800 different arrangements for the 11 players.

2. If you were to determine how many arrangements for planting 10 distinct trees, there would be 10! arrangements. However, not all the trees are distinct.

If each tree is represented by its first letter (P for pine, E for elm, B for birch, and W for willow), the arrangement consists of 3 Ps, 4 Es, 2 Bs, and 1 W. For example, one possible order to plant the trees would be EEBWPEPPBE, and another order would be PEPEEEBPBW.

The number of possible arrangements for planting the trees is $\frac{10!}{3!4!2!1!}=12600$.

3. **Step 1**
Substitute 12 for n and 5 for r in the formula ${}_nP_r=\frac{n!}{(n-r)!}$.

$${}_nP_r=\frac{n!}{(n-r)!}$$
$${}_{12}P_5=\frac{12!}{(12-5)!}$$
$${}_{12}P_5=\frac{12!}{7!}$$

Step 2
Rewrite $\frac{12!}{7!}$ as a quotient of products, and then reduce the numerator and denominator.

$${}_{12}P_5=\frac{12!}{7!}$$
$${}_{12}P_5=\frac{12\times11\times10\times9\times8\times7!}{7!}$$
$${}_{12}P_5=12\times11\times10\times9\times8$$
$${}_{12}P_5=95040$$

4. There are 6 children, and the number of different seating arrangements of 4 children is required. This problem is a permutation problem. The number of different seating arrangements of 4 children is given by ${}_6P_4$. Using a calculator, the value of ${}_6P_4$ is 360. Therefore, there are 360 different seating arrangements.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. **Step 1**
Substitute 11 for n and 3 for r in the formula ${}_nP_r=\frac{n!}{(n-r)!}$.

$${}_nP_r=\frac{n!}{(n-r)!}$$
$${}_{11}P_3=\frac{11!}{(11-3)!}$$
$${}_{11}P_3=\frac{11!}{8!}$$

Step 2
Rewrite $\frac{11!}{8!}$ as a quotient of products, and then reduce the numerator and denominator.

$${}_{11}P_3=\frac{11!}{8!}$$
$${}_{11}P_3=\frac{11\times10\times9\times8!}{8!}$$
$${}_{11}P_3=11\times10\times9$$
$${}_{11}P_3=990$$

2. **Step 1**
Substitute 17 for n and 4 for r in the formula ${}_nP_r=\frac{n!}{(n-r)!}$.

$${}_nP_r=\frac{n!}{(n-r)!}$$
$${}_{17}P_4=\frac{17!}{(17-4)!}$$
$${}_{17}P_4=\frac{17!}{13!}$$

Step 2
Rewrite $\frac{17!}{13!}$ as a quotient of products, and then reduce the numerator and denominator.

$${}_{17}P_4=\frac{17!}{13!}$$
$${}_{17}P_4=\frac{17\times16\times15\times14\times13!}{13!}$$
$${}_{17}P_4=57120$$

3. **Step 1**

Apply the formula ${}_nP_r = \dfrac{n!}{(n-r)!}$.

In the equation ${}_{n+1}P_2 = 2$, rewrite ${}_nP_2$ as $\dfrac{(n+1)!}{((n+1)-2)!}$.

$$
\begin{aligned}
{}_{n+1}P_2 &= 2\\
\frac{(n+1)!}{((n+1)-2)!} &= 2\\
\frac{(n+1)!}{(n-1)!} &= 2
\end{aligned}
$$

Step 2

Rewrite $(n+1)!$ as $(n+1)n(n-1)!$.

$$
\begin{aligned}
\frac{(n+1)!}{(n-1)!} &= 2\\
\frac{(n+1)n(n-1)!}{(n-1)!} &= 2
\end{aligned}
$$

Step 3

Divide out common factors, and solve for *n*.

$$
\begin{aligned}
\frac{(n+1)n(n-1)!}{(n-1)!} &= 2\\
(n+1)n &= 2\\
n^2+n &= 2\\
n^2+n-2 &= 0
\end{aligned}
$$

Factor $n^2+n-2=0$.

$$
\begin{aligned}
n^2+n-2 &= 0\\
(n-1)(n+2) &= 0
\end{aligned}
$$

$$
\begin{aligned}
n-1 &= 0 \qquad & n+2 &= 0\\
n &= 1 & n &= -2
\end{aligned}
$$

Since $n \geq 2$, the value of *n* is 1.

4. **Step 1**

Apply the formula ${}_nP_r = \dfrac{n!}{(n-r)!}$.

In the equation ${}_nP_2 = 240$, rewrite ${}_nP_2$ as $\dfrac{n!}{(n-2)!}$.

$$
\begin{aligned}
{}_nP_2 &= 240\\
\frac{n!}{(n-2)!} &= 240
\end{aligned}
$$

Step 2

Rewrite *n*! as $n(n-1)(n-2)!$.

$$
\begin{aligned}
\frac{n!}{(n-2)!} &= 240\\
\frac{n(n-1)(n-2)!}{(n-2)!} &= 240
\end{aligned}
$$

Step 3

Divide out common factors, and solve for *n*.

$$
\begin{aligned}
\frac{n(n-1)(n-2)!}{(n-2)!} &= 240\\
n(n-1) &= 240\\
n^2-n &= 240\\
n^2-n-240 &= 0
\end{aligned}
$$

Factor $n^2 - n - 240 = 0$.

$$
\begin{aligned}
n^2-n-240 &= 0\\
(n-16)(n+15) &= 0
\end{aligned}
$$

$$
\begin{aligned}
n-16 &= 0 \qquad & n+15 &= 0\\
n &= 16 & n &= -15
\end{aligned}
$$

Since $n \geq 2$, the value of *n* is 16.

5. Kyla can listen to 4 CDs in 4! different arrangements in which order matters.

Evaluate 4!.

$$
\begin{aligned}
&4!\\
&= 4\times 3\times 2\times 1\\
&= 2
\end{aligned}
$$

Kyla can listen to 4 CDs in 24 different arrangements.

6. Since the first three positions have already been determined, consider how the last three runners could finish. Stevie, Ray, and Vaughan can finish the race in 3! different ways. Therefore, the race has 3 × 2× 1, or 6, different outcomes.

7. Sarah has 9 gifts that comprise of 3 identical music books, 4 identical tuning forks, a Mozart CD, and a box of chocolates.

Therefore, in the formula $\dfrac{n!}{a!b!c!d!}$, substitute 9 for *n*, 3 for *a*, 4 for *b*, 1 for *c*, and 1 for *d*, and simplify.

$$
\begin{aligned}
&\frac{n!}{a!b!c!d!}\\
&= \frac{9!}{3!4!1!1!}\\
&= \frac{9\times 8\times 7\times 6\times 5\times 4!}{3\times 2\times 1\times 4!}\\
&= \frac{9\times 8\times 7\times 6\times 5}{3\times 2\times 1}\\
&= 2520
\end{aligned}
$$

Therefore, there are 2 520 ways of distributing the gifts.

8. **Step 1**
Determine the possible arrangements for the restricted leg positions.

Curt and Adam have to run the first 2 legs of the relay race in either order. Therefore, the first 2 legs can be arranged in $2\times1=2$ different ways. Since Ben has to run the last leg of the relay race, there is only 1 arrangement for the last leg.

Step 2
Determine the number of possible arrangements for the other 5 runners.

The remaining 5 runners must be arranged in the third, fourth, fifth, sixth, and seventh leg positions.

Arranging 5 people for 5 distinct positions can be represented as 5!. Therefore, the remaining 5 runners can be arranged in $5\times4\times3\times2\times1=120$ different ways.

Step 3
Apply the fundamental counting principle.
$2\times120\times1=240$

Therefore, the number of different arrangements of the 8 runners in the relay race is 240.

9. There are 8 competitors, and the number of different outcomes for the top 3 positions in the race is required. Therefore, to arrange 3 competitors out of 8, use the notation ${}_8P_3$.

Use the permutation formula ${}_nP_r=\dfrac{n!}{(n-r)!}$ to solve the problem.

Substitute 8 for *n* and 3 for *r*, and then evaluate.

$$\begin{aligned}
{}_nP_r &= \frac{n!}{(n-r)!}\\
{}_8P_3 &= \frac{8!}{(8-3)!}\\
{}_8P_3 &= \frac{8!}{5!}\\
{}_8P_3 &= \frac{8\times7\times6\times5!}{5!}\\
{}_8P_3 &= 8\times7\times6\\
{}_8P_3 &= 336
\end{aligned}$$

Therefore, there are 336 different possible outcomes for the top 3 positions.

10. There are 8 distinct letters in the word KITCHENS, and 6 letters are arranged at a time. The order in which the letters are arranged matters, so this is a permutation problem.

Use the permutation formula ${}_nP_r=\dfrac{n!}{(n-r)!}$ to solve the problem.

Substitute 8 for *n* and 6 for *r*, and evaluate.

$$\begin{aligned}
{}_nP_r &= \frac{n!}{(n-r)!}\\
{}_8P_6 &= \frac{8!}{(8-6)!}\\
{}_8P_6 &= \frac{8!}{2!}\\
{}_8P_6 &= \frac{8\times7\times6\times5\times4\times3\times2!}{2!}\\
{}_8P_6 &= 8\times7\times6\times5\times4\times3\\
{}_8P_6 &= 20160
\end{aligned}$$

There are 20 160 different six-letter arrangements that can be made.

Lesson 3—Combinations

CLASS EXERCISES ANSWERS AND SOLUTIONS

1. The order in which the numbers are selected for a raffle ticket does not matter. In other words, a raffle ticket with the selected numbers 3, 45, 63, 1, and 33 is the same as a raffle ticket with the selected numbers 33, 45, 1, 63, and 3. Therefore, this is a combination problem.

2. **Step 1**
Apply the formula ${}_nC_r=\dfrac{n!}{(n-r)!r!}$.

In the equation ${}_nC_2=28$, rewrite ${}_nC_2$ as $\dfrac{n!}{(n-2)!2!}$.

$$\begin{aligned}
{}_nC_2 &= 28\\
\frac{n!}{(n-2)!2!} &= 28
\end{aligned}$$

Step 2
Rewrite $n!$ as $n(n-1)(n-2)!$.

$$\begin{aligned}
\frac{n!}{(n-2)!2!} &= 28\\
\frac{(n-1)(n-2)!}{(n-2)!2!} &= 28
\end{aligned}$$

Step 3
Divide out common factors.

$$\frac{n(n-1)(n-2)!}{(n-2)!2!}=28$$
$$\frac{n(n-1)}{2!}=28$$

Step 4
Solve for *n*.

$$\frac{n(n-1)}{2!}=28$$
$$\frac{n^2-n}{2\times1}=28$$
$$n^2-n=56$$
$$n^2-n-56=0$$
$$(n+7)(n-8)=0$$
$$n+7=0 \qquad n-8=0$$
$$n=-7 \qquad n=8$$

Since $n\geq 2$, the solution is $n=8$.

3. The order in which the members of the committee are selected does not matter. Therefore, this is a combination problem. Since there are 10 people available for the committee, and 6 members are required, the number of possible committees is given by ${}_{10}C_6$.

Step 1
In the formula ${}_nC_r=\frac{n!}{(n-r)!r!}$, substitute 10 for *n* and 6 for *r*.

$${}_nC_r=\frac{n!}{(n-r)!r!}$$
$${}_{10}C_6=\frac{10!}{(10-6)!6!}$$
$${}_{10}C_6=\frac{10!}{4!6!}$$

Step 2
Evaluate $\frac{10!}{4!6!}$.

$$\frac{10!}{4!6!}$$
$$=\frac{10\times9\times8\times7\times6!}{4!6!}$$
$$=\frac{10\times9\times8\times7}{4!}$$
$$=\frac{10\times9\times8\times7}{4\times3\times2\times1}$$
$$=\frac{5040}{624}$$
$$=210$$

Therefore, the number of possible six-member committees is 210.

4. A hospital team is made of 2 doctors and 5 nurses. There are 3 doctors to choose from and 12 nurses to choose from. The order in which the doctors and nurses are chosen does not matter.

Step 1
Determine the number of possible selections of doctors.

A hospital team consists of 2 doctors, and there are 3 doctors to choose from.

Therefore, the number of possible selections of doctors is ${}_3C_2=3$.

Step 2
Determine the number of possible selections of nurses.

A hospital team consists of 5 nurses, and there are 12 nurses to choose from.

Therefore, the number of possible selections of nurses is ${}_{12}C_5=792$.

Step 3
Determine the number of different hospital teams. Applying the fundamental counting principle, the total number of different hospital teams possible is ${}_3C_2\times{}_{12}C_5=3\times792=2376$.

PRACTICE EXERCISES ANSWERS AND SOLUTIONS

1. In the formula ${}_nC_r=\frac{n!}{(n-r)!r!}$, substitute 7 for *n* and 5 for *r*.

Step 1

$${}_nC_r=\frac{n!}{(n-r)!r!}$$
$${}_7C_5=\frac{7!}{(7-5)!5!}$$
$${}_7C_5=\frac{7!}{2!5!}$$

Step 2
Evaluate $\frac{7!}{2!5!}$.

$$\frac{7!}{2!5!}$$
$$=\frac{7\times6\times5!}{2!5!}$$
$$=\frac{7\times6}{2!}$$
$$=\frac{7\times6}{2\times1}$$
$$=21$$

Therefore, ${}_7C_5=21$.

2. Step 1

In the formula ${}_nC_r = \dfrac{n!}{(n-r)!r!}$, substitute 15 for *n* and 4 for *r*.

$$\begin{aligned}{}_nC_r &= \frac{n!}{(n-r)!r!}\\ {}_{15}C_4 &= \frac{15!}{(15-4)!4!}\\ {}_{15}C_4 &= \frac{15!}{11!4!}\end{aligned}$$

Step 2

Evaluate $\dfrac{15!}{11!4!}$.

$$\begin{aligned}&\frac{15!}{11!4!}\\ &= \frac{15\times14\times13\times12\times11!}{11!4!}\\ &= \frac{15\times14\times13\times12}{4!}\\ &= \frac{15\times14\times13\times12}{4\times3\times2\times1}\\ &= \frac{32760}{24}\\ &= 1365\end{aligned}$$

Therefore, ${}_{15}C_4 = 1365$.

3. Step 1

Apply the formula ${}_nC_r = \dfrac{n!}{(n-r)!r!}$.

In the equation ${}_nC_2 = 66$, rewrite ${}_nC_2$ as $\dfrac{n!}{(n-2)!2!}$.

$$\begin{aligned}{}_nC_2 &= 66\\ \frac{n!}{(n-2)!2!} &= 66\end{aligned}$$

Step 2

Rewrite $n!$ as $n(n-1)(n-2)!$.

$$\begin{aligned}\frac{n!}{(n-2)!2!} &= 66\\ \frac{n(n-1)(n-2)!}{(n-2)!2!} &= 66\end{aligned}$$

Step 3

Divide out common factors.

$$\begin{aligned}\frac{n(n-1)(n-2)!}{(n-2)!2!} &= 66\\ \frac{n(n-1)}{2!} &= 66\end{aligned}$$

Step 4

Solve for *n*.

$$\begin{aligned}\frac{n(n-1)}{2!} &= 66\\ \frac{n^2-n}{2\times1} &= 66\\ n^2-n &= 132\\ n^2-n-132 &= 0\\ (n-12)(n+11) &= 0\end{aligned}$$

$$\begin{aligned}n-12 &= 0 \qquad & n+11 &= 0\\ n &= 12 & n &= -11\end{aligned}$$

Since $n \ge 2$, the solution is $n = 12$.

4. Step 1

Apply the formula ${}_nC_r = \dfrac{n!}{(n-r)!r!}$.

In the equation ${}_{n-1}C_2 = 6$, rewrite ${}_{n-1}C_2$ as $\dfrac{(n-1)!}{((n-1)-2)!2!}$.

$$\begin{aligned}{}_{n-1}C_2 &= 6\\ \frac{(n-1)!}{((n-1)-2)!2!} &= 6\\ \frac{(n-1)!}{(n-3)!2!} &= 6\end{aligned}$$

Step 2

Rewrite $(n-1)!$ as $(n-1)(n-2)(n-3)!$.

$$\begin{aligned}\frac{(n-1)!}{(n-3)!2!} &= 6\\ \frac{(n-1)(n-2)(n-3)!}{(n-3)!2!} &= 6\end{aligned}$$

Step 3

Divide out common factors.

$$\begin{aligned}\frac{(n-1)(n-2)(n-3)!}{(n-3)!2!} &= 6\\ \frac{(n-1)(n-2)}{2!} &= 6\end{aligned}$$

Step 4
Solve for *n*.

$$\frac{(n-1)(n-2)}{2!}=6$$
$$\frac{(n-1)(n-2)}{2\times 1}=6$$
$$(n-1)(n-2)=12$$
$$n^2-3n+2=12$$
$$n^2-3n-10=0$$
$$(n-5)(n+2)=0$$
$$n-5=0 \qquad n+2=0$$
$$n=5 \qquad n=-2$$

Since $n\geq 2$, the solution is $n=5$.

5. A group 40 people is to be formed from 3 members, and the order does not matter. Therefore, the number of different boards is given by ${}_{40}C_3$.

Step 1
In the formula ${}_nC_r=\frac{n!}{(n-r)!r!}$, substitute 40 for *n* and 3 for *r*.

$${}_nC_r=\frac{n!}{(n-r)!r!}$$
$${}_{40}C_3=\frac{40!}{(40-3)!3!}$$
$${}_{40}C_3=\frac{40!}{37!3!}$$

Step 2
Evaluate $\frac{40!}{37!3!}$.

$$\frac{40!}{37!3!}$$
$$=\frac{40\times 39\times 38\times 37!}{37!3!}$$
$$=\frac{40\times 39\times 38}{3!}$$
$$=9880$$

Therefore, 9 880 different boards can be formed.

6. The order in which the two flavours are chosen does not matter. Since there are 31 flavours in total, and 2 flavours are chosen at a time, the number of different two-scoop ice-cream cones is given by ${}_{31}C_2$.

Step 1
In the formula ${}_nC_r=\frac{n!}{(n-r)!r!}$, substitute 31 for *n* and 2 for *r*.

$${}_nC_r=\frac{n!}{(n-r)!r!}$$
$${}_{31}C_2=\frac{31!}{(31-2)!2!}$$
$${}_{31}C_2=\frac{31!}{29!2!}$$

Step 2
Evaluate $\frac{31!}{29!2!}$.

$$\frac{31!}{29!2!}$$
$$=\frac{31\times 30\times 29!}{29!2!}$$
$$=\frac{31\times 30}{2!}$$
$$=\frac{31\times 30}{2\times 1}$$
$$=\frac{930}{2}$$
$$=465$$

Therefore, 465 different two-scoop ice-cream cones are possible from a selection of 31 flavours.

7. Two sculptures must be selected from 25 sculptures, and the order of selection does not matter. Therefore, the number of different donations is given by ${}_{25}C_2$.

Step 1
In the formula ${}_nC_r=\frac{n!}{(n-r)!r!}$, substitute 25 for *n* and 2 for *r*.

$${}_nC_r=\frac{n!}{(n-r)!r!}$$
$${}_{25}C_2=\frac{25!}{(25-2)!2!}$$
$${}_{25}C_2=\frac{25!}{23!2!}$$

Step 2

Evaluate $\frac{25!}{23!2!}$.

$$\begin{aligned}&\frac{25!}{23!2!}\\&=\frac{25\times24\times23!}{23!2!}\\&=\frac{25\times24}{2!}\\&=\frac{25\times24}{2\times1}\\&=\frac{600}{2}\\&=300\end{aligned}$$

Mr. Fraser can make 300 different donations.

8. If Terry is one of the 3 members of the group, then there are 2 other members to be selected from the group of 19. The order of selection does not matter.

Step 1

In the formula ${}_nC_r=\frac{n!}{(n-r)!r!}$, substitute 19 for *n* and 2 for *r*.

$$\begin{aligned}{}_nC_r&=\frac{n!}{(n-r)!r!}\\{}_{19}C_2&=\frac{19!}{(19-2)!2!}\\{}_{19}C_2&=\frac{19!}{17!2!}\end{aligned}$$

Step 2

Evaluate ${}_{19}C_2=\frac{19!}{17!2!}$.

$$\begin{aligned}&\frac{19!}{17!2!}\\&=\frac{19\times18\times17!}{17!2!}\\&=\frac{19\times18}{2!}\\&=\frac{19\times18}{2\times1}\\&=\frac{342}{2}\\&=171\end{aligned}$$

Therefore, there are 171 groups that can go on the trip and include Terry.

9. The order that the men and women are chosen for the committee does not matter.

Step 1
Determine the number of selections of 2 women from a group of 4.

In the formula ${}_nC_r=\frac{n!}{(n-r)!r!}$, substitute 4 for *n* and 2 for *r*, and simplify.

$$\begin{aligned}{}_nC_r&=\frac{n!}{(n-r)!r!}\\{}_4C_2&=\frac{4!}{(4-2)!2!}\\{}_4C_2&=\frac{4!}{2!2!}\\{}_4C_2&=\frac{4\times3\times2!}{2!2!}\\{}_4C_2&=\frac{4\times3}{2!}\\{}_4C_2&=\frac{4\times3}{2\times1}\\{}_4C_2&=6\end{aligned}$$

Step 2
Determine the number of selections of 4 men from a group of 5.In the formula ${}_nC_r=\frac{n!}{(n-r)!r!}$, substitute 5 for *n* and 4 for *r*, and simplify.

$$\begin{aligned}{}_nC_r&=\frac{n!}{(n-r)!r!}\\{}_5C_4&=\frac{5!}{(5-4)!4!}\\{}_5C_4&=\frac{5!}{1!4!}\\{}_5C_4&=\frac{5\times4!}{1!4!}\\{}_5C_4&=\frac{5}{1!}\\{}_5C_4&=5\end{aligned}$$

Step 3
Determine the total number of possible committees.

Applying the fundamental counting principle, the number of possible committees with 4 men and 2 women is $6\times5=30$.

10. The chef can choose from 3 types of meat, 5 types of vegetables, and 2 types of noodles. The order in which the items are chosen for each meal does not matter.

Step 1
Determine the number of possible selections of meat.

The chef needs to select 1 type of meat from 3 options.

In the formula ${}_nC_r = \dfrac{n!}{(n-r)!r!}$, substitute 3 for *n* and 1 for *r*, and simplify.

$$
\begin{aligned}
{}_nC_r &= \frac{n!}{(n-r)!r!} \\
{}_3C_1 &= \frac{3!}{(3-1)!1!} \\
{}_3C_1 &= \frac{3!}{2!1!} \\
{}_3C_1 &= \frac{3\times 2!}{2!1!} \\
{}_3C_1 &= \frac{3}{1!} \\
{}_3C_1 &= 3
\end{aligned}
$$

Step 2
Determine the number of possible selections of vegetables.

The chef needs to select 3 vegetables from 5 options.

In the formula ${}_nC_r = \dfrac{n!}{(n-r)!r!}$, substitute 5 for *n* and 3 for *r*, and simplify.

$$
\begin{aligned}
{}_nC_r &= \frac{n!}{(n-r)!r!} \\
{}_5C_3 &= \frac{5!}{(5-3)!3!} \\
{}_5C_3 &= \frac{5!}{2!3!} \\
{}_5C_1 &= \frac{5\times 4\times 3!}{2!3!} \\
{}_5C_1 &= \frac{5\times 4}{2!} \\
{}_5C_1 &= 10
\end{aligned}
$$

Step 3
Determine the number of possible selections of noodles. The chef needs to select 1 type of noodle from 2 options.

In the formula ${}_nC_r = \dfrac{n!}{(n-r)!r!}$, substitute 2 for *n* and 1 for *r*, and simplify.

$$
\begin{aligned}
{}_nC_r &= \frac{n!}{(n-r)!r!} \\
{}_2C_1 &= \frac{2!}{(2-1)!1!} \\
{}_2C_1 &= \frac{2!}{1!1!} \\
{}_2C_1 &= 2
\end{aligned}
$$

Step 4
Determine the number of different stir-fry meals the chef can create.

Applying the fundamental counting principle, the number of different stir-fry meals the chef can create is $3\times 10\times 2 = 60$.

Lesson 4—Binomial Expansions and Pascal's Triangle

CLASS EXERCISES
ANSWERS AND SOLUTIONS

1. **Step 1**

Apply the binomial theorem, and substitute 5 for *n*, $\frac{x}{3}$ for *x*, and –9 for *y*.

$$(x+y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_ny^n$$
$$\left(\frac{x}{3}-9\right)^5 = {}_5C_0\left(\frac{x}{3}\right)^5 + {}_5C_1\left(\frac{x}{3}\right)^{5-1}(-9) + {}_5C_2\left(\frac{x}{3}\right)^{5-2}(-9)^2 + {}_5C_3\left(\frac{x}{3}\right)^{5-3}(-9)^3 + {}_5C_4\left(\frac{x}{3}\right)^{5-4}(-9)^4 + {}_5C_5(-9)^5$$

Step 2
Simplify.

$$
\begin{aligned}
&{}_5C_0\left(\frac{x}{3}\right)^5 + {}_5C_1\left(\frac{x}{3}\right)^{5-1}(-9) + {}_5C_2\left(\frac{x}{3}\right)^{5-2}(-9)^2 + {}_5C_3\left(\frac{x}{3}\right)^{5-3}(-9)^3 + {}_5C_4\left(\frac{x}{3}\right)^{5-4}(-9)^4 + {}_5C_5(-9)^5\\
&=1\left(\frac{x}{3}\right)^5 + 5\left(\frac{x}{3}\right)^4(-9) + 10\left(\frac{x}{3}\right)^3(-9)^2 + 10\left(\frac{x}{3}\right)^2(-9)^3 + 5\left(\frac{x}{3}\right)^1(-9)^4 + 1(-9)^5\\
&=\frac{x^5}{243} + 5\left(\frac{x^4}{81}\right)(-9) + 10\left(\frac{x^3}{27}\right)(81) + 10\left(\frac{x^2}{9}\right)(-729) + 5\left(\frac{x}{3}\right)(6561) - 59049\\
&=\frac{x^5}{243} - \frac{5x^4}{9} + 30x^3 - 810x^2 + 10935x - 59049
\end{aligned}
$$

2. **Step 1**
Determine the general term of the expansion.

In the equation $t_{k+1} = {}_nC_kx^{n-k}y^k$, let $n=6$, $x=x$, and $y=\frac{2}{x}$.

$$t_{k+1} = {}_nC_kx^{n-k}y^k$$
$$t_{k+1} = {}_{16}C_k\left(x^{16-k}\right)\left(\frac{2}{x}\right)^k$$

Step 2
Apply the laws of exponents, and simplify the general term.

$$t_{k+1} = {}_{16}C_k\left(x^{16-k}\right)\left(\frac{2}{x}\right)^k$$
$$t_{k+1} = {}_{16}C_k\left(x^{16-k}\right)\left(\frac{2^k}{x^k}\right)$$
$$t_{k+1} = {}_{16}C_k\left(\frac{x^{16-k}}{x^k}\right)\left(2^k\right)$$
$$t_{k+1} = {}_{16}C_k\left(x^{16-2k}\right)\left(2^k\right)$$

Step 3
Determine the value of *k*.

The constant term of an expansion has a degree of 0. Therefore, equate the exponent of x^{16-2k} to 0, and solve for *k*.

$$
\begin{aligned}
16-2k&=0\\
16&=2k\\
8&=k
\end{aligned}
$$

Step 4
Determine the constant term.

Substitute 8 for k in the general term $t_{k+1} = {}_{16}C_k\left(x^{16-k}\right)\left(\frac{2}{x}\right)^k$, and simplify.

$$\begin{aligned}
t_{k+1} &= {}_{16}C_k\left(x^{16-k}\right)\left(\frac{2}{x}\right)^k \\
t_{8+1} &= {}_{16}C_8\left(x^{16-8}\right)\left(\frac{2}{x}\right)^8 \\
t_9 &= (12870)\left(x^8\right)\left(\frac{256}{x^8}\right) \\
t_5 &= (12870)(256) \\
t_5 &= 3294720
\end{aligned}$$

Therefore, the constant term in the expansion of $\left(x+\frac{2}{x}\right)^6$ is 3 294 720.

PRACTICE EXERCISES
ANSWERS AND SOLUTIONS

1. **Step 1**
Apply the binomial theorem, and substitute 4 for n, $5a$ for x, and $-3b$ for y.

$$(x+y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_ny^n$$
$$(5a-3b)^4 = {}_4C_0(5a)^4 + {}_4C_1(5a)^{4-1}(-3b) + {}_4C_2(5a)^{4-2}(-3b)^2 + {}_4C_3(5a)^{4-3}(-3b)^3 + {}_4C_4(-3b)^4$$

Step 2
Simplify.

$$\begin{aligned}
&{}_4C_0(5a)^4 + {}_4C_1(5a)^{4-1}(-3b) + {}_4C_2(5a)^{4-2}(-3b)^2 + {}_4C_3(5a)^{4-3}(-3b)^3 + {}_4C_4(-3b)^4 \\
&= {}_4C_0(5a)^4 + {}_4C_1(5a)^3(-3b) + {}_4C_2(5a)^2(-3b)^2 + {}_4C_3(5a)^1(-3b)^3 + {}_4C_4(-3b)^4 \\
&= 1\left(625a^4\right) + 4\left(125a^3\right)(-3b) + 6\left(25a^2\right)\left(9b^2\right) + 4(5a)\left(-27b^3\right) + 1\left(81b^4\right) \\
&= 625a^4 - 1500a^3b + 1350a^2b^2 - 540ab^3 + 81b^4
\end{aligned}$$

2. **Step 1**

Apply the binomial theorem, and substitute 5 for n, $\frac{2}{a}$ for x, and -2 for y.

$$(x+y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_ny^n$$
$$\left(\frac{2}{a}-2\right)^5 = {}_5C_0\left(\frac{2}{a}\right)^5 + {}_5C_1\left(\frac{2}{a}\right)^{5-1}(-2) + {}_5C_2\left(\frac{2}{a}\right)^{5-2}(-2)^2 + {}_5C_3\left(\frac{2}{a}\right)^{5-3}(-2)^3 + {}_5C_4\left(\frac{2}{a}\right)^{5-4}(-2)^4 + {}_5C_5(-2)^5$$

Step 2
Simplify.

$$\begin{aligned}
&{}_5C_0\left(\frac{2}{a}\right)^5 + {}_5C_1\left(\frac{2}{a}\right)^{5-1}(-2) + {}_5C_2\left(\frac{2}{a}\right)^{5-2}(-2)^2 + {}_5C_3\left(\frac{2}{a}\right)^{5-3}(-2)^3 + {}_5C_4\left(\frac{2}{a}\right)^{5-4}(-2)^4 + {}_5C_5(-2)^5 \\
&= {}_5C_0\left(\frac{2}{a}\right)^5 + {}_5C_1\left(\frac{2}{a}\right)^4(-2) + {}_5C_2\left(\frac{2}{a}\right)^3(-2)^2 + {}_5C_3\left(\frac{2}{a}\right)^2(-2)^3 + {}_5C_4\left(\frac{2}{a}\right)^1(-2)^4 + {}_5C_5(-2)^5 \\
&= 1\left(\frac{32}{a^5}\right) + 5\left(\frac{16}{a^4}\right)(-2) + 10\left(\frac{8}{a^3}\right)(4) + 10\left(\frac{4}{a^2}\right)(-8) + 5\left(\frac{2}{a}\right)(16) - 1(32) \\
&= \frac{32}{a^5} - \frac{160}{a^4} + \frac{320}{a^3} - \frac{320}{a^2} + \frac{160}{a} - 32
\end{aligned}$$

Alternate method:

Factor out 2 from the binomial $\left(\frac{2}{a}-2\right)^5$ such that $\left[2\left(\frac{1}{a}-1\right)\right]^5$. Then, apply the power to each term in the bracket such that $2^5\left(\frac{1}{a}-1\right)^5$. Finally, apply the binomial theorem to expand $\left(\frac{1}{a}-1\right)^5$, and multiply each term by 2^5.

3. Step 1

Apply the binomial theorem, and substitute 4 for n, x^2 for x, and $-4y$ for y.

$$(x+y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_ny^n$$
$$\left(x^2-4y\right)^4 = {}_4C_0\left(x^2\right)^4 + {}_4C_1\left(x^2\right)^{4-1}(-4y) + {}_4C_2\left(x^2\right)^{4-2}(-4y)^2 + {}_4C_3\left(x^2\right)^{4-3}(-4y)^3 + {}_4C_4(-4y)^4$$

Step 2

Simplify.

$$\begin{aligned}&{}_4C_0\left(x^2\right)^4 + {}_4C_1\left(x^2\right)^{4-1}(-4y) + {}_4C_2\left(x^2\right)^{4-2}(-4y)^2 + {}_4C_3\left(x^2\right)^{4-3}(-4y)^3 + {}_4C_4(-4y)^4\\
&= {}_4C_0\left(x^2\right)^4 + {}_4C_1\left(x^2\right)^3(-4y) + {}_4C_2\left(x^2\right)^2(-4y)^2 + {}_4C_3\left(x^2\right)(-4y)^3 + {}_4C_4(-4y)^4\\
&= 1\left(x^8\right) + 4\left(x^6\right)(-4y) + 6\left(x^4\right)\left(16y^2\right) + 4\left(x^2\right)\left(-64y^3\right) + 1\left(256y^4\right)\\
&= x^8 - 16x^6y + 96x^4y^2 - 256x^2y^3 + 256y^4\end{aligned}$$

4. Step 1

Apply the binomial theorem, and substitute 6 for n, $\frac{1}{x}$ for x, and $\frac{1}{y}$ for y.

$$(x+y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_ny^n$$
$$\begin{aligned}\left(\frac{1}{x}+\frac{1}{y}\right)^6 = {}&{}_6C_0\left(\frac{1}{x}\right)^6 + {}_6C_1\left(\frac{1}{x}\right)^{6-1}\left(\frac{1}{y}\right) + {}_6C_2\left(\frac{1}{x}\right)^{6-2}\left(\frac{1}{y}\right)^2 + {}_6C_3\left(\frac{1}{x}\right)^{6-3}\left(\frac{1}{y}\right)^3 + {}_6C_4\left(\frac{1}{x}\right)^{6-4}\left(\frac{1}{y}\right)^4\\
&+ {}_6C_5\left(\frac{1}{x}\right)^{6-5}\left(\frac{1}{y}\right)^5 + {}_6C_6\left(\frac{1}{y}\right)^6\end{aligned}$$

Step 2

Simplify.

$$\begin{aligned}&{}_6C_0\left(\frac{1}{x}\right)^6 + {}_6C_1\left(\frac{1}{x}\right)^{6-1}\left(\frac{1}{y}\right) + {}_6C_2\left(\frac{1}{x}\right)^{6-2}\left(\frac{1}{y}\right)^2 + {}_6C_3\left(\frac{1}{x}\right)^{6-3}\left(\frac{1}{y}\right)^3 + {}_6C_4\left(\frac{1}{x}\right)^{6-4}\left(\frac{1}{y}\right)^4 + {}_6C_5\left(\frac{1}{x}\right)^{6-5}\left(\frac{1}{y}\right)^5 + {}_6C_6\left(\frac{1}{y}\right)^6\\
&= {}_6C_0\left(\frac{1}{x}\right)^6 + {}_6C_1\left(\frac{1}{x}\right)^5\left(\frac{1}{y}\right) + {}_6C_2\left(\frac{1}{x}\right)^4\left(\frac{1}{y}\right)^2 + {}_6C_3\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)^3 + {}_6C_4\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)^4 + {}_6C_5\left(\frac{1}{x}\right)^1\left(\frac{1}{y}\right)^5 + {}_6C_6\left(\frac{1}{y}\right)^6\\
&= 1\left(\frac{1}{x^6}\right) + 6\left(\frac{1}{x^5}\right)\left(\frac{1}{y}\right) + 15\left(\frac{1}{x^4}\right)\left(\frac{1}{y^2}\right) + 20\left(\frac{1}{x^3}\right)\left(\frac{1}{y^3}\right) + 15\left(\frac{1}{x^2}\right)\left(\frac{1}{y^4}\right) + 6\left(\frac{1}{x}\right)\left(\frac{1}{y^5}\right) + 1\left(\frac{1}{y^6}\right)\\
&= \frac{1}{x^6} + \frac{6}{x^5y} + \frac{15}{x^4y^2} + \frac{20}{x^3y^3} + \frac{15}{x^2y^4} + \frac{6}{xy^5} + \frac{1}{y^6}\end{aligned}$$

5. Step 1

Apply the binomial theorem, and substitute 6 for n and $2\sqrt{x}$ for y.

$$(x+y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_ny^n$$
$$\begin{aligned}\left(x+2\sqrt{x}\right)^6 = {}&{}_6C_0(x)^6 + {}_6C_1(x)^{6-1}\left(2\sqrt{x}\right) + {}_6C_2(x)^{6-2}\left(2\sqrt{x}\right)^2 + {}_6C_3(x)^{6-3}\left(2\sqrt{x}\right)^3\\
&+ {}_6C_4\left(\frac{1}{x}\right)^{6-4}\left(2\sqrt{x}\right)^4 + {}_6C_5(x)^{6-5}\left(2\sqrt{x}\right)^5 + {}_6C_6\left(2\sqrt{x}\right)^6\end{aligned}$$

Step 2
Simplify.

$$
\begin{aligned}
&{}_6C_0\left(x\right)^6+{}_6C_1\left(x\right)^{6-1}\left(2\sqrt{x}\right)+{}_6C_2\left(x\right)^{6-2}\left(2\sqrt{x}\right)^2+{}_6C_3\left(x\right)^{6-3}\left(2\sqrt{x}\right)^3+{}_6C_4\left(\frac{1}{x}\right)^{6-4}\left(2\sqrt{x}\right)^4\\
&\quad+{}_6C_5\left(x\right)^{6-5}\left(2\sqrt{x}\right)^5+{}_6C_6\left(2\sqrt{x}\right)^6\\
&={}_6C_0\left(x\right)^6+{}_6C_1\left(x\right)^5\left(2\sqrt{x}\right)+{}_6C_2\left(x\right)^4\left(2\sqrt{x}\right)^2+{}_6C_3\left(x\right)^3\left(2\sqrt{x}\right)^3+{}_6C_4\left(x\right)^2\left(2\sqrt{x}\right)^4\\
&\quad+{}_6C_5\left(x\right)\left(2\sqrt{x}\right)^5+{}_6C_6\left(2\sqrt{x}\right)^6\\
&=1x^6+6\left(x^5\right)\left(2\sqrt{x}\right)+15\left(x^4\right)\left(4x\right)+20\left(x^3\right)\left(8x^{\frac{3}{2}}\right)+15\left(x^2\right)\left(16x^2\right)+6\left(x\right)\left(32x^{\frac{5}{2}}\right)+1\left(64x^3\right)\\
&=x^6+12x^5\sqrt{x}+60x^4x+160x^3x^{\frac{3}{2}}+240x^2x^2+192xx^{\frac{5}{2}}+64x^3\\
&=x^6+12x^{\frac{11}{2}}+60x^5+160x^{\frac{9}{2}}+240x^4+192x^{\frac{7}{2}}+64x^3
\end{aligned}
$$

6. **Step 1**

Determine the general term of the expansion of $\left(x+\frac{3}{y}\right)^{11}$.

In the equation $t_{k+1}={}_nC_kx^{n-k}y^k$, let $n=11$ and $y=\frac{3}{y}$.

$$
\begin{aligned}
t_{k+1}&={}_nC_kx^{n-k}y^k\\
t_{k+1}&={}_{11}C_k\left(x\right)^{11-k}\left(\frac{3}{y}\right)^k
\end{aligned}
$$

The general term of the expansion is $t_{k+1}={}_{11}C_k\left(x\right)^{11-k}\left(\frac{3}{y}\right)^k$.

Step 2
Determine the value of *k*.

Since the term containing $\frac{1}{y^5}$ is required, the value of *k* must be 5.

Step 3

Determine the term in the expansion that contains $\frac{1}{y^5}$.

Let $k=5$ in equation $t_{k+1}={}_{11}C_k\left(x\right)^{11-k}\left(\frac{3}{y}\right)^k$, and simplify.

$$
\begin{aligned}
t_{k+1}&={}_{11}C_k\left(x\right)^{11-k}\left(\frac{3}{y}\right)^k\\
t_{5+1}&={}_{11}C_5\left(2\right)^{11-5}\left(\frac{3}{y}\right)^5\\
t_6&={}_{11}C_5\left(2\right)^6\left(\frac{3}{y}\right)^5\\
t_6&=462\left(64\right)\left(\frac{243}{y^5}\right)\\
t_6&=\frac{7185024}{y^5}
\end{aligned}
$$

Therefore, term in the expansion of $\left(x+\frac{3}{y}\right)^{11}$ that contains $\frac{1}{y^5}$ is $\frac{7185024}{y^5}$.

7. Step 1
Determine the general term of the expansion of
$\left(2x^4 - \frac{13}{y}\right)^8$.

In the equation $t_{k+1} = {}_nC_k x^{n-k} y^k$, let $n = 8$, $x = 2x^4$, and $y = -\frac{13}{y}$.

$$t_{k+1} = {}_nC_k x^{n-k} y^k$$
$$t_{k+1} = {}_8C_k \left(2x^4\right)^{8-k}\left(-\frac{13}{y}\right)^k$$

The general term of the expansion is
$$t_{k+1} = {}_8C_k \left(2x^4\right)^{8-k}\left(-\frac{13}{y}\right)^k.$$

Step 2
Apply the exponent rules, and simplify the general term.

$$t_{k+1} = {}_8C_k \left(2x^4\right)^{8-k}\left(-\frac{13}{y}\right)^k$$
$$t_{k+1} = {}_8C_k \left(2^{8-k}\right)x^{4(8-k)}\left(-\frac{13^k}{y^k}\right)$$
$$t_{k+1} = {}_8C_k \left(2^{8-k}\right)x^{32-4k}\left(-\frac{13^k}{y^k}\right)$$

Step 3
Determine the value of *k*.

Since the term containing x^{16} is required, equate the exponent of $x^{32-4}k$ to the exponent of x^{16}, and solve for *k*.
$$32 - 4k = 16$$
$$-4k = -16$$
$$k = 4$$

Step 4
Determine the term in the expansion that contains x^{16}.

Let $k = 4$ in equation $t_{k+1} = {}_8C_k \left(2x^4\right)^{8-k}\left(-\frac{13}{y}\right)^k$,

and simplify.

$$t_{k+1} = {}_8C_k \left(2x^4\right)^{8-k}\left(-\frac{13}{y}\right)^k$$
$$t_{4+1} = {}_8C_4 \left(2x^4\right)^{8-4}\left(-\frac{13}{y}\right)^4$$
$$t_5 = {}_8C_4 \left(2x^4\right)^4\left(-\frac{13}{y}\right)^4$$
$$t_5 = 70\left(16x^{16}\right)\left(\frac{28561}{y^4}\right)$$
$$t_5 = \frac{31988320x^{16}}{y^4}$$

Therefore, the term in the expansion that contains x^{16} is $\frac{31988320x^{16}}{y^4}$.

8. Step 1
In the equation $t_{k+1} = {}_nC_k x^{n-k} y^k$, let $n = 9$, $x = y^2$, and $y = -2$.

$$t_{k+1} = {}_nC_k x^{n-k} y^k$$
$$t_{k+1} = {}_9C_k \left(y^2\right)^{9-k}(-2)^k$$

Step 2
Determine the value of *k*.

Since the coefficient of y^4 is required, $\left(y^2\right)^{9-k}$ must be y^4, which implies that $9-k$ must be 2, and $k = 7$.

Step 3
Determine the coefficient of y^4.

Substitute 7 for *k* in the equation $t_{k+1} = {}_9C_k y^{18-2k}(-2)^k$, and simplify.

$$t_{k+1} = {}_9C_k y^{18-2k}(-2)^k$$
$$t_{7+1} = {}_9C_7 y^{18-2(7)}(-2)^7$$
$$t_8 = (36)y^4(-128)$$
$$t_8 = -4608y^4$$

The coefficient of y^4 in the expansion of $\left(y^2-2\right)^9$ is –4 608.

9. Step 1
In the equation $t_{k+1} = {}_nC_k x^{n-k} y^k$, let $n = 8$, $x = \frac{1}{a^3}$, and $y = -2a$.

$$t_{k+1} = {}_nC_k x^{n-k} y^k$$
$$t_{k+1} = {}_8C_k \left(\frac{1}{a^3}\right)^{8-k}(-2a)^k$$

Step 2
Apply the laws of exponents, and simplify.

$$t_{k+1} = {}_8C_k \left(\frac{1}{a^3}\right)^{8-k}(-2a)^k$$
$$t_{k+1} = {}_8C_k \left(\frac{1}{a^{3(8-k)}}\right) a^k$$
$$t_{k+1} = {}_8C_k \left(\frac{a^k}{a^{24-3k}}\right)$$
$$t_{k+1} = {}_8C_k a^{k-(24-3k)}\left(-2^k\right)$$
$$t_{k+1} = {}_8C_k a^{-24+4k}\left(-2^k\right)$$

Step 3
Determine the value of k.

Since the constant term is required, equate the exponent of a^{-24+4k} to 0, and find k.

$$\begin{aligned} -24+4k &= 0 \\ 4k &= 24 \\ k &= 6 \end{aligned}$$

Step 4
Determine the constant term.

Substitute 6 for k in the equation

$t_{k+1} = {}_8C_k\left(\frac{1}{a^3}\right)^{8-k}(-2a)^k$, and simplify.

$$\begin{aligned} t_{k+1} &= {}_8C_k\left(\frac{1}{a^3}\right)^{8-k}(-2a)^k \\ t_{6+1} &= {}_8C_6\left(\frac{1}{a^3}\right)^{8-6}(-2a)^6 \\ t_7 &= {}_8C_6\left(\frac{1}{a^3}\right)^{2}(-2a)^6 \\ t_7 &= 28\left(\frac{1}{a^6}\right)\left(64a^6\right) \\ t_7 &= 28(64) \\ t_7 &= 1792 \end{aligned}$$

The constant term in the expansion of $\left(\frac{1}{a^3}-2a\right)^8$ is 1 792.

10. **Step 1**
In the equation $t_{k+1} = {}_nC_k x^{n-k}y^k$, let $n=8$ and $y=-2$.

$$\begin{aligned} t_{k+1} &= {}_nC_k x^{n-k}y^k \\ t_{k+1} &= {}_8C_k x^{8-k}(-2)^k \end{aligned}$$

Step 2
Determine the value of k.

Since the term with a degree of 5 is required, equate the exponent of x^{8-k} to 5, and solve for k.

$$\begin{aligned} 8-k &= 5 \\ -k &= -3 \\ k &= 3 \end{aligned}$$

Step 3
Determine the coefficient of the term with a degree of 5.

Substitute 3 for k in the equation $t_{k+1} = {}_8C_k x^{8-k}(-2)^k$, and simplify.

$$\begin{aligned} t_{k+1} &= {}_8C_k x^{8-k}(-2)^k \\ t_4 &= {}_8C_3 x^{8-3}(-2)^3 \\ t_4 &= (56)x^5(-8) \\ t_4 &= -448x^5 \end{aligned}$$

The coefficient is –448.

Practice Test

ANSWERS AND SOLUTIONS

1. **Step 1**
List the stages of the task. The task is to put together an outfit.

This task has three stages:

1. Select a top. There are 5 possible choices for the top.
2. Select a skirt. There are 4 possible choices for the skirt.
3. Select a pair of shoes. There are 3 possible choices for the shoes.

Step 2
Determine the number of different outfits Sally can put together.

Apply the fundamental counting principle. Multiply the number of possible choices for each part of the outfit.
$5\times4\times3=60$

Therefore, Sally can put together 60 different outfits.

2. Since Jen is keeping 1 specific item, there are only 7 items to give to 7 people. Therefore, the number of possible distribution of gifts is given by 7!.
7! = 7 × 6 ×5 ×4 ×3 ×2 ×1 = 5 40

There are 5 040 different distributions of gifts.

3. **Step 1**
Determine how many ways the first letter can be assigned.

The first letter must be N. Since there are two identical Ns, there is $\frac{2}{2!}=1$ way to assign the first letter.

Step 2
Determine the number of possible arrangements of the other six letters.

For the other six places, there are 6 letters to arrange. Two of these letters are identical *A*s.

Therefore, in the formula $\frac{n!}{a!}$, substitute 6 for *n* and 2 for *a*, and simplify.

$$\begin{aligned}&\frac{6!}{2!}\\&=\frac{6\times5\times4\times3\times2!}{2!}\\&=6\times5\times4\times3\\&=360\end{aligned}$$

Therefore, the number of different letter arrangements that begin with the letter N is 360.

4. Since the order in which the vehicles are lined up matters, this is a permutation problem.

There are 2 sports cars, 3 sport utility vehicles, and 4 trucks. Therefore, the total number of arrangements is given by $\frac{9!}{2!3!4!}$.

$$\begin{aligned}&\frac{9!}{2!3!4!}\\&=\frac{9\times8\times7\times6\times5\times4!}{2!3!4!}\\&=\frac{9\times8\times7\times6\times5}{2!3!}\\&=\frac{9\times8\times7\times6\times5}{(2\times1)(3\times2\times1)}\\&=\frac{15120}{12}\\&=1260\end{aligned}$$

Therefore, the dealership can arrange the 9 vehicles in 1 260 ways.

5. There are 12 women auditioning, and 4 are to be assigned. Since the order in which the roles are assigned matters, this is a permutation problem. To assign 4 women out of 12, use the notation ${}_{12}P_4$.

Step 1
Substitute 12 for *n* and 4 for *r* in the formula ${}_nP_r=\frac{n!}{(n-r)!}$.

$$\begin{aligned}{}_nP_r&=\frac{n!}{(n-r)!}\\{}_{12}P_4&=\frac{12!}{(12-4)!}\\{}_{12}P_4&=\frac{12!}{8!}\end{aligned}$$

Step 2
Evaluate $\frac{12!}{8!}$.

$$\begin{aligned}&\frac{12!}{8!}\\&=\frac{12\times11\times10\times9\times8!}{8!}\\&=12\times11\times10\times9\\&=11880\end{aligned}$$

Therefore, 11 880 different assignments are possible for the 4 roles.

6. The order in which the toppings are chosen does not matter. Therefore, this is a combination problem. Since there are 9 sundae toppings, and a customer can select exactly 3, the number of possible sundaes is given by ${}_9C_3$.

Step 1
In the formula ${}_nC_r=\frac{n!}{(n-r)!r!}$, substitute 9 for *n* and 3 for *r*.

$$\begin{aligned}{}_nC_r&=\frac{n!}{(n-r)!r!}\\{}_9C_3&=\frac{9!}{(9-3)!3!}\\{}_9C_3&=\frac{9!}{6!3!}\end{aligned}$$

Step 2
Evaluate $\frac{9!}{6!3!}$.

$$\begin{aligned}&\frac{9!}{6!3!}\\&=\frac{9\times8\times7\times6!}{6!3!}\\&=\frac{9\times8\times7}{3!}\\&=\frac{9\times8\times7}{3\times2\times1}\\&=\frac{504}{6}\\&=84\end{aligned}$$

Therefore, a customer can order 84 different three-topping sundaes.

7. The order in which the pizza toppings are chosen does not matter. Therefore, this is a combination problem.

Step 1
Determine an equation that represents the number of selections of 2 toppings out of *n* toppings.

In the formula ${}_nC_r = \frac{n!}{(n-r)!r!}$, substitute 2 for *r*.

$$\begin{aligned} {}_nC_r &= \frac{n!}{(n-r)!r!} \\ 190 &= \frac{n!}{(n-2)!2!} \end{aligned}$$

Step 2
Determine how many toppings are used at Vinny's Pizzeria.

Since Vinny can make 190 different two-topping specials, solve for *n* in the equation $190 = \frac{n!}{(n-2)!2!}$.

$$\begin{aligned} 190 &= \frac{n!}{(n-2)!2!} \\ 190 &= \frac{n(n-1)(n-2)!}{(n-2)!2!} \\ 190 &= \frac{n(n-1)}{2!} \\ 380 &= n(n-1) \\ 380 &= n^2 - n \\ 0 &= n^2 - n - 380 \end{aligned}$$

Factor $0 = n^2 - n - 380$.

$$\begin{aligned} 0 &= n^2 - n - 380 \\ 0 &= (n-20)(n+19) \end{aligned}$$

$$n - 20 = 0 \qquad n + 19 = 0$$
$$n = 20 \qquad n = -19$$

Since *n* cannot be negative, there are 20 toppings used at Vinny's Pizzeria.

8. The order that the boys and girls are chosen for the team does not matter. Therefore, this is a combination problem.

Step 1
Determine the number of selections of 3 girls from a group of 7.

In the formula ${}_nC_r = \frac{n!}{(n-r)!r!}$, substitute 3 for *n* and 7 for *r*, and simplify.

$$\begin{aligned} {}_nC_r &= \frac{n!}{(n-r)!r!} \\ {}_7C_3 &= \frac{7!}{(7-3)!3!} \\ {}_7C_3 &= \frac{7!}{4!3!} \\ {}_7C_3 &= \frac{7\times6\times5\times4!}{4!3!} \\ {}_7C_3 &= \frac{7\times6\times5}{3!} \\ {}_7C_3 &= \frac{7\times6\times5}{3\times2\times1} \\ {}_7C_3 &= \frac{210}{6} \\ {}_7C_3 &= 35 \end{aligned}$$

Step 2
Determine the number of selections of 3 boys from a group of 8.

In the formula ${}_nC_r = \frac{n!}{(n-r)!r!}$, substitute 3 for *n* and 8 for *r*, and simplify.

$$\begin{aligned} {}_nC_r &= \frac{n!}{(n-r)!r!} \\ {}_8C_3 &= \frac{8!}{(8-3)!3!} \\ {}_8C_3 &= \frac{8!}{5!3!} \\ {}_8C_3 &= \frac{8\times7\times6\times5!}{5!3!} \\ {}_8C_3 &= \frac{8\times7\times6}{3\times2\times1} \\ {}_8C_3 &= \frac{336}{6} \\ {}_8C_3 &= 56 \end{aligned}$$

Step 3
Determine the total number of different selections for the volleyball team.

Applying the fundamental counting principle, the total number of different selections for the volleyball team is $35\times56 = 1960$.

9. **Step 1**
Divide both sides by $(n + 4)!$.

$$(n+6)! = 72(n+4)!$$
$$\frac{(n+6)!}{(n+4)!} = 72$$

Step 2
Rewrite $(n + 6)!$ as $(n + 6)(n + 5)(n + 4)!$, and divide out common factors.

$$\frac{(n+6)!}{(n+4)!} = 72$$
$$\frac{(n+6)(n+5)(n+4)!}{(n+4)!} = 72$$
$$(n+6)(n+5) = 72$$

Step 3
Solve for n.

$$(n+6)(n+5) = 72$$
$$n^2 + 11n + 30 = 72$$
$$n^2 + 11n - 42 = 0$$

Factor $n^2 + 11n - 42 = 0$.

$$n^2 + 11n - 42 = 0$$
$$(n+14)(n-3) = 0$$
$$n + 14 = 0 \qquad n - 3 = 0$$
$$n = -14 \qquad n = 3$$

Since n cannot be negative, the value of n is 3.

10. **Step 1**

Apply the formula ${}_nP_r = \frac{n!}{(n-r)!}$.

In the equation ${}_nP_2 = {}_nC_3$, rewrite ${}_nP_2$ as $\frac{n!}{(n-2)!}$.

$${}_nP_2 = {}_nC_3$$
$$\frac{n!}{(n-2)!} = {}_nC_3$$

Step 2

Apply the formula ${}_nC_r = \frac{n!}{(n-r)!r!}$.

In the equation $\frac{n!}{(n-2)!} = {}_nC_3$, rewrite ${}_nC_3$ as

$\frac{n!}{(n-3)!3!}$.

$$\frac{n!}{(n-2)!} = {}_nC_3$$
$$\frac{n!}{(n-2)!} = \frac{n!}{(n-3)!3!}$$

Step 3
Solve for n.

$$\frac{n!}{(n-2)!} = \frac{n!}{(n-3)!3!}$$
$$n!(n-3)!3! = n!(n-2)!$$
$$(n-3)!3! = (n-2)!$$
$$(n-3)!3! = (n-2)(n-3)!$$
$$3! = (n-2)$$
$$6 = n-2$$
$$8 = n$$

The value of n is 8.

11. **Step 1**

Apply the formula ${}_nP_r = \frac{n!}{(n-r)!}$.

In the equation ${}_nP_4 = 12({}_nC_3)$, rewrite ${}_nP_4$ as $\frac{n!}{(n-4)!}$.

$${}_nP_4 = 12({}_nC_3)$$
$$\frac{n!}{(n-4)!} = 12({}_nC_3)$$

Step 2

Apply the formula ${}_nC_r = \frac{n!}{(n-r)!r!}$.

In the equation $\frac{n!}{(n-4)!} = 4({}_nC_3)$, rewrite ${}_nC_3$

as $\frac{n!}{(n-3)!3!}$.

$$\frac{n!}{(n-4)!} = 12({}_nC_3)$$
$$\frac{n!}{(n-4)!} = 12\left(\frac{n!}{(n-3)!3!}\right)$$
$$\frac{n!}{(n-4)!} = \frac{12n!}{(n-3)!3!}$$

Step 3
Solve for n.

$$\frac{n!}{(n-4)!} = \frac{12n!}{(n-3)!3!}$$
$$n!(n-3)!3! = 12n!(n-4)!$$
$$(n-3)!3! = 12(n-4)!$$
$$(n-3)(n-4)!3! = 12(n-4)!$$
$$(n-3)3! = 12$$
$$(n-3)6 = 12$$
$$n-3 = 2$$
$$n = 5$$

The value of n is 5.

12. Step 1

Apply the binomial theorem, and substitute 4 for *n*, 6*a* for *x*, and $-\frac{5}{b}$ for *y*.

$$\begin{aligned}(x+y)^n &= {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_ny^n \\ \left(6a-\frac{5}{b}\right)^4 &= {}_4C_0(6a)^4 + {}_4C_1(6a)^{4-1}\left(-\frac{5}{b}\right)^1 \\ &\quad + {}_4C_2(6a)^{4-2}\left(-\frac{5}{b}\right)^2 + {}_4C_3(6a)^{4-3}\left(-\frac{5}{b}\right)^3 + {}_4C_4\left(-\frac{5}{b}\right)^4\end{aligned}$$

Step 2
Simplify.

$$\begin{aligned}&{}_4C_0(6a)^4 + {}_4C_1(6a)^{4-1}\left(-\frac{5}{b}\right)^1 + {}_4C_2(6a)^{4-2}\left(-\frac{5}{b}\right)^2 + {}_4C_3(6a)^{4-3}\left(-\frac{5}{b}\right)^3 + {}_4C_4\left(-\frac{5}{b}\right)^4 \\ &= 1(6a)^4 + 4(6a)^3\left(-\frac{5}{b}\right)^1 + 6(6a)^2\left(-\frac{5}{b}\right)^2 + 4(6a)^1\left(-\frac{5}{b}\right)^3 + 1\left(-\frac{5}{b}\right)^4 \\ &= 1296a^4 + 4\left(216a^3\right)\left(-\frac{5}{b}\right) + 6\left(36a^2\right)\left(\frac{25}{b^2}\right) + 4(6a)\left(-\frac{125}{b^3}\right) + \frac{625}{b^4} \\ &= 1296a^4 - \frac{4320a^3}{b} + \frac{5400a^2}{b^2} - \frac{3000a}{b^3} + \frac{625}{b^4}\end{aligned}$$

13. Step 1
Apply the binomial theorem, and substitute 5 for *n*, –8*x* for *x*, and y^2 for *y*.

$$\begin{aligned}(x+y)^n &= {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_ny^n \\ \left(-8x+y^2\right)^5 &= {}_5C_0(-8x)^5 + {}_5C_1(-8x)^{5-1}\left(y^2\right) + {}_5C_2(-8x)^{5-2}\left(y^2\right)^2 + {}_5C_3(-8x)^{5-3}\left(y^2\right)^3 \\ &\quad + {}_5C_4(-8x)^{5-4}\left(y^2\right)^4 + {}_5C_5\left(y^2\right)^5\end{aligned}$$

Step 2
Simplify.

$$\begin{aligned}&{}_5C_0(-8x)^5 + {}_5C_1(-8x)^{5-1}\left(y^2\right) + {}_5C_2(-8x)^{5-2}\left(y^2\right)^2 + {}_5C_3(-8x)^{5-3}\left(y^2\right)^3 + {}_5C_4(-8x)^{5-4}\left(y^2\right)^4 + {}_5C_5\left(y^2\right)^5 \\ &= 1(-8x)^5 + 5(-8x)^4\left(y^2\right) + 10(-8x)^3\left(y^2\right)^2 + 10(-8x)^2\left(y^2\right)^3 + 5(-8x)^1\left(y^2\right)^4 + 1\left(y^2\right)^5 \\ &= -32768x^5 + 5\left(4096x^4\right)\left(y^2\right) + 10\left(-512x^3\right)\left(y^4\right) + 10(64x)^2\left(y^6\right) + 5(-8x)\left(y^8\right) + y^{10} \\ &= -32768x^5 + 20480x^4y^2 - 5120x^3y^4 + 640x^2y^6 - 40xy^8 + y^{10}\end{aligned}$$

14. Step 1

Determine the general term of the expansion of $\left(2a^3-b\right)^7$.

In the equation $t_{k+1} = {}_nC_kx^{n-k}y^k$, let $n=7$, $x=2a^3$, and $y=-b$.

$$\begin{aligned}t_{k+1} &= {}_nC_kx^{n-k}y^k \\ t_{k+1} &= {}_7C_k\left(2a^3\right)^{7-k}(-b)^k\end{aligned}$$

The general term of the expansion is $t_{k+1} = {}_7C_k\left(2a^3\right)^{7-k}(-b)^k$.

Step 2
Determine the value of *k*.

Since the term that contains b^2 is required, $(-b)^k$ must be b^2, which makes $k=2$.

Step 3
Determine the term in the expansion that contains b^2.

Let $k = 2$ in equation $t_{k+1} = {}_7C_k\left(2a^3\right)^{7-k}(-b)^k$, and simplify.

$$t_{k+1} = {}_7C_k\left(2a^3\right)^{7-k}(-b)^k$$
$$t_{2+1} = {}_7C_3\left(2a^3\right)^{7-2}(-b)^2$$
$$t_3 = 21\left(2a^3\right)^5(-b)^2$$
$$t_3 = 21\left(32a^{15}\right)b^2$$
$$t_3 = 672a^{15}b^2$$

Therefore, the term in the expansion that contains b^2 is $672a^{15}b^2$.

15. Step 1

In the equation $t_{k+1} = {}_nC_kx^{n-k}y^k$, let $n = 10$, $x = p^3$, and $y = -\frac{1}{p^2}$.

$$t_{k+1} = {}_nC_kx^{n-k}y^k$$
$$t_{k+1} = {}_{10}C_k\left(p^3\right)^{10-k}\left(-\frac{1}{p^2}\right)^k$$

Step 2
Apply the laws of exponents, and simplify.

$$t_{k+1} = {}_{10}C_k\left(p^3\right)^{10-k}\left(-\frac{1}{p^2}\right)^k$$
$$t_{k+1} = {}_{10}C_kp^{30-3k}\left(-\frac{1}{p^{2k}}\right)$$
$$t_{k+1} = -{}_{10}C_k\frac{p^{30-3k}}{p^{2k}}$$
$$t_{k+1} = -{}_{10}C_kp^{30-5k}$$

Step 3
Determine the value of *k*.

Since the term containing p^5 is required, equate the exponent of p^5 to the exponent of p^{30-5k}, and solve for *k*.

$$30 - 5k = 5$$
$$-5k = -25$$
$$k = 5$$

Step 4
Determine the coefficient of the term
containing p^5.

Substitute 5 for *k* in the equation $t_{k+1} = -{}_{10}C_kp^{30-5k}$, and simplify.

$$t_{k+1} = -{}_{10}C_kp^{30-5k}$$
$$t_{5+1} = -{}_{10}C_5p^{30-5(5)}$$
$$t_6 = -252p^5$$

The coefficient of the term containing p^5 is –252.

